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Concentration Phenomena for (p, N)-Laplace Equation Under Discontinuous Nonlinearities and Penalization Method

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Abstract

In this paper, we investigate the existence and concentration of solutions to a (p, N)-Laplace equation in $\mathbb{R}^N$ involving a discontinuous nonlinearity and critical exponential growth. To establish the existence of solutions, we employ a penalization technique in the sense of Del Pino and Felmer adapted to a locally Lipschitz functional. Furthermore, by combining variational methods with Moser-type iteration techniques, we obtain the concentration behavior of the solutions. Our results contribute to the study of nonlinear elliptic problems with irregular nonlinearities and critical growth phenomena.

Keywords (p, N)-Laplace · Cerami sequence · Nonsmooth analysis · Discontinuous nonlinearity · Penalization method · Concentration phenomenon

Mathematics Subject Classification 35J60 · 35A15 · 35B38 · 35J62 · 49J52

Contents

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1 Introduction

Our aim is to study the following problem

$$\begin{cases} -\epsilon^p \Delta_p u - \epsilon^N \Delta_N u + V(x)(|u|^{p-2}u + |u|^{N-2}u) = H(u-\beta)f(u) \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x < +\infty, \int_{\mathbb{R}^N} V(x)|u|^N \mathrm{d}x < +\infty, \text{ and} \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N), \end{cases} \tag{$\mathscr{P}_{\epsilon,\beta}$}$$

where $2 < p < N$, $\epsilon, \beta > 0$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ for $r \in \{p, N\}$, and H is the Heaviside function. More assumptions on f and V will be followed. Many authors in recent decades have focused on Schrödinger equation

$$\iota\epsilon \frac{\partial \psi}{\partial t} = -\epsilon^2 \Delta\psi + (V(x) + E)\psi - f(\psi) \text{ in } \mathbb{R}^N,$$

where $\epsilon > 0$. We note the corresponding steady-state problem be described as

$$-\epsilon^2 \Delta u + W(x)u = f(u) \text{ in } \mathbb{R}^N,$$

which is equivalently written as under the change of variable $x \mapsto \epsilon x$:

$$-\Delta u + W(\epsilon x)u = f(u) \text{ in } \mathbb{R}^N.$$

In del Pino and Felmer (1996), for $p = 2 < N$, the authors explored that the solutions of the equation concentrate on the local minimum of $V(x)$ in a bounded domain. Later, as a general case, many authors have also studied the quasilinear problem

$$-\epsilon^p \Delta_p u + W(x)|u|^{p-2}u = f(u) \text{ in } \mathbb{R}^N. \tag{1.1}$$

In the case when $\epsilon = 1$, the problem (1.1) models several steady state cases for non-Newtonian fluids and pseudo-plastic fluids. When $p = 2$, i.e., in the problem (1.1) when $\epsilon \to 0$, the solutions concentrate at global minimum points of W (see Wang (1993)). In Alves and Figueiredo (2005, 2006), the authors discussed the existence, multiplicity and concentration of positive solutions when $1 < p < N$. Moreover, in Alves and Figueiredo (2009), the authors considered the following equation

$$-\epsilon^N \Delta_N u + W(x)|u|^{N-2}u = f(u) \text{ in } \mathbb{R}^N. \tag{1.2}$$

Where the source term arises from the Moser-Trudinger type inequality (Do 1997). For continuous nonlinearities with (p, N)-Laplace operator, we also refer to Li et al. (2025). In addition to other significant works related to the concentration of solutions involving the (p, q)-Laplace operator, we also refer to Zhang et al. (2022). We also refer to Chen and Tang (2025) for a comprehensive review on Schrödinger equation. So far, the literature reviewed above has mainly treated the source term as continuous.

Now, in the direction where the source term could exhibit discontinuous nonlinearity, we recall some notable works. In Gazzola and Rădulescu (2000), the authors have considered the existence of a solution for the following class of problem

$$\begin{aligned} Lu + W(x)u &= f(x, u) \text{ in } \mathbb{R}^N \\ u &> 0 \text{ in } \mathbb{R}^N, \end{aligned}$$

where L is a general second-order elliptic operator, W is coercive and continuous, and f is discontinuous function with subcritical growth. Among the other notable works, with discontinuous nonlinearities we refer to Alves and Nascimento (2013), Alves et al. (2002, 2011, 2014, 2021), Badiale (1993), Ambrosio (2023, 2024a), Alves and Mukherjee (2021), Kim (2018), dos Santos et al. (2020), Zhang and Jia (2021), Liu et al. (2022), Ambrosio and Di Donato (2023) and Ankit and Sarkar (2026) and the references therein.

Motivated by the works mentioned above, we consider the problem ($\mathscr{P}_{\epsilon,\beta}$) and study the existence and concentration phenomena of solutions corresponding to the information available about V.

The function $V : \mathbb{R}^N \to \mathbb{R}$ verifies the following conditions:

(V1) V is positive continuous function and $V(x) \geq V_0 > 0$, for all $x \in \mathbb{R}^N$.
(V2) There exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 = \inf_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

Furthermore, the source-term f satisfies following assumptions:

(f1) f is continuous and has critical exponential growth, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{|t| \to +\infty} |f(t)| \exp(-\alpha |t|^{\frac{N}{N-1}}) = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty & \text{if } \alpha < \alpha_0. \end{cases}$$

(f2) There exists $\zeta > 0$ such that $F(t) \geq \zeta |t|^{N+1}$ for all $t \in \mathbb{R}$, where $F(s) = \int_0^s f(t)\mathrm{d}t$.

(f3) $\limsup_{t\to 0}\frac{f(t)}{t^{N-1}} = 0.$

(f4) The map $t \mapsto \frac{f(t)}{|t|^{N-2}t}$ is an increasing function for all $t > 0$ and decreasing for all $t < 0$.

(f5) There exists $\delta > 1$ such that $\delta\mathscr{G}(t) \geq \mathscr{G}(st)$ for all $t \in \mathbb{R}$, $s \in [0, 1]$ and $\mathscr{G}(s) = sf(s) - NF(s)$.

Remark 1.1 Note that, **(f2)** implies $\lim_{|t|\to\infty}\frac{F(t)}{|t|^N} = +\infty$, where $F(s) = \int_0^s f(t)\mathrm{d}t$.

Example 1.2 Here, we give two examples of the functions that satisfy all the conditions **(f1)**–**(f5)**. We define $f : \mathbb{R} \to \mathbb{R}$ as follows:

(i) $f(t) = \operatorname{sgn}(t)|t|^{N-1+a}\exp\left(\alpha_0|t|^{\frac{N}{N-1}}\right),\ a \in (0, 1],\ \alpha_0 > 0.$

(ii) $f(t) = t|t|^{N-2}(\sqrt{t} + \Phi_1(t))$, where Φ_1 defined in (1.3).

Definition 1.3 (*Weak Solution*) We say $u \in \mathbf{X}_\epsilon$ (defined in Sect. 3) is a weak solution of problem $(\mathscr{P}_{\epsilon,\beta})$, if there is $\rho_0 \in L^{\tilde{\Phi}_1}(\mathbb{R}^N)$(defined in Sect. 2) such that

$$\sum_{t\in\{p,N\}}\int_{\mathbb{R}^N}(|\nabla u|^{t-2}\nabla u\cdot\nabla v + V(\epsilon x)|u|^{t-2}uv)\mathrm{d}x - \int_{\mathbb{R}^N}\rho_0 v\mathrm{d}x = 0, \quad \forall\, v \in \mathbf{X}_\epsilon$$

and $\rho_0(x) \in [\underline{f_H}(u(x)), \overline{f_H}(u(x))]$ a.e. in $\mathbb{R}^N$ where $f_H(t) = H(t-\beta)f(t)$ and

$$\underline{f_H}(t) = \lim_{\delta\to 0}\operatorname*{ess\,inf}_{|s-t|<\delta} f_H(s) \text{ and } \overline{f_H}(t) = \lim_{\delta\to 0}\operatorname*{ess\,sup}_{|s-t|<\delta} f_H(s),$$

with

$$\Phi_1(t) = \exp(|t|^{\frac{N}{N-1}}) - \sum_{j=0}^{N-2}\frac{|t|^{\frac{Nj}{N-1}}}{j!}, \tag{1.3}$$

and $\tilde{\Phi}_1$ is defined in Definition 2.5.

Now we state our main theorem.

Theorem 1.4 (Existence and Concentration) *Assume that conditions* **(V1)**–**(V2)** *and* **(f1)**–**(f5)** *are fulfilled. Then, there exists positive* $\tilde{\epsilon}, \tilde{\beta}$ *such that for all* $\epsilon \in (0, \tilde{\epsilon})$ *and* $\beta \in (0, \tilde{\beta})$, *problem* $(\mathscr{P}_{\epsilon,\beta})$ *possesses a weak solution* $u_{\epsilon,\beta} \in \mathbf{X}_\epsilon$. *Furthermore, if* $x_{\epsilon,\beta} \in \mathbb{R}^N$ *denotes a point at which* $u_{\epsilon,\beta}$ *achieves its maximum, then* $\lim_{(\epsilon,\beta)\to(0,0)} V(\epsilon x_{\epsilon,\beta}) = V_0$.

The present work establishes new existence and concentration results for a class of (p, N)-Laplace equations with discontinuous nonlinearities and critical exponential growth in $\mathbb{R}^N$. By combining Clarke's nonsmooth critical point theory with variational tools associated with the Moser–Trudinger inequality, we develop a unified analytical framework for treating mixed-growth (p, N) operators under critical exponential conditions. In addition, we demonstrate that the solutions concentrate near the global minima of the potential. To the best of our knowledge, this is the first contribution addressing such a mixed-growth, discontinuous, critically exponential problem in the absence of the Ambrosetti–Rabinowitz condition.

This type of result remains new even when replacing the (p, N)-Laplace operator by the pure N-Laplacian. In this sense, our theorem extends and complements the study carried out in Alves et al. (2014), as we consider a more general operator and employ new arguments that are necessary due to the absence of the Ambrosetti–Rabinowitz condition, the generality of the operator, and the presence of critical exponential growth.

There has been a growing interest in the analysis of nonlinear partial differential equations featuring discontinuous nonlinearities, due to their significance in several free-boundary problems in mathematical physics. Key examples of these issues are the obstacle problem, the seepage surface problem, and the Elenbaas equation. We refer to Chang (1981) and the references cited there for more such applications.

This article is organized as follows. In Sect. 2, we recall some results that are crucial in our proofs. The Sect. 3 deals with the existence of solutions for the auxiliary problem $(\mathscr{P}^a_{\epsilon,\beta})$. Section 4 is concerned with the study of the autonomous problem related to $(\mathscr{P}_{\epsilon,\beta})$. Finally, Sect. 5 contains the proof of the main theorem.

Notations: Throughout the paper, we will use the following notations:

- $X \hookrightarrow Y$ denotes the continuous embedding of X into Y.
- $X \hookrightarrow\hookrightarrow Y$ denotes compact embedding of X into Y.
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to +\infty$.
- $C_1, C_2, C_3, \cdots$ all are positive constants and may have different values at different places.
- $\rightharpoonup$ denotes weak convergence, $\overset{*}{\rightharpoonup}$ denotes weak* convergence and $\to$ denotes strong convergence.
- $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is volume of $N-1$ dimensional unit sphere.
- $[u > d] := \{x \in \mathbb{R}^N : u(x) > d\}$.
- $\partial f(x)$ denote Clarke's generalized gradient of function f at x.
- $\|u\|_{L^p} = \left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \ 1 \le p < \infty$.
- $B_R \subset \mathbb{R}^N$ denote the ball of radius $R > 0$ centered at origin.
- $\mathrm{m}(S)$ denotes the N-dimensional Lebesgue measure for $S \subset \mathbb{R}^N$.
- X^* denotes the dual space of X.

2 Preliminaries

In this section, we recall some important existing results useful for our arguments.

Definition 2.1 (*Generalized Directional Derivative*) The generalized directional derivative of J at x in the direction h is given by

$$J^o(x; h) = \limsup_{y \to x, \lambda \to 0} \frac{J(y + \lambda h) - J(y)}{\lambda}.$$

The function $h \mapsto J^0(x, h)$ is a subadditive, continuous, and convex function. Hence, its generalized gradient in the Clark sense is given by

$$\partial J(x) = \{x^* \in X^* : \langle x^*, h\rangle_X \leq J^0(x; h), \ \forall\, h \in X\}.$$

For each $x \in X$, $\partial J(x)$ is non empty, convex and weak*-compact subset of X^*. Moreover, the function

$$\lambda(x) = \min_{x^* \in \partial J(x)} \|x^*\|_{X^*} \tag{2.1}$$

exists and is lower semi-continuous. If $J \in C^1(X, \mathbb{R})$, then $\partial J(x) = \{J'(x)\}$. A point $x_0 \in X$ is called a critical point for J if $0 \in \partial J(x)$. For more details on this topic, we refer to the monograph (Clarke, 1983).

Definition 2.2 (*Cerami Condition–Non Smooth Version* Stuart 2011) Let X be a Banach space and $\phi : X \to \mathbb{R}$ be a locally Lipschitz functional. Then one say that ϕ satisfies the non smooth Cerami condition at level $c \in \mathbb{R}$, denoted as $(C)_c$, if every sequence $\{x_n\} \subseteq X$ such that

$$\phi(x_n) \to c \text{ and } (1 + \|x_n\|_X)\lambda(x_n) \to 0,$$

has a strongly convergent subsequence.

Definition 2.3 (*Orlicz space*) The Orlicz space associated with Θ (an N-function) as

$$L^{\Theta}(\mathbb{R}^N) = \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \Theta\left(\frac{|u|}{\lambda}\right) \mathrm{d}x < +\infty \text{ for some } \lambda > 0 \right\}.$$

The space $L^{\Theta}(\mathbb{R}^N)$ is Banach space endowed with following norm

$$\|u\|_{\Theta} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \Theta\left(\frac{|u|}{\lambda}\right) \mathrm{d}x \leq 1 \right\}.$$

For more on Orlicz spaces, we refer to Rao and Ren (1991).

Definition 2.4 We have $E_{\Theta}(\mathbb{R}^N)$ as a subspace of $L^{\Theta}(\mathbb{R}^N)$ in the following way,

$$E_{\Theta}(\mathbb{R}^N) = \overline{\left\{u \in L^1_{loc}(\mathbb{R}^N) : u \text{ having bounded support on } \mathbb{R}^N\right\}}^{\|\cdot\|_{\Theta}}.$$

Then, equivalently, we also have

$$E_{\Theta}(\mathbb{R}^N) = \overline{C_c^{\infty}(\mathbb{R}^N)}^{\|\cdot\|_{\Theta}}.$$

Definition 2.5 (*Complementary/conjugate of a function*) The Complementary function $\tilde{\Theta}$ associated with Θ is given by

$$\tilde{\Theta}(s) = \sup_{t\geq 0}\{st - \Theta(t)\},\ s \geq 0.$$

Next, we recall some Lemmas without proof, which will be crucial for establishing our arguments.

Lemma 2.1 (Moser-Trudinger Inequality Do 1997, Lemma 1) *For any $\alpha > 0$, $N \geq 2$ and $u \in W^{1,N}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \Phi(\alpha|u|^{\frac{N}{N-1}})\mathrm{d}x < +\infty.$$

Moreover, if $\|\nabla u\|_N \leq 1$ and $\|u\|_N \leq M < +\infty$ and $\alpha < \alpha_N$ there exist a constant $C > 0$ such that

$$\sup_{\|\nabla u\|_N\leq 1, \|u\|_N\leq M} \int_{\mathbb{R}^N} \Phi\left(\alpha|u|^{\frac{N}{N-1}}\right)\mathrm{d}x \leq C,$$

where $\Phi(t) = \exp(t) - \sum_{j=0}^{N-2} \frac{t^j}{j!}$, $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and $\|u\|_N = (\int_{\mathbb{R}^N} |u|^N\mathrm{d}x)^{\frac{1}{N}}$.

Here we would like to note that $\Phi_1(t) = \Phi(t^{\frac{N}{N-1}})$.

Lemma 2.2 (Ambrosio 2024b) *Let $\boldsymbol{Y}$ be a Banach space and $I \in C^1(\boldsymbol{Y}, \mathbb{R})$. Let $\{u_n\}_{n\geq 1} \in \mathcal{N}_0$ such that $I(u_n) \to u_0$. Then $\{u_n\}_{n\geq 1}$ has a strongly convergent subsequence in $\boldsymbol{Y}$, where $\mathcal{N}_0$ denote Nehari manifold associated with functional I, defined as*

$$\mathcal{N}_0 := \{u \in W_{V_0}^{1,p}(\mathbb{R}^N) \cap W_{V_0}^{1,N}(\mathbb{R}^N) \setminus \{0\} :\ \langle I_{V_0}'(u), u\rangle = 0\}.$$

Lemma 2.3 (Alves et al. 2014, Proposition 2.2) *Let X be a real Banach space. Suppose that $\varphi \in Lip_{loc}(X, \mathbb{R})$. Let $\{x_n\}_{n\geq 1} \subset X$ and $\{\rho_n\}_{n\geq 1} \subset X^*$ with $\rho_n \in \partial\varphi(x_n)$. If $x_n \to x$ in X and $\rho_n \overset{*}{\rightharpoonup} \rho_0$ in X^*, then $\rho_0 \in \partial\varphi(x)$.*

Lemma 2.4 (Ankit and Sarkar 2026, Lemma 2.5) *Let $\varsigma(t) = \max\{t, t^N\}$ and $\tilde{\Phi}_1$ be conjugate function associated with Φ_1. Then, the following inequalities are satisfied:*

(i) $\tilde{\Phi}_1(\frac{\Phi_1(r)}{r}) \leq \Phi_1(r),\ \forall\, r > 0.$
(ii) $\tilde{\Phi}_1(tr) \leq \varsigma(t)\tilde{\Phi}_1(r),\ \forall\, t, r \geq 0.$

Hence, $\tilde{\Phi}_1 \in \Delta_2$ and $E_{\tilde{\Phi}_1}(\mathbb{R}^N) = L^{\tilde{\Phi}1}(\mathbb{R}^N)$.

Lemma 2.5 (Ankit and Sarkar 2026, Lemma 2.6) *Let $E_{\Phi_1}(\mathbb{R}^N)$(defined in Definition 2.4) be a subspace of an Orlicz space $L^{\Phi_1}(\mathbb{R}^N)$. Then, the following embeddings hold:*

(i) $\mathbf{X}_\epsilon \hookrightarrow E_{\Phi_1}(\mathbb{R}^N)$, *and*
(ii) $E_{\Phi_1}(\mathbb{R}^N) \hookrightarrow L^N(\mathbb{R}^N)$.

3 Auxiliary Problem

Now, in this section, we will study the auxiliary problem instead of the ($\mathscr{P}_{\epsilon,\beta}$). The auxiliary problem is found under the change of variable $x \mapsto \epsilon x$. Our motivation to study under these circumstances is based on the findings of del Pino and Felmer (1996) and Alves et al. (2014).

To begin with, first, we fix $k, a, \beta > 0$ such that $\beta < a < k$ and $\frac{f(a)}{a^{N-1}} = \frac{V_0}{k}$, and we define

$$\tilde{f}(s) = \begin{cases} f(s)\ , s < a \\ \frac{V_0 s^{N-1}}{k},\ s \geq a. \end{cases}$$

Note that $\tilde{f}$ is a continuous function on $\mathbb{R}$. Define

$$g(x,t) = \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t). \tag{3.1}$$

One can easily observe that the function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function such that it satisfy following properties:

(g1) $g(x,t)$ also having critical exponential growth i.e. there exist $\alpha_0 > 0$ such that

$$\lim_{|t| \to +\infty} |g(x,t)| \exp(-\alpha |t|^{\frac{N}{N-1}}) = \begin{cases} 0 \ \text{if}\ \alpha > \alpha_0, +\infty \ \text{if}\ \alpha < \alpha_0. \end{cases} \quad \text{uniformly a.e. in } x \in \mathbb{R}^N.$$

(g2) There exists $\zeta > 0$ such that $G(x,t) \geq \zeta |t|^{N+1}$, for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^N$, where $G(x,t) = \int_0^t g(x,s)\mathrm{d}s$.

(g3) $\limsup_{t \to 0} \frac{g(x,t)}{t^{N-1}} = 0$, uniformly a.e. in $x \in \mathbb{R}^N$.

(g4) The map $t \mapsto \frac{g(x,t)}{|t|^{N-2}t}$ is an increasing function for all $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ and decreasing for all $(x,t) \in \mathbb{R}^N \times (-\infty, 0)$.

(g5) There exists $\delta > 1$ such that $\delta \mathscr{G}(x,t) \geq \mathscr{G}(x,st)$, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $s \in [0,1]$, where $\mathscr{G}(x,t) = tf(x,t) - NG(x,t)$.

Remark 3.1 We note that **(g2)** implies $\lim_{|t| \to \infty} \frac{G(x,t)}{|t|^N} = +\infty$ uniformly a.e. in $x \in \mathbb{R}^N$, where $G(x,t) = \int_0^t g(x,s)\mathrm{d}s$.

So the auxiliary problem is reduced to the following problem under the transformation $x \mapsto \epsilon x$

$$\begin{cases} -\Delta_p u - \Delta_N u + V(\epsilon x)(|u|^{p-2}u + |u|^{N-2}u) = H(u-\beta) g(\epsilon x, u) \ \text{in}\ \mathbb{R}^N, \\ \int_{\mathbb{R}^N} V(\epsilon x)|u|^p \mathrm{d}x < +\infty\ , \int_{\mathbb{R}^N} V(\epsilon x)|u|^N \mathrm{d}x < +\infty,\ \text{and} \\ u \in W^{1,p}_{V_\epsilon}(\mathbb{R}^N) \cap W^{1,N}_{V_\epsilon}(\mathbb{R}^N), \end{cases} \tag{$\mathscr{P}^a_{\epsilon,\beta}$}$$

where $V_\epsilon(x) := V(\epsilon x)$. For the auxiliary problem, our working space will be

$$\mathbf{X}_\epsilon = W^{1,p}_{V_\epsilon}(\mathbb{R}^N) \cap W^{1,N}_{V_\epsilon}(\mathbb{R}^N)$$

equipped with the norm

$$\|u\|_{\mathbf{X}_\epsilon} = \|u\|_{W^{1,p}_{V_\epsilon}} + \|u\|_{W^{1,N}_{V_\epsilon}},$$

where

$$\|u\|_{W^{1,r}_{V_\epsilon}} = \left(\int_{\mathbb{R}^N} |\nabla u|^r \,\mathrm{d}x + \int_{\mathbb{R}^N} V(\epsilon x)|u|^r \,\mathrm{d}x \right)^{\frac{1}{r}} \text{ for } r \in \{p, N\}.$$

We say that $u \in \mathbf{X}_\epsilon$ is weak solution to ($\mathscr{P}^a_{\epsilon,\beta}$), if there is $\rho_0 \in L^{\Phi_1}(\mathbb{R}^N)$ such that

$$\sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} (|\nabla u|^{t-2}\nabla u \cdot \nabla v + V(\epsilon x)|u|^{t-2}uv)\mathrm{d}x - \int_{\mathbb{R}^N} \rho_0 v \mathrm{d}x = 0, \quad \forall\, v \in \mathbf{X}_\epsilon,$$

and $\rho_0(x) \in [\underline{g_H}(\epsilon x, u(x)), \overline{g_H}(\epsilon x, u(x))]$ a.e. in $\mathbb{R}^N$ with $g_H(x,t) = H(t-\beta)g(x,t)$, where

$$\underline{g_H}(x,t) = \lim_{\delta\to 0} \operatorname*{ess\,inf}_{|s-t|<\delta} g_H(x,s) \text{ and } \overline{g_H}(x,t) = \lim_{\delta\to 0} \operatorname*{ess\,sup}_{|s-t|<\delta} g_H(x,s).$$

Remark 3.2 If $u \in \mathbf{X}_\epsilon$ is weak solution of ($\mathscr{P}^a_{\epsilon,\beta}$) with $u(x) < a$ for all $x \in \Lambda^c_\epsilon$, where $\Lambda_\epsilon = \{x \in \mathbb{R}^N : \epsilon x \in \Lambda\}$, then u is weak solution to ($\mathscr{P}_{\epsilon,\beta}$).

3.1 Mountain Pass Geometry of the Energy Functional

Define the functional $I_{\epsilon,\beta} : \mathbf{X}_\epsilon \to \mathbb{R}$

$$\begin{aligned} I_{\epsilon,\beta}(u) = &\frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\epsilon x)|u|^p)\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N} (|\nabla u|^N \\ &+ V(\epsilon x)|u|^N)\mathrm{d}x - \int_{\mathbb{R}^N} G_H(\epsilon x, u)\mathrm{d}x, \end{aligned} \tag{3.2}$$

where $G_H(x,t) = \int_0^t H(s-\beta)g(x,s)\mathrm{d}s$.

Next, we will consider

$$I_{\epsilon,\beta}(u) = Q_\epsilon(u) - \Upsilon_{\epsilon,\beta}(u),$$

where

$$Q_\epsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V(\epsilon x)|u|^p)\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla u|^N + V(\epsilon x)|u|^N)\mathrm{d}x$$

and

$$\Upsilon_{\epsilon,\beta}(u) = \int_{\mathbb{R}^N} G_H(\epsilon x, u)\mathrm{d}x.$$

In the next Lemma, we discuss the mountain pass geometry for the energy functional.

Lemma 3.1 (Mountain Pass Geometry) *Assume that (g1)–(g3) hold. Then*

(i) $I_{\epsilon,\beta}(0) = 0$ *and the functional* $I_{\epsilon,\beta} \in Lip_{loc}(\mathbf{X}_\epsilon, \mathbb{R})$.
(ii) there exist $\varrho > 0$ *and* $r > 0$ *such that* $I_{\epsilon,\beta}(u) \geq r$ *for all* $\|u\|_{\mathbf{X}_\epsilon} = \varrho$.
(iii) there is $\vartheta \in \mathbf{X}_\epsilon$ *such that* $I_{\epsilon,\beta}(\vartheta) < 0$.

Proof (i) $I_{\epsilon,\beta}(0) = 0$ and second subpart follows from Lemma 3.1 of Ankit and Sarkar (2026).

(ii) From (g1) and (g3), it follows that for any $\tau > 0$ and $\nu > N$, there exist a constant $C_1(\tau)$ (depends only on τ) and $\alpha_0 > 0$ such that

$$|g(x,t)| \leq \tau|t|^{N-1} + C_1(\tau)|t|^{\nu-1}\Phi_{\alpha_0,N-2}(t),$$

and therefore, we get

$$|G(x,t)| \leq \frac{\tau}{N}|t|^N + C_1(\tau)|t|^\nu\Phi_{\alpha_0,N-2}(t), \tag{3.3}$$

where

$$\Phi_{\alpha,N-2}(t) = \exp(\alpha|t|^{\frac{N}{N-1}}) - \sum_{j=0}^{N-2}\frac{(\alpha|t|^{\frac{N}{N-1}})^j}{j!}.$$

Using (3.3), we have

$$\int_{\mathbb{R}^N} G_H(\epsilon x, u)\mathrm{d}x \leq \int_{\mathbb{R}^N} G(\epsilon x, u)\mathrm{d}x \leq \frac{\tau}{N}\int_{\mathbb{R}^N}|u|^N\mathrm{d}x + C_1(\tau)\int_{\mathbb{R}^N}|u|^\nu\Phi_{\alpha_0,N-2}(u)\mathrm{d}x. \tag{3.4}$$

Let $\kappa \in (0,1)$ such that $\|u\|_{\mathbf{X}_\epsilon} \leq \kappa$. Choose $\alpha > \alpha_0$ close to α_0 such that $\alpha\|u\|_{\mathbf{X}_\epsilon}^{\frac{N}{N-1}} < \alpha_N$. So using (Do et al., 2009, Lemma 2.3) and Sobolev embedding, we obtain

$$C_1(\tau)\int_{\mathbb{R}^N}|u|^\nu\Phi_{\alpha_0,N-2}(u)\mathrm{d}x \leq \tilde{C}\|u\|_{\mathbf{X}_\epsilon}^\nu. \tag{3.5}$$

From (3.4) and (3.5), we get

$$\int_{\mathbb{R}^N} G_H(\epsilon x, u)\mathrm{d}x \le \frac{\tau}{N}\|u\|^N_{\mathbf{X}_\epsilon} + C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon}. \tag{3.6}$$

Finally, combining (3.2) and (3.6), we have

$$\begin{aligned}
I_{\epsilon,\beta}(u) &\ge \frac{1}{p}\|u\|^p_{W^{1,p}_{V_\epsilon}} + \frac{1}{N}\|u\|^N_{W^{1,N}_{V_\epsilon}} - \frac{\tau}{N}\|u\|^N_{\mathbf{X}_\epsilon} - C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon}\\
&\ge \frac{\left(\|u\|^p_{W^{1,p}_{V_\epsilon}} + \|u\|^N_{W^{1,N}_{V_\epsilon}}\right)}{N} - \frac{\tau}{N}\|u\|^N_{\mathbf{X}_\epsilon} - C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon}\\
&\ge \frac{2^{1-N}}{N}\left(\|u\|_{W^{1,p}_{V_\epsilon}} + \|u\|_{W^{1,N}_{V_\epsilon}}\right)^N - \frac{\tau}{N}\|u\|^N_{\mathbf{X}_\epsilon} - C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon}\\
&= \frac{2^{1-N}}{N}\|u\|^N_{\mathbf{X}_\epsilon} - \frac{\tau}{N}\|u\|^N_{\mathbf{X}_\epsilon} - C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon}\\
&= \left(\frac{2^{1-N}}{N} - \frac{\tau}{N}\right)\|u\|^N_{\mathbf{X}_\epsilon} - C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon}.
\end{aligned}$$

Choose $\tau = \frac{1}{2^N}$, then

$$I_{\epsilon,\beta}(u) \ge \frac{1}{N2^N}\|u\|^N_{\mathbf{X}_\epsilon} - C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon} \ge \frac{1}{N2^{N+1}}\|u\|^N_{\mathbf{X}_\epsilon} - C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon}.$$

Using the basic calculus, we obtain that $\frac{1}{N2^{N+1}}\|u\|^N_{\mathbf{X}_\epsilon} - C_1(\tau)\|u\|^\nu_{\mathbf{X}_\epsilon}$ attain its maxima at some point say $\varrho \in (0, \kappa]$. So for all $u \in \mathbf{X}_\epsilon$ such that $\|u\|_{\mathbf{X}_\epsilon} = \varrho$, we have

$$I_{\epsilon,\beta}(u) \ge r, \ \forall\ \|u\|_{\mathbf{X}_\epsilon} = \varrho.$$

(iii) Define $\Lambda_\epsilon = \{x \in \mathbb{R}^N : \epsilon x \in \Lambda\}$. Choose $\eta \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ such that $\mathrm{supp}(\eta) \subset \Lambda_\epsilon$. For $t > 0$, we set $u = t\eta$, and then we obtain

$$I_{\epsilon,\beta}(t\eta) = \frac{t^p}{p}\|\eta\|^p_{W^{1,p}_{V_\epsilon}} + \frac{t^N}{N}\|\eta\|^N_{W^{1,N}_{V_\epsilon}} - \int_{\mathbb{R}^N \cap\ \mathrm{supp}(\eta)} G_H(\epsilon x, t\eta)\mathrm{d}x.$$

Due to **(g2)** (Remark 3.1), for any $\tilde{M} > 0$ there is $C_{\tilde{M}} > 0$ such that

$$G(x, t) \ge \tilde{M}|t|^N - C_{\tilde{M}}, \ \forall\ (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Therefore, we get the following estimate

$$I_{\epsilon,\beta}(t\eta) \le \frac{t^p}{p}\|\eta\|^p_{W^{1,p}_{V_\epsilon}} + \frac{t^N}{N}\|\eta\|^N_{W^{1,N}_{V_\epsilon}} - \tilde{M}t^N \int_{\mathbb{R}^N \cap \mathrm{supp}(\eta)} \eta^N \mathrm{d}x + C_{\tilde{M}} m(\mathrm{supp}(\eta) \cap \mathbb{R}).$$

Choose $\tilde{M} > 0$ such that $\frac{1}{N}\|\eta\|^N_{W^{1,N}_{V_\epsilon}} - \tilde{M} \int_{\mathbb{R}^N \cap \text{supp}(\eta)} \eta^N \mathrm{d}x < 0$. Now for large enough t, we have $I_{\epsilon,\beta}(t\eta) \to -\infty$ and hence the proof of (iii) follows.

□

3.2 Compactness of the Energy Functional–Cerami Condition

In this subsection, we discuss the compactness of the energy functional $I_{\epsilon,\beta}$. Before we begin, we first recall a Lemma that ensures the existence of a Cerami sequence.

Lemma 3.2 (Livrea and Marano 2009, Theorem 3.1) *Let X be a real Banach space and $\phi : X \to \mathbb{R}$ be a locally Lipschitz functional. Suppose we have*

$$\max\{\phi(0), \phi(x_1)\} < \inf_{\|x\|_X \leq \rho} \phi(x)$$

for some $x_1 \in X$ with $\|x_1\|_X > \rho$. Let

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where Γ is the collection of paths joining 0 and x_1. Then there is a non-smooth Cerami sequence for the functional ϕ.

By the virtue of Lemmas 3.1 and 3.2, we obtain a Cerami sequence $\{u_n\} \subset \mathbf{X}_\epsilon$ such

$$I_{\epsilon,\beta}(u_n) \to c_{\epsilon,\beta} \text{ and } (1 + \|u_n\|_{\mathbf{X}_\epsilon})\lambda(u_n) \to 0, \tag{3.7}$$

where λ defined in (2.1) and

$$c_{\epsilon,\beta} = \inf_{\gamma \in \Gamma_{\epsilon,\beta}} \max_{t \in [0,1]} I_{\epsilon,\beta}(\gamma(t)),$$

and

$$\Gamma_{\epsilon,\beta} = \{\gamma \in (C[0,1], \mathbf{X}_\epsilon) : I_{\epsilon,\beta}(0) = 0, I_{\epsilon,\beta}(\gamma(1)) < 0\}.$$

The next lemma establishes that such a sequence is bounded. We are influenced by the methods discussed in (Fang and Liu, 2009, Lemma 2.2) and (Lam and Lu, 2013, Lemma 3.2) to prove the preceding lemma.

Lemma 3.3 *Let* **(g1)***,* **(g2)** *and* **(g5)** *hold, then the Cerami sequence $\{u_n\}$ is bounded in $\mathbf{X}_\epsilon$.*

Proof On contrary suppose that the sequence $\{u_n\}$ is not bounded in $\mathbf{X}_\epsilon$. This means

$$\|u_n\|_{\mathbf{X}_\epsilon} \to +\infty \text{ as } n \to +\infty.$$

Define $v_n = \frac{u_n}{\|u_n\|_{\mathbf{X}_\epsilon}}$. Then v_n is bounded in $\mathbf{X}_\epsilon$ as $\|u_n\|_{\mathbf{X}_\epsilon} = 1$, for all $n \in \mathbb{N}$. So, up to a subsequence still denoted by itself, we can assume that

$$\begin{cases} v_n \rightharpoonup v \quad \text{in } \mathbf{X}_\epsilon, \\ v_n(x) \to v(x) \text{ a.e. in } \mathbb{R}^N. \end{cases}$$

Claim: $v = 0$ a.e. in $\mathbb{R}^N$.
Consider the set $S = \{x \in \mathbb{R}^N : v(x) \neq 0\}$ such that $m(S) \neq 0$. If $x \in S$, then, we have $|u_n(x)| = |v_n(x)|\|u_n\|_{\mathbf{X}_\epsilon} \to +\infty$, for $x \in S$ as $n \to +\infty$. From assumption **(g2)** (Remark 3.1), for each $x \in S$, we get

$$\lim_{n\to+\infty} \frac{G(x, u_n(x))}{|u_n(x)|^N} \frac{|u_n(x)|^N}{\|u_n\|^N_{\mathbf{X}_\epsilon}} = \lim_{n\to+\infty} \frac{G(x, u_n(x))}{|u_n(x)|^N}|v_n(x)|^N = +\infty \tag{3.8}$$

and $G(x, t) \geq K_1$, for all $(x, t) \in S \times \mathbb{R}$, for some constant K_1. So

$$\frac{G(x, u_n) - K_1}{\|u_n\|^N_{\mathbf{X}_\epsilon}} \geq 0, \ \forall\, x \in S \text{ and } \forall\, n \in \mathbb{N}.$$

That is

$$\frac{G(x, u_n)|v_n(x)|^N}{|u_n(x)|^N} - \frac{K_1}{\|u_n\|^N_{\mathbf{X}_\epsilon}} \geq 0, \ \forall\, x \in S \text{ and } \forall\, n \in \mathbb{N}.$$

From (3.7), it follows that as $n \to \infty$, we obtain

$$\begin{aligned} c_{\epsilon,\beta} &= I_{\epsilon,\beta}(u_n) + o_n(1), \\ &\geq \frac{1}{N}\left(\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \|u_n\|^N_{W^{1,N}_{V_\epsilon}}\right) - \int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x + o_n(1). \end{aligned}$$

Hence

$$\int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x \geq \frac{1}{N}\left(\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \|u_n\|^N_{W^{1,N}_{V_\epsilon}}\right) - c_{\epsilon,\beta} + o_n(1) \text{ as } n \to +\infty. \tag{3.9}$$

Similarly,

$$\int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x \leq \frac{1}{p}\left(\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \|u_n\|^N_{W^{1,N}_{V_\epsilon}}\right) - c_{\epsilon,\beta} + o_n(1) \text{ as n} \to +\infty. \tag{3.10}$$

We have $G_H(\epsilon x, u_n) \leq G(\epsilon x, u_n)$, for large n, and then by (3.8) and Fatou's Lemma,

$$+\infty = \int_S \liminf_{n\to+\infty} \frac{G(\epsilon x, u_n)}{|u_n|^N}|v_n|^N \mathrm{d}x \leq \liminf_{n\to+\infty} \int_S \frac{G(\epsilon x, u_n)}{\|u_n\|^N_{\mathbf{X}_\epsilon}}\mathrm{d}x. \tag{3.11}$$

Since

$$\|u_n\|_{\mathbf{X}_\epsilon} = \|u_n\|_{W^{1,p}_{V_\epsilon}} + \|u_n\|_{W^{1,N}_{V_\epsilon}} \to +\infty, \text{ as } n \to +\infty.$$

Hence, three cases arise:

(i) Both $\|u_n\|_{W^{1,p}_{V_\epsilon}}$ and $\|u_n\|_{W^{1,N}_{V_\epsilon}}$ are unbounded.
(ii) $\|u_n\|_{W^{1,p}_{V_\epsilon}}$ is bounded and $\|u_n\|_{W^{1,N}_{V_\epsilon}}$ are unbounded.
(iii) $\|u_n\|_{W^{1,p}_{V_\epsilon}}$ is unbounded and $\|u_n\|_{W^{1,N}_{V_\epsilon}}$ are bounded.

Suppose case(i) holds. Then there exists $N_0 \in \mathbb{N}$ such that $\|u_n\|_{W^{1,p}_{V_\epsilon}} > 1$ and $\|u_n\|_{W^{1,N}_{V_\epsilon}} > 1$ for all $n \geq N_0$. This implies $\|u_n\|^p_{W^{1,p}_{V_\epsilon}} \leq \|u_n\|^N_{W^{1,N}_{V_\epsilon}}$. As $n \to \infty$, from (3.9), we have

$$\begin{aligned} c_{\epsilon,\beta} &\geq \frac{1}{N}\left(\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \|u_n\|^N_{W^{1,N}_{V_\epsilon}}\right) - \int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x + o_n(1) \\ &\geq \frac{1}{N2^{p-1}}\|u_n\|^p_{\mathbf{X}_\epsilon} - \int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x + o_n(1). \end{aligned}$$

From (3.11), and $\|u_n\|^p_{W^{1,p}_{V_\epsilon}} \leq \|u_n\|^N_{W^{1,N}_{V_\epsilon}}$ for all $n \geq N_0$, we have

$$\begin{aligned} +\infty \leq \liminf_{n\to+\infty}\int_S \frac{G(\epsilon x, u_n)}{\|u_n\|^N_{\mathbf{X}_\epsilon}}\mathrm{d}x &\leq \liminf_{n\to+\infty}\int_S \frac{G(\epsilon x, u_n)}{\|u_n\|^p_{\mathbf{X}_\epsilon}}\mathrm{d}x \\ &= \liminf_{n\to+\infty} \frac{\int_S G(\epsilon x, u_n)\mathrm{d}x}{N2^{p-1}(c_{\epsilon,\beta} + \int_S G(\epsilon x, u_n)\mathrm{d}x - o_n(1))}. \end{aligned}$$

As $n \to +\infty$, we have

$$+\infty \leq \frac{1}{N2^{p-1}}.$$

which is not possible. So $v = 0$ a.e. in $\mathbb{R}^N$.

Now, consider the case (ii) holds. This means there exists $M > 0$ such that $\|u_n\|_{W^{1,p}_{V_\epsilon}} \leq M$. Then, as $n \to \infty$, we get

$$\begin{aligned} c_{\epsilon,\beta} &\leq \frac{1}{p}\left(\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \|u_n\|^N_{W^{1,N}_{V_\epsilon}}\right) - \int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x + o_n(1), \\ &\leq \frac{M^p}{p} + \frac{1}{p}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} - \int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x + o_n(1). \end{aligned} \tag{3.12}$$

Therefore, as $n \to \infty$, we derive

$$\frac{1}{p}\|u_n\|_{\mathbf{X}_\epsilon}^N \geq c_{\epsilon,\beta} - \frac{M^p}{p} + \int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x + o_n(1).$$

As $n \to \infty$, we have

$$\|u_n\|_{\mathbf{X}_\epsilon}^N \geq pc_{\epsilon,\beta} - M^p + p\int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x + o_n(1).$$

Again, using a similar argument for case (i), we have

$$+\infty \leq \frac{1}{p},$$

which is absurd. So $v = 0$ a.e. in $\mathbb{R}^N$. One can repeat this argument for the case (iii) also. Therefore, we have proven that $v = 0$ almost everywhere in $\mathbb{R}^N$ in all cases.

Next, we define a continuous function $K_n : [0, 1] \to \mathbb{R}$ such that $K_n(t) = I_{\epsilon,\beta}(tu_n)$. Since $I_{\epsilon,\beta}$ is locally Lipschitz is also continuous, K_n is also continuous on the compact interval [0, 1].

For each $n \in \mathbb{N}$, there is a t (depends on n), say t_n such that

$$I_{\epsilon,\beta}(t_n u_n) = \max_{t\in[0,1]} I_{\epsilon,\beta}(tu_n). \tag{3.13}$$

Note that, $t_n \in (0, 1]$ for all n. Indeed, if $t_n = 0$ for some $n \in \mathbb{N}$, as $I_{\epsilon,\beta}(u_n) \to c_{\epsilon,\beta}$, so there is $n_0 \in \mathbb{N}$ such that $I_{\epsilon,\beta}(u_n) \geq \frac{c_{\epsilon,\beta}}{2}$ for all $n \geq n_0$, and there holds

$$0 < \frac{c_{\epsilon,\beta}}{2} < I_{\epsilon,\beta}(u_n) \leq \max_{t\in[0,1]} I_{\epsilon,\beta}(tu_n) = \max_{t\in[0,1]} I_{\epsilon,\beta}(t_n u_n) = I_{\epsilon,\beta}(0) = 0,$$

which is a contradiction. So $t_n \in (0, 1]$ for all n. Let $\{r_k\}$ be a sequence of positive real numbers such that $r_k > 1$ and $\lim_{k\to+\infty} \mathrm{r}_k = +\infty$. Then $\|r_k v_n\|_{\mathbf{X}_\epsilon} = r_k$ for any k and n. For each $k \in \mathbb{N}$ and due to **(g1)** and **(g2)** (Remark 3.1), we have

$$|G_H(\epsilon x, r_k v_n)| \leq \tau r_k^N |v_n|^N + C_1(\tau)|r_k|^\nu |v_n|^\nu \Phi_{\alpha,N-2}(r_k^{\frac{N}{N-1}} v_n^{\frac{N}{N-1}}).$$

For fixed $k \in \mathbb{N}$, the following hold a.e. in $\mathbb{R}^{\mathbb{N}}$, as $k \to \infty$

$$r_k v_n(x) \to 0 \quad \text{and } G_H(\epsilon x, r_k v_n) \to 0.$$

So, by the Lebesgue dominated convergence theorem, we deduce

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} G_H(\epsilon x, r_k v_n)\mathrm{d}x = 0 \text{ for each fixed } k \in \mathbb{N}. \tag{3.14}$$

For large n, we note $\frac{r_k}{\|u_n\|_{\mathbf{X}_\epsilon}} \in (0,1)$ because $\|u_n\|_{X_\epsilon} \to +\infty$ as $n \to +\infty$. By the definition in (3.13), we have

$$\begin{aligned}
I_{\epsilon,\beta}(t_n u_n) &\geq I_{\epsilon,\beta}\left(\frac{r_k u_n}{\|u_n\|_{\mathbf{X}_\epsilon}}\right) = I_{\epsilon,\beta}(r_k v_n) \\
&= \frac{1}{p}\|r_k v_n\|^p_{W^{1,p}_{V_\epsilon}} + \frac{1}{N}\|r_k v_n\|^N_{W^{1,N}_{V_\epsilon}} - \int_{\mathbb{R}^N} G_H(\epsilon x, r_k v_n)\mathrm{d}x \\
&\geq \frac{1}{N}\left(\frac{N}{p}\|r_k v_n\|^p_{W^{1,p}_{V_\epsilon}} + \|r_k v_n\|^N_{W^{1,N}_{V_\epsilon}}\right) - \int_{\mathbb{R}^N} G_H(\epsilon x, r_k v_n)\mathrm{d}x \\
&\geq \frac{1}{N} r_k^p \left(\|v_n\|^N_{W^{1,p}_{V_\epsilon}} + \|r_k v_n\|^N_{W^{1,N}_{V_\epsilon}}\right)^N - \int_{\mathbb{R}^N} G_H(\epsilon x, r_k v_n)\mathrm{d}x \\
&\geq \frac{1}{N2^{N-1}} r_k^p - \int_{\mathbb{R}^N} G_H(\epsilon x, r_k v_n)\mathrm{d}x. \qquad (3.15)
\end{aligned}$$

Now by using (3.14) and (3.15), we deduce

$$\limsup_{n\to+\infty} I_{\epsilon,\beta}(t_n u_n) = +\infty. \qquad (3.16)$$

For $\omega_n \in \partial I_{\epsilon,\beta}(t_n u_n)$, we obtain

$$\begin{aligned}
I_{\epsilon,\beta}(t_n u_n) &= I_{\epsilon,\beta}(t_n u_n) - \frac{1}{N}\langle \omega_n, t_n u_n\rangle + o_n(1) \\
&= \frac{t_n^p}{p}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} - \int_{\mathbb{R}^N} G_H(\epsilon x, t_n u_n)\mathrm{d}x \\
&\quad - \frac{t_n^p}{N}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} - \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} + \frac{1}{N}\int_{\mathbb{R}^N} \rho_n(t_n u_n)\mathrm{d}x + o_n(1),
\end{aligned}$$

where

$$\rho_n(x) \in [\underline{g_H}(\epsilon x, u_n(x)), \overline{g_H}(\epsilon x, u_n(x))] = \begin{cases} 0, & u_n(x) < \beta \\ [0, g(\epsilon x, \beta)], & u_n(x) = \beta \\ \{g(\epsilon x, u_n(x))\}, & u_n(x) > \beta \end{cases}, \text{ a.e. in } \mathbb{R}^N.$$

Now thanks to **(g5)** and for large enough n, we get

$$I_{\epsilon,\beta}(t_n u_n) = \frac{t_n^p}{p}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} - \int_{\mathbb{R}^N} G(\epsilon x, t_n u_n)\mathrm{d}x - \frac{t_n^p}{N}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} - \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}}$$

$$+\frac{1}{N}\int_{\mathbb{R}^N} g(\epsilon x, u_n(x)t_n)(t_n u_n)\mathrm{d}x + o_n(1)$$
$$= \frac{t_n^p}{p}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} - \frac{1}{N}\int_{\mathbb{R}^N}[NG(\epsilon x, t_n u_n) - g(\epsilon x, u_n(x)t_n)t_n u_n]\mathrm{d}x$$
$$- \frac{t_n^p}{N}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} - \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} + o_n(1)$$
$$\le \frac{t_n^p}{p}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} - \delta\int_{\mathbb{R}^N} G_H(\epsilon x, u_n)\mathrm{d}x - \frac{t_n^p}{N}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} - \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}}$$
$$+ \frac{\delta}{N}\int_{\mathbb{R}^N} g(\epsilon x, u_n(x))(u_n)\mathrm{d}x + o_n(1)$$
$$\le I_{\epsilon,\beta}(u_n) - \left(\frac{t_n^p}{N}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} - \frac{\delta}{N}\int_{\mathbb{R}^N} g(\epsilon x, u_n(x))(u_n)\mathrm{d}x\right) + o_n(1).$$

Due to condition $(1 + \|u_n\|_{\mathbf{X}_\epsilon})\lambda(u_n) \to 0$ as $n \to +\infty$, the sequence

$$\left\{\frac{t_n^p}{N}\|u_n\|^p_{W^{1,p}_{V_\epsilon}} + \frac{t_n^N}{N}\|u_n\|^N_{W^{1,N}_{V_\epsilon}} - \frac{\delta}{N}\int_{\mathbb{R}^N} g(\epsilon x, u_n(x))u_n\mathrm{d}x\right\}$$

is bounded. So as $n \to +\infty$, we have

$$\lim_{n\to+\infty} I_{\epsilon,\beta}(t_n u_n) \le C_2,$$

for some constant $C_2 > 0$, which contradicts to (3.16). Hence sequence $\{u_n\}_{n\ge1}$ is bounded in $\mathbf{X}_\epsilon$. □

Lemma 3.4 *Assume that (g1)–(g3) and (g5) hold. Let $\{u_n\}$ be a Cerami sequence such that $\limsup_{n\to+\infty}\|u_n\|^{N'}_{W^{1,N}} < \frac{\alpha_N}{\alpha_0}$. Then, there exists $u_{\epsilon,\beta} \in \mathbf{X}_\epsilon$ such that $\nabla u_n \to \nabla u_{\epsilon,\beta}$ a.e. in $\mathbb{R}^N$.*

Proof Using Lemma 3.3, up to a subsequence still denoted by itself, we can assume that

$$\begin{cases} u_n \rightharpoonup u_{\epsilon,\beta} \text{ in } \mathbf{X}_\epsilon, \\ u_n(x) \to u_{\epsilon,\beta}(x) \text{ a.e. in } \mathbb{R}^N, \\ u_n \to u_{\epsilon,\beta} \text{ in } L^q(B_R),\ \forall\, q \in [1,\infty) \text{ and for some R>0.} \end{cases}$$

Due to the boundedness of the sequence $\{u_n\}$, we obtain$(1 + \|u_n\|_{\mathbf{X}_\epsilon})\lambda(u_n) \to 0$, as $n \to +\infty$ which is equivalent to assume $\lambda(u_n) \to 0$ as $n \to +\infty$. Fix $R > 0$. Let $\psi \in C_c^\infty(\mathbb{R}^N)$, $0 \le \psi \le 1$ and

$$\psi \equiv 1 \text{ in } B_R \text{ and } \psi \equiv 0 \text{ in } B_{2R}^c$$

and $\|\nabla\psi\|_\infty \leq C_3$, where C_3 does not depend on R. Now for each u_n, there is a $\omega_n \in \partial I_{\epsilon,\beta}(u_n)$ such that as $n \to +\infty$, we get

$$\begin{aligned} \langle \omega_n, (u_n - u_{\epsilon,\beta})\psi \rangle &= o_n(1) \\ \langle Q'_{\epsilon,\beta}(u_n) - \rho_n, (u_n - u_{\epsilon,\beta})\psi \rangle &= o_n(1) \end{aligned} \tag{3.17}$$

where $\rho_n(x) \in [\underline{g_H}(\epsilon x, u_n(x)), \overline{g_H}(\epsilon x, u_n(x))]$ a.e. in $\mathbb{R}^N$. Due to Simon's inequality (Bahrouni and Rădulescu, 2018, Lemma 4.2), we get

$$\begin{aligned} C_N^{-1} \int_{B_{2R}} |\nabla u_n - \nabla u_{\epsilon,\beta}|^N \mathrm{d}x &\leq C_N^{-1} \left(\int_{B_{2R}} |\nabla u_n - \nabla u_{\epsilon,\beta}|^N dx + V(\epsilon x)|u_n - u_{\epsilon,\beta}|^N \mathrm{d}x \right) \\ &\leq \sum_{t \in \{p,N\}} \int_{B_{2R}} \Big[\left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u_{\epsilon,\beta}|^{t-2}\nabla u_{\epsilon,\beta} \right) \cdot (\nabla u_n - \nabla u_{\epsilon,\beta}) \\ &\quad + V(\epsilon x) \left(|u_n|^{t-2}u_n - |u_{\epsilon,\beta}|^{t-2}u_{\epsilon,\beta} \right) (u_n - u_{\epsilon,\beta}) \Big] \mathrm{d}x. \end{aligned} \tag{3.18}$$

From Eqs. (3.17) and (3.18), we have

$$\begin{aligned} C_N^{-1} \int_{B_{2R}} |\nabla u_n - \nabla u_{\epsilon,\beta}|^N \mathrm{d}x &\leq - \sum_{t \in \{p,N\}} \Big[\int_{\mathbb{R}^N} \left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u_{\epsilon,\beta}|^{t-2}\nabla u_{\epsilon,\beta} \right) (u_n - u_{\epsilon,\beta}) \nabla\psi \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \rho_n \psi (u_n - u_{\epsilon,\beta}) \mathrm{d}x \Big] + o_n(1). \end{aligned} \tag{3.19}$$

Now using the Hölder's inequality, compact embedding $W^{1,N}(B_{2R}) \hookrightarrow\hookrightarrow L^p(B_{2R})$ for all $p \in [1, +\infty)$, and boundedness of u_n, for $t \in \{p, N\}$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u_{\epsilon,\beta}|^{t-2}\nabla u_{\epsilon,\beta} \right) (u_n - u_{\epsilon,\beta}) \nabla\psi \mathrm{d}x \right| &\leq \|\nabla\psi\|_\infty \left(\|\nabla u_n\|_t^{t-1} + \|\nabla u_{\epsilon,\beta}\|_t^{t-1} \right) \\ &\quad \times \left(\int_{B_{2R}} |u_n - u_{\epsilon,\beta}|^{t'} \mathrm{d}x \right)^{t'}. \end{aligned} \tag{3.20}$$

We observe that the RHS of (3.20) tends to 0 as n tends to ∞. Now, we also have

$$\left| \int_{\mathbb{R}^N} \rho_n \psi (u_n - u_{\epsilon,\beta}) \mathrm{d}x \right| \leq \int_{B_{2R}} |\rho_n||u_n - u_{\epsilon,\beta}| \mathrm{d}x. \tag{3.21}$$

From **(g1)** and **(g3)**, it follows that for any $\tau > 0$ and $\nu > N$ there exist a constant $C_1(\tau)$ (depends on τ) and α_0 such that

$$|g(x,t)| \leq \tau |t|^{N-1} + C_1(\tau)|t|^{\nu-1}\Phi_{\alpha_0,N-2}(t). \tag{3.22}$$

Then, from (3.21) and (3.22), we get

$$\int_{B_{2R}} |\rho_n||u_n - u_{\epsilon,\beta}|\mathrm{d}x \le \int_{B_{2R}} \left(\tau|u_n|^{N-1} + C_1(\tau)|u_n|^{\nu-1}\Phi_{\alpha_0,N-2}(u_n)\right)|u_n - u_{\epsilon,\beta}|\mathrm{d}x =: I_1 + I_2 \tag{3.23}$$

where

$$I_1 = \int_{B_{2R}} \left(\tau|u_n|^{N-1}\right)|u_n - u_{\epsilon,\beta}|\mathrm{d}x \text{ and } I_2 = \int_{B_{2R}} C_1(\tau)|u_n|^{\nu-1}\Phi_{\alpha_0,N-2}(u_n)|u_n - u_{\epsilon,\beta}|\mathrm{d}x.$$

Using Hölder's inequality one can easily show that $I_1 \to 0$ as $n \to +\infty$. Now for I_2, Choose $t > 1$ and $\alpha > \alpha_0$ such that $\alpha t\|u_n\|^{N'}_{W^{1,N}} < \alpha_N$ for all $n \ge N_0$, and $\frac{1}{t} + \frac{1}{t'} = 1$. This is possible only due to the assumption $\limsup_{n\to+\infty}\|u_n\|^{N'}_{W^{1,N}} < \frac{\alpha_N}{\alpha_0}$. Next, we estimate I_2 and we obtain

$$\begin{aligned}
I_2 &= \int_{B_{2R}} C_1(\tau)|u_n|^{\nu-1}\Phi_{\alpha,N-2}(u_n)|u_n - u_{\epsilon,\beta}|\mathrm{d}x \\
&\le C_1(\tau)\left(\int_{B_{2R}} |u_n|^{(\nu-1)t'}|u_n - u_{\epsilon,\beta}|^{t'}\mathrm{d}x\right)^{\frac{1}{t'}} \left(\int_{B_{2R}} \left(\Phi_{\alpha,N-2}(u_n)\right)^t \mathrm{d}x\right)^{\frac{1}{t}} \\
&\le C_1(\tau)\left(\int_{B_{2R}} |u_n|^{(\nu-1)t'}|u_n - u_{\epsilon,\beta}|^{t'}\mathrm{d}x\right)^{\frac{1}{t'}} \left(\int_{B_{2R}} \left(\Phi_{\alpha t\|u_n\|^{N'}_{W^{1,N}},N-2}\left(\frac{u_n}{\|u_n\|_{W^{1,N}}}\right)\right)\mathrm{d}x\right)^{\frac{1}{t}} \\
&\le C_3\left(\int_{B_{2R}} |u_n|^{(\nu-1)t'}|u_n - u_{\epsilon,\beta}|^{t'}\mathrm{d}x\right)^{\frac{1}{t'}}.
\end{aligned}$$

Choose $\nu = t'N + 1$ and apply again Hölder's inequality, we obtain

$$\begin{aligned}
I_2 &\le C_3\left(\int_{B_{2R}} |u_n|^{t'^2 N}|u_n - u_{\epsilon,\beta}|^{t'}\mathrm{d}x\right)^{\frac{1}{t'}} \\
&\le C_3\left(\left(\int_{B_{2R}} |u_n|^{\frac{t'^2N^2}{N-1}}\mathrm{d}x\right)^{\frac{N-1}{N}} \left(\int_{B_{2R}} |u_n - u_{\epsilon,\beta}|^{Nt'}\mathrm{d}x\right)^{\frac{1}{N}}\right)^{\frac{1}{t'}} \\
&\le C_4\left(\left(\int_{B_{2R}} |u_n - u_{\epsilon,\beta}|^{Nt'}\mathrm{d}x\right)^{\frac{1}{N}}\right)^{\frac{1}{t'}} \to 0 \text{ as } n \to +\infty.
\end{aligned} \tag{3.24}$$

From equations (3.18), (3.20)–(3.24), we have

$$\lim_{n\to+\infty}\int_{B_{2R}}|\nabla u_n-\nabla u_{\epsilon,\beta}|^N\mathrm{d}x=0.$$

Since $R>0$ was arbitrary, we get $\nabla u_n(x)\to\nabla u_{\epsilon,\beta}(x)$ a.e. in $\mathbb{R}^N$. □

Lemma 3.5 *Assume that* **(V1)** *and* **(g2)** *hold. Then mountain pass level $c_{\epsilon,\beta}$ satisfies*

$$0<c_{\epsilon,\beta}<c_0=:\frac{N-p}{pN}\left(\frac{\alpha_N}{\alpha_0}(\min\{1,V_0\})^{\frac{N^2}{N-1}}-M^{\frac{N}{N-1}}\right)^{\frac{p(N-1)}{N}}\tag{3.25}$$

where, $M=\max\{M_1,M_2\}$ such that $\|u_n\|_{\mathbf{X}_\epsilon}\le M_1$ and we choose M_2 such that $\left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}(\min\{1,V_0\})^N>M_2$. Moreover the sequence $\{u_n\}$ satisfies

$$\limsup_{n\to+\infty}\|u_n\|_{W^{1,N}}^{N'}<\frac{\alpha_N}{\alpha_0}.\tag{3.26}$$

Proof Let $\xi\in C_c^\infty(\mathbb{R}^N)$ such that $0\le\xi\le1$,

$$\xi\equiv1\text{ in }B_{\frac{1}{2}}\text{ and }\xi\equiv0\text{ in }B_1^c,$$

and $m(\{t\xi>\beta\})>0$ for $t>0$. For such ξ, we get

$$I_{\epsilon,\beta}(t\xi)=\frac{t^p}{p}\|\xi\|^p_{W^{1,p}_{V_\epsilon}}+\frac{t^N}{N}\|\xi\|^N_{W^{1,N}_{V_\epsilon}}-\int_{\{t\xi>\beta\}}G_H(\epsilon x,t\xi)\mathrm{d}x.$$

From assumption **(g2)**, we have $\zeta>0$ such that

$$I_{\epsilon,\beta}(t\xi)\le\frac{t^p}{p}\|\xi\|^p_{W^{1,p}_{V_\epsilon}}+\frac{t^N}{N}\|\xi\|^N_{W^{1,N}_{V_\epsilon}}-\zeta t^{N+1}\int_{\mathbb{R}^N}|\xi|^{N+1}\mathrm{d}x.$$

There is ζ_1 such that $t\xi\in\Gamma$ for all $\zeta>\zeta_1$ and using the elementary calculus, we obtain

$$0\le c_{\epsilon,\beta}\le\max_{t\in[0,1]}I_{\epsilon,\beta}(t\xi)=A\left(\frac{A}{B(N+1)\zeta}\right)^{\frac{p}{N-p+1}}\left(\frac{1}{p}-\frac{1}{N+1}\right).$$

where $A = \|\xi\|^p_{W^{1,p}_{V_\epsilon}} + \|\xi\|^N_{W^{1,N}_{V_\epsilon}}$ and $B = \int_{\mathbb{R}^N} |\xi|^{N+1}dx$. Hence $A\left(\frac{A}{B(N+1)\zeta}\right)^{\frac{p}{N-p+1}}$ $\left(\frac{1}{p} - \frac{1}{N+1}\right) \to 0$ as $\zeta \to +\infty$. Due to this, there is a $\zeta_0 > 0$ such that for all $\zeta > \zeta_0 > \zeta_1$, (3.25) holds. As $n \to +\infty$, we have

$$c_{\epsilon,\beta} + o_n(1) \geq \left(\frac{1}{p} - \frac{1}{N}\right) \|u_n\|^p_{W_{V_\epsilon}},$$

and

$$\limsup_{n\to+\infty} \|u_n\|_{W^{1,p}_{V_\epsilon}} \leq \left(\frac{c_{\epsilon,\beta} Np}{N-p}\right)^{\frac{1}{p}}.$$

Since the sequence $\{u_n\}$ is bounded in $\mathbf{X}_\epsilon$, we deduce

$$\|u_n\|_{W^{1,p}_{V_\epsilon}} + \|u_n\|_{W^{1,N}_{V_\epsilon}} \leq M \text{ and } \limsup_{n\to+\infty} \|u_n\|_{W^{1,N}_{V_\epsilon}} \leq M - \left(\frac{c_{\epsilon,\beta} Np}{N-p}\right)^{\frac{1}{p}}.$$

Due to assumption **(V1)**, by embedding $W^{1,N}_{V_\epsilon}(\mathbb{R}^N) \hookrightarrow W^{1,N}(\mathbb{R}^N)$, it follows that

$$\begin{aligned}
\limsup_{n\to+\infty} \|u_n\|^{\frac{N}{N-1}}_{W^{1,N}} &\leq \left((\min\{1, V_0\})^{-N} \limsup_{n\to+\infty} \|u_n\|_{W^{1,N}_{V_\epsilon}}\right)^{\frac{N}{N-1}} \\
&= \left((\min\{1, V_0\})^{\frac{-N^2}{N-1}} \left(M - \left(\frac{c_{\epsilon,\beta} Np}{N-p}\right)^{\frac{1}{p}}\right)\right)^{\frac{N}{N-1}} \\
&\leq (\min\{1, V_0\})^{\frac{-N^2}{N-1}} \left(M^{\frac{N}{N-1}} + \left(\frac{c_{\epsilon,\beta} Np}{N-p}\right)^{\frac{N}{p(N-1)}}\right).
\end{aligned}$$

Hence, we obtain (3.25) and conclude the proof.

□

Lemma 3.6 *Let $\{u_n\}$ be a Cerami sequence for $I_{\epsilon,\beta}$. Then for all $\xi > 0$, there exist $R = R(\xi) > 0$ such that*

$$\limsup_{n\to+\infty} \sum_{t\in\{p,N\}} \left(\int_{B_R^c} |\nabla u_n|^t + V(\epsilon x)|u_n|^t dx\right) < \xi. \tag{3.27}$$

Proof For $R > 0$, consider $\tilde{\psi}_R \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \tilde{\psi}_R \leq 1$ with

$$\tilde{\psi}_R \equiv 0 \text{ in } B_{\frac{R}{2}} \text{ and } \tilde{\psi}_R \equiv 1 \text{ in } B_R^c,$$

and $|\nabla\tilde{\psi}_R| \leq \frac{C}{R}$, where $C > 0$ is a constant independent of $R > 0$. Next, choose $R > 0$ such that $\Lambda_\epsilon \subset B_{\frac{R}{2}}$. The boundedness of $\{u_n\}_{n\geq 1}$ in $\boldsymbol{X_\epsilon}$ implies that $\{u_n\tilde{\psi}_R\}_{n\geq 1}$ is also bounded in $\boldsymbol{X_\epsilon}$. As $n \to +\infty$, from 3.7, it follows that

$$\begin{aligned} \langle w_n, u_n\tilde{\psi}_R\rangle &= o_n(1),\\ \langle Q'(u_n) - \rho_n, u_n\tilde{\psi}_R\rangle &= o_n(1),\\ \langle Q'(u_n), u_n\tilde{\psi}_R\rangle - \langle \rho_n, u_n\tilde{\psi}_R\rangle &= o_n(1), \end{aligned}$$

where $w_n \in \partial I_{\epsilon,\beta}(u_n) = Q'(u_n) - \rho_n$. Further, we deduce

$$\sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}\nabla u_n\cdot\nabla(\tilde{\psi}_R u_n) + V(\epsilon x)|u_n|^{t-2}u_n(\tilde{\psi}_R u_n)\mathrm{d}x = o_n(1) + \int_{\mathbb{R}^N} \rho_n u_n\tilde{\psi}_R\mathrm{d}x,$$

and this implies

$$\begin{aligned} &\sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}(\nabla u_n\cdot\nabla u_n)\tilde{\psi}_R + |\nabla u_n|^{t-2}u_n\nabla u_n\cdot\nabla\tilde{\psi}_R + V(\epsilon x)|u_n|^{t-2}u_n(\tilde{\psi}_R u_n)\mathrm{d}x\\ &= o_n(1) + \int_{\mathbb{R}^N} \rho_n u_n\tilde{\psi}_R\mathrm{d}x. \end{aligned}$$

Further, we have

$$\begin{aligned} &\sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}(\nabla u_n\cdot\nabla u_n)\tilde{\psi}_R\mathrm{d}x + \sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}u_n\nabla u_n\cdot\nabla\tilde{\psi}_R\\ &+ V(\epsilon x)|u_n|^{t-2}u_n(\tilde{\psi}_R u_n)\mathrm{d}x = o_n(1) + \int_{\mathbb{R}^N} \rho_n u_n\tilde{\psi}_R\mathrm{d}x. \end{aligned}$$

Since

$$\int_{\mathbb{R}^N} \rho_n u_n\tilde{\psi}_R\mathrm{d}x \leq \int_{\mathbb{R}^N} \overline{g_H}(\epsilon x, u_n)u_n\tilde{\psi}_R\mathrm{d}x \leq \int_{\mathbb{R}^N} g(\epsilon x, u_n)u_n\tilde{\psi}_R\mathrm{d}x,$$

we obtain

$$\begin{aligned} &\sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}(\nabla u_n\cdot\nabla u_n)\tilde{\psi}_R + V(\epsilon x)|u_n|^{t-2}u_n(\tilde{\psi}_R u_n)\mathrm{d}x\\ &\leq \sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}|u_n||\nabla u_n||\nabla\tilde{\psi}_R|\mathrm{d}x\\ &+ \int_{\mathbb{R}^N} g(\epsilon x, u_n)u_n\tilde{\psi}_R\mathrm{d}x + o_n(1)\\ &\leq \frac{C}{R}\sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} |\nabla u_n|^{t-1}|u_n|\mathrm{d}x + \int_{\mathbb{R}^N} g(\epsilon x, u_n)u_n\tilde{\psi}_R\mathrm{d}x \end{aligned}$$

$$+ o_n(1).$$

Using the fact that $\Lambda_\epsilon \subset B_{\frac{R}{2}}$ with definition in 3.1 and boundedness of $u_n \in \boldsymbol{X_\epsilon}$ with Hölder's inequality, there exist a constant $C_1 > 0$ such that we have

$$\left(1 - \frac{1}{k}\right) \sum_{t \in \{p,N\}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^t + V(\epsilon x)|u_n|^t \right) \tilde{\psi}_R \mathrm{d}x \leq \frac{C_1}{R}.$$

So choose $R > 0$ large enough such that

$$\limsup_{n \to +\infty} \sum_{t \in \{p,N\}} \left(\int_{B_R^c} \left(|\nabla u_n|^t + V(\epsilon x)|u_n|^t\right) \tilde{\psi}_R \mathrm{d}x \right) \leq \frac{C_1}{R} < \xi.$$

This completes the proof. □

3.3 Existence of Critical Point of the Energy Functional Corresponding to the Auxiliary Problem

We begin this subsection by proving the existence of nontrivial critical points for the energy functional corresponding to the problem $(\mathscr{P}^a_{\epsilon,\beta})$.

Lemma 3.7 *Assume that* (**V1**)*,* (**g1**)*–*(**g3**)*, and* (**g5**) *hold. Then the sequence* $\{u_n\}$ *satisfies non-smooth Cerami c-condition for* $0 < c < c_0$ *and* $u_{\epsilon,\beta}$ *is a non trivial weak solution for problem* $(\mathscr{P}^a_{\epsilon,\beta})$*. Here* c_0 *is the same number defined in Lemma 3.5.*

Proof By the virtue of Lemma 3.4 and 3.5, it follows that $\nabla u_n \to \nabla u_{\epsilon,\beta}$ a.e. in $\mathbb{R}^N$. For $t \in \{p, N\}$, the sequence $\{|\nabla u_n|^{t-2}\nabla u_n\}$ is bounded in $L^{\frac{t}{t-1}}(\mathbb{R}^N)$. So,

$$|\nabla u_n|^{t-2}\nabla u_n \rightharpoonup |\nabla u_{\epsilon,\beta}|^{t-2}\nabla u_{\epsilon,\beta} \text{ in } L^{\frac{t}{t-1}}(\mathbb{R}^N), \quad |\nabla u_n|^{t-2}\nabla u_n \to |\nabla u_{\epsilon,\beta}|^{t-2}\nabla u_{\epsilon,\beta} \text{ a.e. in } \mathbb{R}^N.$$

Using density arguments for $t \in \{p, N\}$, one can prove that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}\nabla u_n \cdot \nabla v \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u_\epsilon|^{t-2}\nabla u_{\epsilon,\beta} \cdot \nabla v \mathrm{d}x, \ \forall v \in \mathbf{X}_\epsilon, \tag{3.28}$$

and

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} V(x)|u_n|^{t-2}u_n v \mathrm{d}x = \int_{\mathbb{R}^N} V(x)|u_{\epsilon,\beta}|^{t-2}u_{\epsilon,\beta} v \mathrm{d}x, \ \ \forall v \in \mathbf{X}_\epsilon. \tag{3.29}$$

Claim: For each $\rho_n \in \partial_t G(\epsilon x, u_n(x))$, there is $\rho_0 \in L^{\tilde{\Phi}_1}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \rho_n v \mathrm{d}x \to \int_{\mathbb{R}^N} \rho_0 v \mathrm{d}x, \ \forall v \in \mathbf{X}_\epsilon. \tag{3.30}$$

Note that Φ_1 is an increasing and convex function and there is $\alpha > \alpha_0$ such that

$$\int_{\mathbb{R}^N} \tilde{\Phi}_1(\rho_n)\mathrm{d}x \leq \int_{\mathbb{R}^N} \tilde{\Phi}_1 \left(\tau_1 |u_n|^{N-1} + C_1(\tau)|u_n|^\nu \Phi_{N-2,\alpha}(u)\right) \mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \tilde{\Phi}(\rho_n)\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{R}^N} \varsigma(2\tau)\tilde{\Phi}_1(|u_n|^{N-1})dx + \frac{\varsigma(2C_1(\tau))}{2}\int_{\mathbb{R}^N} \tilde{\Phi}_1 \left(|u_n|^\nu \Phi_1\left(\alpha^{\frac{N-1}{N}}|u_n|\right)\right) \mathrm{d}x.$$

By Lemma 2.4, there exist constant $C_1 > 0$ and $C_2 > 0$ such that

$$\int_{\mathbb{R}^N} \tilde{\Phi}(\rho_n)\mathrm{d}x \leq \frac{C_1}{2}\int_{\mathbb{R}^N} \varsigma(|u_n|^{N-1})\mathrm{d}x + \frac{C_2}{2}\int_{\mathbb{R}^N} \varsigma(|u_n|^\nu \Phi_1(\alpha^{\frac{N}{N-1}}|u_n|))\mathrm{d}x.$$

By using $\varsigma(t) = \max\{t, t^N\}$. Now two cases arise:
Case 1:

$$\int_{\mathbb{R}^N} \tilde{\Phi}_1(\rho_n)\mathrm{d}x \leq C_1 \left(\int_{\mathbb{R}^N} |u_n|^{N-1}\mathrm{d}x\right) + C_2 \left(\int_{\mathbb{R}^N} \left(|u_n|^\nu\right) \Phi_1\left(\alpha^{\frac{N-1}{N}}|u_n|\right) \mathrm{d}x\right). \tag{3.31}$$

Case 2:

$$\int_{\mathbb{R}^N} \tilde{\Phi}_1(\rho_n)\mathrm{d}x \leq C_1 \left(\int_{\mathbb{R}^N} |u_n|^{N(N-1)}\mathrm{d}x\right) + C_2 \left(\int_{\mathbb{R}^N} |u_n|^{N\nu} \Phi_1\left(\alpha^{\frac{N-1}{N}}|u_n|\right) \mathrm{d}x\right). \tag{3.32}$$

By Hölder's inequality and by Lemma 3.5, we obtain

$$\left(\int_{\mathbb{R}^N} \left(|u_n|^\nu\right) \Phi_1\left(\alpha^{\frac{N-1}{N}}|u_n|\right) \mathrm{d}x\right) \leq C_3, \quad \forall n \in \mathbb{N}, \tag{3.33}$$

and

$$\left(\int_{\mathbb{R}^N} \left(|u_n|^{N\nu}\right) \Phi_1\left(\alpha^{\frac{N-1}{N}}|u_n|\right) \mathrm{d}x\right) \leq C_4, \quad \forall n \in \mathbb{N}. \tag{3.34}$$

Suppose case 1 holds, then by the definition of ς, $0 < |u_n(x)| < 1$. We have embedding

$$\mathbf{X}_\epsilon \hookrightarrow W^{1,p}_{V_\epsilon} \hookrightarrow L^q(\mathbb{R}^N),\ \forall q \in [p, p^*].$$

If $N-1 < p^*$, we have

$$\|u_n\|_{N-1}^{N-1} \leq \|u_n\|_{\mathbf{X}_\epsilon}^{N-1} \leq C_5, \tag{3.35}$$

and if $N-1 > p^*$, we have

$$\|u_n\|_{N-1}^{N-1} \leq \|u_n\|_{p^*}^{p*} \leq C_6. \tag{3.36}$$

For Case 1, from equation (3.31), (3.33)–(3.36), we conclude that

$$\int_{\mathbb{R}^N} \tilde{\Phi}_1(\rho_n)\mathrm{d}x \leq C_7, \quad \forall n \in \mathbb{N}. \tag{3.37}$$

For Case 2, from equation (3.32) and embedding of $\mathbf{X}_\epsilon$ in $L^s(\mathbb{R})^{\mathbb{N}}$ for $s \in [N, +\infty)$, we conclude that

$$\int_{\mathbb{R}^N} \tilde{\Phi}_1(\rho_n)\mathrm{d}x \leq C_8,\ \forall n \in \mathbb{N}. \tag{3.38}$$

From equation (3.37) and (3.38), it follows that $\{\rho_n\}_{n\geq 1}$ is bounded in $L^{\tilde{\Phi}_1}(\mathbb{R}^N)$. So the sequence of functionals $\tilde{\rho}_n \subset \partial\Upsilon(u_n) \subset (E_{\Phi_1}(\mathbb{R}^N))^*$ corresponding to $\{\rho_n\}$ is also bounded in $(E_{\Phi_1}(\mathbb{R}^N))^*$. So there is $\tilde{\rho}_0 \in (E_{\Phi_1}(\mathbb{R}^N))^*$ such that $\tilde{\rho}_n \overset{*}{\rightharpoonup} \tilde{\rho}_0$ in $(E_{\Phi_1}(\mathbb{R}^N))^*$ i.e.,

$$\int_{\mathbb{R}^N} \rho_n v\mathrm{d}x = \langle \tilde{\rho}_n, v\rangle \to \langle \tilde{\rho}_0, v\rangle = \int_{\mathbb{R}^N} \rho_0 v\mathrm{d}x, \quad \forall v \in \mathbf{X}_\epsilon, \tag{3.39}$$

for some $\rho_0 \in L^{\tilde{\Phi}_1}(\mathbb{R}^N)$. From equation (3.28), (3.29) and (3.39), we have

$$\sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} |\nabla u_\epsilon|^{t-2}\nabla u_{\epsilon,\beta}\cdot\nabla v\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u_{\epsilon,\beta}|^{t-2}u_{\epsilon,\beta}v\mathrm{d}x = \int_{\mathbb{R}^N} \rho_0 v\mathrm{d}x,\ \forall v \in \mathbf{X}_\epsilon. \tag{3.40}$$

Set $\upsilon = u_n - u_{\epsilon,\beta}$. Due to Brézis-Lieb-type results (see Mahanta et al. (2025a)), we have

$$\begin{cases} \|u_n\|^p_{W^{1,p}_{V_\epsilon}} = \|u_{\epsilon,\beta}\|^p_{W^{1,p}_{V_\epsilon}} + \|\upsilon_n\|^p_{W^{1,p}_{V_\epsilon}} + o_n(1), \\ \|u_n\|^N_{W^{1,N}_{V_\epsilon}} = \|u_{\epsilon_\beta}\|^N_{W^{1,N}_{V_\epsilon}} + \|\upsilon_n\|^N_{W^{1,N}_{V_\epsilon}} + o_n(1). \end{cases} \tag{3.41}$$

So,

$$\|u_{\epsilon,\beta}\|^p_{W^{1,p}_{V_\epsilon}} + \|\upsilon_n\|^p_{W^{1,p}_{V_\epsilon}} + \|u_{\epsilon_\beta}\|^N_{W^{1,N}_{V_\epsilon}} + \|\upsilon_n\|^N_{W^{1,N}_{V_\epsilon}} - \int_{\mathbb{R}^N} \rho_n u_n\mathrm{d}x = o_n(1), \tag{3.42}$$

and

$$\|u_{\epsilon,\beta}\|^p_{W^{1,p}_{V_\epsilon}} + \|u_{\epsilon_\beta}\|^N_{W^{1,N}_{V_\epsilon}} - \int\limits_{\mathbb{R}^N} \rho_0 u_{\epsilon,\beta} \mathrm{d}x = 0. \tag{3.43}$$

From equation (3.42) and (3.43), we get

$$o_n(1) = \|\upsilon_n\|^p_{W^{1,p}_{V_\epsilon}} + \|\upsilon_n\|^N_{W^{1,N}_{V_\epsilon}} - \int\limits_{\mathbb{R}^N} \rho_n u_n \mathrm{d}x + \int\limits_{\mathbb{R}^N} \rho_0 u_{\epsilon,\beta} \mathrm{d}x.$$

Now define $R := \max\{R_1, R_2\}$ where R_1 and R_2 are taken in (3.47) and (3.48), respectively. For large enough $R > 0$, we get

$$o_n(1) = \|\upsilon_n\|^p_{W^{1,p}_{V_\epsilon}} + \|\upsilon_n\|^N_{W^{1,N}_{V_\epsilon}} - \int\limits_{B_R} (\rho_n u_n - \rho_0 u_{\epsilon,\beta}) \mathrm{d}x - \int\limits_{B_R^c} (\rho_n u_n - \rho_0 u_{\epsilon,\beta}) \mathrm{d}x. \tag{3.44}$$

We have

$$\int\limits_{B_R} (\rho_n u_n - \rho_0 u_{\epsilon,\beta}) \mathrm{d}x = \int\limits_{B_R} \rho_n (u_n - u_{\epsilon,\beta}) \mathrm{d}x + \int\limits_{B_R} (\rho_n - \rho_0) u_{\epsilon,\beta} \mathrm{d}x.$$

Due to (3.39), we deduce

$$\int\limits_{B_R} (\rho_n - \rho_0) u_{\epsilon,\beta} \mathrm{d}x \to 0 \text{ as } n \to \infty. \tag{3.45}$$

Due to assumption **(g1)**, Hölder's inequality, boundedness of $\{u_n\}$ in $\mathbf{X}_\epsilon$ and Compact Embedding of Sobolev spaces for a bounded domain, we have

$$\int\limits_{B_R} \rho_n (u_n - u_{\epsilon,\beta}) \mathrm{d}x \to 0 \text{ as } n \to \infty. \tag{3.46}$$

Since $\{\rho_0 u_{\epsilon,\beta}\} \in L^1(\mathbb{R}^N)$, so for large $R_1 > 0$ such that for any $\xi_1 > 0$, we have

$$\int\limits_{B_{R_1}^c} \rho_0 u_{\epsilon,\beta} \mathrm{d}x < \xi_1. \tag{3.47}$$

From Lemma 3.6, one can easily prove the tightness of the Cerami sequence $\{u_n\}_{n\geq 1}$ for $I_{\epsilon,\beta}$. This implies that for any $\xi_2 > 0$, there is $R_2 > 0$, and there holds

$$\int\limits_{B_{R_2}^c} \rho_n u_n \mathrm{d}x < \xi_2. \tag{3.48}$$

From equations (3.44)–(3.48), it follows that

$$\|\upsilon_n\|^p_{W^{1,p}_{V_\epsilon}} + \|\upsilon_n\|^N_{W^{1,N}_{V_\epsilon}} \to 0 \text{ as } n \to +\infty.$$

This implies $\upsilon_n \to 0$ in $\mathbf{X}_\epsilon$. Hence, $u_n \to u_{\epsilon,\beta}$ in $\mathbf{X}_\epsilon$. So,

$$0 \in \partial I_{\epsilon,\beta}(u_{\epsilon,\beta}). \tag{3.49}$$

From equations (3.43), (3.49) and Lemma 2.3, it follows that $u_{\epsilon,\beta}$ is a non-trivial weak solution for $(\mathscr{P}^a_{\epsilon,\beta})$. □

4 Autonomous Problem

In this section, we consider an autonomous problem related to $(\mathscr{P}_{\epsilon,\beta})$ and compare the critical level of both problems. We are motivated by the methods discussed in (Fiscella and Pucci, 2021, Fiscella and Pucci) and (Costa and Figueiredo, 2022, Costa and Figueiredo). The autonomous problem related to $(\mathscr{P}_{\epsilon,\beta})$ is as follows:

$$\begin{cases} -\Delta_p u - \Delta_N u + V_0(|u|^{p-2}u + |u|^{N-2}u) = f(u) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}_{V_0}(\mathbb{R}^N) \cap W^{1,N}_{V_0}(\mathbb{R}^N). \end{cases} \tag{$\mathscr{P}_{V_0}$}$$

Denote $\boldsymbol{Y} = W^{1,p}_{V_0}(\mathbb{R}^N) \cap W^{1,N}_{V_0}(\mathbb{R}^N)$, with norm

$$\|u\|_{\boldsymbol{Y}} = \|u\|_{W^{1,p}_{V_0}} + \|u\|_{W^{1,N}_{V_0}}$$

and $F(t) = \int_0^t f(s)\mathrm{d}s$. The energy functional associated with the autonomous problem, denoted as $I_{V_0} : W^{1,p}_{V_0}(\mathbb{R}^N) \cap W^{1,N}_{V_0}(\mathbb{R}^N) \to \mathbb{R}$, is defined as

$$I_{V_0}(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V_0|u|^p)\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla u|^N + V_0|u|^N)\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

Hence

$$I_{V_0}(u) = \frac{1}{p}\|u\|^p_{W^{1,p}_{V_0}} + \frac{1}{N}\|u\|^N_{W^{1,N}_{V_0}} - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

I_{V_0} is well defined and belongs to $C^1(\boldsymbol{Y}, \mathbb{R})$. Using the similar strategy in (Lam and Lu, 2013, Lemma 2) and by invoking the Mountain Pass theorem for C^1 functional, we obtained a Cerami sequence $\{\hat{u}_n\}_{n\geq 1} \subset \boldsymbol{Y}$ i.e.

$$I_{V_0}(\hat{u}_n) \to c_{V_0} \text{ and } \|1 + \hat{u}_n\|_{\boldsymbol{Y}}\, I'_{V_0}(\hat{u}_n) \to 0 \text{ as } n \to +\infty, \tag{4.1}$$

where

$$c_{V_0} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{V_0}(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0,1], \boldsymbol{Y}) : \gamma(0) = 0 \ , I_{\boldsymbol{V_0}}(\gamma(1)) < 0\}.$$

By adapting the strategy found in (Lam and Lu, 2013, Lemma 5), one can prove the boundedness of $\{\hat{u}_n\}_{n\geq 1}$ in $\boldsymbol{Y}$. So up to a subsequence still denoted by itself, we can assume that

$$\begin{cases} \hat{u}_n \rightharpoonup u_0 \text{ in } \boldsymbol{Y}, \\ \hat{u}_n(x) \to u_0(x) \text{ a.e. in } \mathbb{R}^N. \end{cases}$$

4.1 Existence of Critical Point

In the following Lemma, we will show the existence of critical point for the functional corresponding to $\mathscr{P}_{V_0}$.

Lemma 4.1 *Let the assumption* **(V1)** *and* **(f1)–(f5)** *hold. Then the function u_0 is a non-trivial critical point for the I_{V_0}.*

Proof As $n \to \infty$, from (4.1), it follows that

$$I_{V_0}(\hat{u}_n) = c_{V_0} + o_n(1) \text{ and } \langle I'_{V_0}(\hat{u}_n), \hat{u}_n \rangle = o_n(1). \tag{4.2}$$

Using arguments as in Sect. 3, as $n \to \infty$, for $v \in \boldsymbol{Y}$, it implies

$$\nabla \hat{u}_n \to \nabla u_0 \ \text{ a.e. in } \mathbb{R}^N$$

$$\sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} |\nabla \hat{u}_n|^{t-2} \nabla \hat{u}_n \cdot \nabla v \mathrm{d}x \to \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} |\nabla u_0|^{t-2} \nabla u_0 \cdot \nabla v \mathrm{d}x, \tag{4.3}$$

and

$$\sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} V_0 |\hat{u}_n|^{t-2} \hat{u}_n v \mathrm{d}x \to \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} V_0 |u_0|^{t-2} u_0 v \mathrm{d}x. \tag{4.4}$$

Claim: $\int_{\mathbb{R}^N} f(\hat{u}_n) v \mathrm{d}x \to \int_{\mathbb{R}^N} f(u_0) v \mathrm{d}x$ **for all** $v \in \boldsymbol{Y}$**.**

Due to assumption **(f1)**, **(f2)**, Remark 1.1 and Lemma 3.5, the sequence $\{f(\hat{u}_n)v\}_{n\geq 1}$ is bounded in $L^1(\mathbb{R}^N)$. This also implies that the sequence $\{f(\hat{u}_n)v\}$ is uniformly integrable over $\mathbb{R}^N$ and $f(\hat{u}_n)v \to f(u_0)v$ a.e. in $\mathbb{R}^N$. Let $R > 0$ such that $\psi_R \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \psi_R \leq 1$ in $\mathbb{R}^N$ and

$$\psi_R \equiv 0 \text{ in } B_{\frac{R}{2}} \text{ and } \psi_R \equiv 1 \text{ in } B_R^c, \tag{4.5}$$

with $|\nabla \psi_R| \leq \frac{C}{R}$, where C is constant independent of R. Consider $U = v\psi_R$, from equation (4.2), as $n \to \infty$, we have

$$\begin{aligned}
\left| \int_{\mathbb{R}^N} f(\hat{u}_n) v \psi_R \mathrm{d}x \right| &\leq \left| \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} |\nabla \hat{u}_n|^{t-2} \nabla \hat{u}_n \cdot \nabla (v\psi_R) \mathrm{d}x + \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} V_0 |\hat{u}_n|^{t-2} \hat{u}_n (v\psi_R) \mathrm{d}x \right| \\
&\quad + o_n(1) \\
&= \left| \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} |\nabla \hat{u}_n|^{t-2} \nabla \hat{u}_n \cdot (\psi_R \nabla v + v \nabla \psi_R) \mathrm{d}x \right. \\
&\quad \left. + \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} V_0 |\hat{u}_n|^{t-2} \hat{u}_n (v\psi_R) \mathrm{d}x \right| + o_n(1) \\
&\leq \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} |\nabla \hat{u}_n|^{t-2} |\nabla \hat{u}_n \cdot \nabla v| \mathrm{d}x + \frac{C}{R} \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} |\nabla \hat{u}_n|^{t-2} |\nabla \hat{u}_n| |v| \mathrm{d}x \\
&\quad + \sum_{t \in \{p,N\}} \int_{\mathbb{R}^N} V_0 |\hat{u}_n|^{t-2} |\hat{u}_n| |v| \mathrm{d}x + o_n(1).
\end{aligned}$$

Since $v \in \boldsymbol{Y}$, so for large enough $R > 0$, for any given $\epsilon_1 > 0$ and $\epsilon_2 > 0$ we have

$$\int_{B_R^c} |\nabla v|^t \mathrm{d}x < \epsilon_1 \text{ and} \int_{B_R^c} |v|^t \mathrm{d}x < \epsilon_2.$$

Now using boundedness of $\{\hat{u}_n\}$, we have $\int_{B_R^c} f(\hat{u}_n) v dx < \xi^*$ for large enough $R > 0$ and any $\xi^* > 0$. This concludes that $\{f(\hat{u}_n)v\}_{n \geq 1}$ is tight over $\mathbb{R}^N$. Hence, by Vitali's Convergence theorem,

$$\int_{\mathbb{R}^N} f(\hat{u}_n) v \mathrm{d}x \to \int_{\mathbb{R}^N} f(u_0) v \mathrm{d}x \quad \text{for all } v \in \boldsymbol{Y}. \tag{4.6}$$

From Eqs. (4.3), (4.4) and (4.6) it follows that u_0 is critical point of I_{V_0} and using similar arguments in Lemma 3.7, we can prove that $\hat{u}_n \to u_0$ in $\boldsymbol{Y}$. Hence, u_0 is non trivial critical point. □

4.2 Relationship Between Both Mountain Pass Levels

In the next Lemma, we establish a relation between $c_{\epsilon,\beta}$ and c_{V_0}, which plays crucial role in our arguments for proving the main theorem.

Lemma 4.2 *Assume that* (**V1**) *and* (**f1**)–(**f5**) *hold. Then*

$$\lim_{\epsilon,\beta \to 0} c_{\epsilon,\beta} = c_{V_0}$$

where $c_{\epsilon,\beta}$ and c_{V_0} are mountain pass level for ($\mathscr{P}_{\epsilon,\beta}$) and ($\mathscr{P}_{V_0}$), respectively.

Proof Due to assumption **(V1)** and $H(t) \leq 1$, we obtain

$$\liminf_{\epsilon,\beta\to 0} c_{\epsilon,\beta} \geq c_{V_0}. \tag{4.7}$$

Next, we will prove that

$$\limsup_{\epsilon,\beta\to 0} c_{\epsilon,\beta} \leq c_{V_0}. \tag{4.8}$$

Let u_0 be the solution of $(\mathscr{P}_{V_0})$ with $I_{V_0}(u_0) = c_{V_0}$ and $I'(u_0) = 0$. Consider $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \phi \leq 1$ in $\mathbb{R}^N$ and

$$\phi \equiv 1,\ x \in\ B_1 \text{ and } \phi \equiv 0,\ x \in B_2^c.$$

Let $R > 0$, define $u_R(x) = \phi(\frac{x}{R})u_0$ and $B_{2R} \subset \Lambda_{\epsilon}$. Since $|u_R| \leq u_0$, so by virtue of the Lebesgue dominated convergence theorem, it follows that

$$u_R \to u_0 \ \text{ in } W^{1,t}(\mathbb{R}^N) \text{ as } R \to +\infty \text{ for } t \in \{p, N\}. \tag{4.9}$$

It follows that $u_R \to u_0$ a.e. in $\mathbb{R}^N$. We now define the function $h : [0, \infty) \to \mathbb{R}$ as follows:

$$h(t) = I_{V_0}(tu_R).$$

It can be proved that $h(t) > 0$ for sufficiently small t and $h(t) < 0$ for large t. As h is a continuously differentiable function so there exist $t_R > 0$ such that $h'(t_R) = I_{V_0}(t_R u_R)u_R = 0$. This implies that for every $R > 0$ there exist $t_R > 0$ such that $t_R u_R \in \mathcal{N}_0$. Now we claim the following.

Claim: The sequence $\{t_R\}$ is bounded and upto a subsequence still denoted by itself we have,

$$\lim_{R\to+\infty} t_R = 1.$$

Suppose, on the contrary, that $t_R \to +\infty$ as $R \to +\infty$. Since $t_R u_R \in \mathcal{N}_0$, this implies that

$$t_R^p \|u_R\|^p_{W^{1,p}_{V_0}} + t_R^N \|u_R\|^N_{W^{1,N}_{V_0}} = \int_{\mathbb{R}^N} f(t_R u_R) t_R u_R \mathrm{d}x \tag{4.10}$$

Using assumptions **(f2)** and **(f5)**, we obtain

$$t_R^p \|u_R\|^p_{W^{1,p}_{V_0}} + t_R^N \|u_R\|^N_{W^{1,N}_{V_0}} = \int_{\mathbb{R}^N} f(t_R u_R) t_R u_R \mathrm{d}x \geq \int_{\mathbb{R}^N} N F(t_R u_R) \mathrm{d}x.$$

That is,

$$\frac{1}{t_R^{N-p}} \|u_R\|^p_{W^{1,p}_{V_0}} + \|u_R\|^N_{W^{1,N}_{V_0}} \geq N\xi t_R \int_{\mathbb{R}^N} |u_R|^{N+1} \mathrm{d}x. \tag{4.11}$$

As $R \to +\infty$, we obtain $\|u_0\|^N_{W^{1,N}_{V_0}} \geq +\infty$, which is not possible. Hence $\{t_R\}$ is bounded.

Again, suppose that $\lim_{R\to+\infty} t_R \neq 1$. This means either $\lim_{R\to+\infty} t_R = t_1 > 1$ or $\lim_{R\to+\infty} t_R = t_1 < 1$. As $R \to +\infty$, we get

$$\frac{1}{t_1^{N-p}}\|u_0\|^p_{W^{1,p}_{V_0}} + \|u_0\|^N_{W^{1,N}_{V_0}} = \int_{\mathbb{R}^N} f(t_1u_0)t_1u_0\mathrm{d}x. \tag{4.12}$$

Since $u_0 \in \mathcal{N}_0$, it follows that

$$\|u_0\|^p_{W^{1,p}_{V_0}} + \|u_0\|^N_{W^{1,N}_{V_0}} = \int_{\mathbb{R}^N} f(u_0)u_0\mathrm{d}x. \tag{4.13}$$

Now, by subtracting the (4.13) from (4.12), we get

$$\begin{aligned}\left(\frac{1}{t_1^{N-p}} - 1\right)\|u_0\|^p_{W^{1,p}_{V_0}} &= \int_{\mathbb{R}^N}\left(\frac{1}{t_1^N} f(t_1u_0)t_1u_0 - f(u_0)u_0\right)\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\left(\frac{1}{t_1^{N-1}} f(t_1u_0) - f(u_0)\right)u_0\mathrm{d}x.\end{aligned}$$

Using assumption **(f4)**, it follows that in both cases we obtained that $\|u_0\|^p_{W^{1,p}_{V_0}} < 0$, which is not possible. Hence, the claim follows, and this implies that $t_Ru_R \to u_0$ as $R \to +\infty$ in $\boldsymbol{Y}$. Similar to Lemma 3.1, there is $\hat{t} > 0$ such that $I_{\epsilon,\beta}(\hat{t}t_Ru_R) < 0$. Define $g(t) = t\hat{t}t_Ru_R$ for $t \in [0, 1]$. Clearly $g \in \Gamma_{\epsilon,\beta}$. Hence, it holds

$$c_{\epsilon,\beta} \le \max_{t\in[0,1]} I_{\epsilon,\beta}(g(t)) \le \max_{t\ge 0} I_{\epsilon,\beta}(\hat{t}tt_Ru_R) = I_{\epsilon,\beta}(t_*t_Ru_R), \tag{4.14}$$

for some $t_* = t_*(\epsilon, \beta, R)$. Without loss of generality, we can take $V(0) = V_0$. Due to which $V(\epsilon x) \to V_0$ as $\epsilon \to 0$ i.e. for any $\epsilon' > 0$, there is $\epsilon_0 >$ and a ball centered at 0 say (B_{2R} in $\mathbb{R}^N$)such that

$$|V(\epsilon x) - V_0| < \epsilon',\ \forall\, \epsilon \in (0, \epsilon_0) \text{ and } x \in B_{2R}. \tag{4.15}$$

The above equation implies that $V(\epsilon x) < \epsilon' + V_0$. From equation (4.14), we get

$$\begin{aligned}c_{\epsilon,\beta} = &\frac{1}{p}\int_{\mathbb{R}^N}\left(t_*^pt_R^p|\nabla u_R|^p + V(\epsilon x)t_*^pt_R^p|u_R|^p\right)\mathrm{d}x\\ &+ \frac{1}{N}\int_{\mathbb{R}^N}\left(t_*^Nt_R^N|\nabla u_R|^N + V(\epsilon x)t_*^Nt_R^N|u_R|^N\right)\mathrm{d}x\\ &- \int_{\mathbb{R}^N} G_H(\epsilon x, t_*t_Ru_R)\mathrm{d}x\end{aligned}$$

$$\leq \sum_{s\in\{p,N\}} \frac{1}{s} \int_{\mathbb{R}^N} \left(t_*^s t_R^s |\nabla u_R|^s + V_0 t_*^s t_R^s |u_R|^s\right) \mathrm{d}x + \sum_{s\in\{p,N\}} \frac{\epsilon' t_*^s t_R^s}{s} \int_{\mathbb{R}^N} |u_R|^s \mathrm{d}x$$
$$- \int_{\mathbb{R}^N} H(t_* t_R u_R - \beta) G(\epsilon x, t_* t_R u_R) \mathrm{d}x.$$

For $\beta \to 0$, we have

$$c_{\epsilon,\beta} \leq I_{V_0}(t_* t_R u_R) + \epsilon' \sum_{s\in\{p,N\}} \frac{t_*^s t_R^s}{s} \int_{\mathbb{R}^N} |u_R|^s \mathrm{d}x.$$

Next, for large enough $R > 0$, the above equation leads to $\limsup_{\epsilon,\beta\to 0} c_{\epsilon,\beta} \leq c_{V_0}$. Therefore, from (4.7) and (4.8), we can conclude that the limit exists and $\lim_{\epsilon,\beta\to 0} c_{\epsilon,\beta} = c_{V_0}$. □

4.3 Compactness Result

In the next lemma, we will prove the compactness result using the Lions Compactness principle Lions (1984).

Lemma 4.3 *Assume that* **(V1)**–**(V2)** *and* **(f1)**–**(f5)** *hold. Let* ϵ_n *and* $\beta_n \to 0$ *as* $n \to \infty$ *and* $u_n = \{u_{\epsilon_n,\beta_n}\}_{n\geq 1} \subseteq \mathbf{X}_\epsilon$ *be a non negative sequence such that* $0 \in \partial I_{\epsilon_n,\beta_n}(u_n)$, $I_{\epsilon_n,\beta_n}(u_n) \to c_{V_0}$ *and* $\limsup_{n\to+\infty} \|u_n\|^{N'}_{W^{1,N}(\mathbb{R}^N)} < \frac{\alpha_N}{\alpha_0}$. *Then there exits a sequence* $\{y_n\}_{n\geq 1} \subseteq \mathbb{R}^N$ *such that sequence*

$$w_n(x) = u_n(x + y_n)$$

has a convergent subsequence in $\boldsymbol{Y}$. *In addition, upto a subsequence* $a_n = \{\epsilon_n y_n\} \to y_0$ *as* $n \to +\infty$ *for some* $y_0 \in \Lambda$ *and* $V(y_0) = V_0$.

Proof From Lemma 3.3 and Assumption **(V1)**, it follows that $\{u_n\}_{n\geq 1}$ is bounded in $\boldsymbol{Y}$.
Claim 1: There exists $\{y_n\} \subseteq \mathbb{R}^N$, $R > 0$ and $\hat{\sigma} > 0$ such that $\liminf_{n\to+\infty} \int_{B_R(y_n)} |u_n|^N \mathrm{d}x \geq \hat{\sigma}$.

Suppose that the above claim doesn't hold. This means $\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^N \mathrm{d}x = 0$. From Lions (1984) and Alves and Figueiredo (2009), it follows that $u_n \to 0$ in $L^\sigma(\mathbb{R}^N)$ for any $\sigma \in (N, \infty)$. Since $0 \in \partial I_{\epsilon_n,\beta_n}(u_n)$, this implies that

$$\|u_n\|^p_{W^{1,p}_{V_{\epsilon_n}}} + \|u_n\|^N_{W^{1,N}_{V_{\epsilon_n}}} = \int_{\mathbb{R}^N} \rho_n u_n \mathrm{d}x + o_n(1) \;\text{ as } n \to +\infty.$$

From assumption **(f1)**, **(f2)** (Remark 1.1) and $\limsup_{n\to+\infty}\|u_n\|^{N'}_{W^{1,N}(\mathbb{R}^N)} < \frac{\alpha_N}{\alpha_0}$, it follows that for any $\tau > 0$

$$\left|\int_{\mathbb{R}^N} \rho_n u_n \mathrm{d}x\right| \leq M_1\tau, \text{ as } n \to +\infty.$$

So for sufficiently small $\tau > 0$ we deduce that $\|u_n\|^p_{W^{1,p}_{V_{\epsilon_n}}} + \|u_n\|^N_{W^{1,N}_{V_{\epsilon_n}}} \to 0$ as $n \to +\infty$.

This means $u_n \to 0$ in $\boldsymbol{X}_{\boldsymbol{\epsilon_n}}$ as $n \to +\infty$. It follows that $I_{\epsilon_n,\beta_n}(u_n) \to 0$ as $n \to +\infty$ which contradicts Lemma 4.2. Hence, our claim follows.

Define $w_n(x) = u_n(x + y_n)$. Since $\|.\|_{\boldsymbol{Y}}$ is invariant under translation. This implies that $\{w_n\}_{n\geq 1}$ is bounded in $\boldsymbol{Y}$. So up to a subsequence

$$\begin{cases} w_n \rightharpoonup w \text{ in } \boldsymbol{Y}, \\ w_n \to w \text{ in } L^s(B_R) \text{ for } s \geq 1, \\ w_n(x) \to w(x) \text{ a.e. in } \mathbb{R}^N. \end{cases}$$

From Claim 1, there holds $\int_{B_R(0)} |w|^N \mathrm{d}x \geq \hat{\sigma} > 0$.

This means $w \not\equiv 0$.

Next, we will show that $\{a_n\} = \{\epsilon_n y_n\}_{n\geq 1}$ is bounded in $\mathbb{R}^N$. If we are able to show that $\lim_{n\to+\infty} \mathrm{dist}(a_n, \overline{\Lambda}) = 0$,

then we can deduce boundedness of $\{a_n\}_{n\geq 1}$. On the contrary, we assume that it is not true, then there exists $\delta^* > 0$ and a subsequence of $\{a_n\}$ such that $\mathrm{dist}(a_n, \overline{\Lambda}) \geq \delta^*$ for all $n \in \mathbb{N}$. This implies there exists $r > 0$, such that $B_r(a_n) \subset \Lambda^c$. Define $\phi_n(x) = \psi(\frac{x}{n})w(x)$, where ψ is defined in Lemma 3.4. Note that, $\phi_n \to w$ in $\boldsymbol{Y}$ as $n \to +\infty$. Since, $0 \in \partial I_{\epsilon_n,\beta_n}(u_n)$ and assumption **(V1)** with a change of variable $z \mapsto x + y_n$, we obtain

$$\begin{aligned}\sum_{t\in\{p,N\}} \int_{\mathbb{R}^N} (|\nabla w_n|^{t-2}\nabla w_n \nabla\phi_n + V_0|w_n|^{t-1}\phi_n)\mathrm{d}x &\leq \int_{\mathbb{R}^N} g(\epsilon_n x, w_n)\phi_n \mathrm{d}x \\ &= \int_{B_{\frac{r}{\epsilon_n}}} g(\epsilon_n x, w_n)\phi_n \mathrm{d}x + \int_{B^c_{\frac{r}{\epsilon_n}}} g(\epsilon_n x, w_n)\phi_n \mathrm{d}x.\end{aligned} \tag{4.16}$$

Since $m(B^c_{\frac{r}{\epsilon_n}}) \to 0$ as $n \to +\infty$, by the Lebesgue dominated convergence theorem

$$\int_{B^c_{\frac{r}{\epsilon_n}}} g(\epsilon_n x, w_n)\phi_n \mathrm{d}x \to 0 \text{ as } n \to +\infty.$$

Now

$$\int_{B_{\frac{r}{\epsilon_n}}} g(\epsilon_n x, w_n)\phi_n \mathrm{d}x = \int_{B_{\frac{r}{\epsilon_n}}} \tilde{f}(w_n)\phi_n \mathrm{d}x \leq \int_{B_{\frac{r}{\epsilon_n}}} f(w_n)\phi_n \mathrm{d}x + \frac{V_0}{k}\int_{B_{\frac{r}{\epsilon_n}}} |w_n|^{N-1}\phi_n \mathrm{d}x.$$

Using Hölder's inequality one can show that

$$\frac{V_0}{k}\int_{B_{\frac{r}{\epsilon_n}}} |w_n|^{N-1}\phi_n \mathrm{d}x \to 0 \text{ as } n \to +\infty.$$

From assumptions **(f1)**, **(f2)** and $\limsup_{n\to+\infty} \|u_n\|^{N'}_{W^{1,N}(\mathbb{R}^N)} < \frac{\alpha_N}{\alpha_0}$, we get

$$\int_{B_{\frac{r}{\epsilon_n}}} f(w_n)\phi_n \mathrm{d}x \to 0 \text{ as } n \to +\infty.$$

Therefore, as $n \to +\infty$ it follows that $w \equiv 0$, which is absurd. Hence $\{a_n\}$ is bounded in $\mathbb{R}^N$. So up to a subsequence, still denoted by itself, such that $a_n \to y_0 \in \overline{\Lambda}$.

Claim 2: The sequence $w_n \to w$ in $\boldsymbol{Y}$.

For each $n \in \mathbb{N}$, there is a $t_n > 0$ such that $t_n w_n \in \mathcal{N}_0$ and using similar arguments as in Lemma 4.2, it can be proved that $\{t_n\}_{n\geq 1}$ is bounded and up to a subsequence we can assume that $t_n w_n \to t_0 w$ a.e. in $\mathbb{R}^N$. So by a change of variable, **(V1)** and using $g_H(x,t) \leq f(t)$, we get,

$$c_{V_0} \leq I_{V_0}(t_n w_n) \leq I_{\varepsilon_n,\beta_n}(t_n u_n) \leq I_{\epsilon_n,\beta_n}(u_n) = c_{V_0} + o_n(1) \text{ as } n \to +\infty. \quad (4.17)$$

Consequently, $I_{V_0}(t_n w_n) \to c_{V_0}$ as $n \to +\infty$. Now using Lemma 2.2, we conclude our claim.

Claim 3: $V(y_0) = V_0$ and $y_0 \in \Lambda$.

Suppose that the above claim is not true. This implies that $V(y_0) > V_0$. Once that $t_n w_n \to t_0 w$ in $\boldsymbol{Y}$, by Fatou's lemma with the invariance of $\mathbb{R}^N$ by translation we obtain that

$$\begin{aligned}
c_{V_0} = I_{V_0}(t_0 w) &< \sum_{s\in\{p,N\}} \frac{1}{s}(|\nabla t_0 w|^s + V(y_0)|t_0 w|^s)\mathrm{d}x - \int_{\mathbb{R}^N} F(t_0 w)\mathrm{d}x \\
&\leq \liminf_{n\to+\infty} \sum_{s\in\{p,N\}} \frac{1}{s}(|\nabla t_n w_n|^s + V(\epsilon_n x + a_n)|t_n w_n|^s)\mathrm{d}x - \int_{\mathbb{R}^N} F(t_n w_n)\mathrm{d}x \\
&\leq \liminf_{n\to+\infty} I_{\epsilon_n,\beta_n}(t_n u_n) \leq c_{V_0}.
\end{aligned}$$

which is not possible. Therefore $V(y_0) = V_0$ and thanks to **(V2)**, it follows that $y_0 \notin \partial\Lambda$. Hence $y_0 \in \Lambda$. □

4.4 Moser Iteration Argument

In the next Lemma, we will show the L^∞ estimate of $\{w_n\}_{n\geq 1}$ using Moser iteration arguments.

Lemma 4.4 *Assume that (V1)–(V2) and (f1)–(f5) hold. Let $\{w_n\}_{n\geq 1}$ be defined in Lemma 4.3. Then there exist a constant $K > 0$ such that*

$$\|w_n\|_{L^\infty(\mathbb{R}^N)} \leq K, \text{ for all } n \in \mathbb{N}.$$

Moreover, we have

$$\lim_{|x|\to+\infty} w_n(x) = 0, \text{ uniformly in } n \in \mathbb{N}.$$

Proof From Lemma 4.2, it follows that $\lim_{n\to+\infty} I_{\epsilon_n,\beta_n}(u_n) = c_{V_0}$. Furthermore, there exist $\{y_n\}_{n\geq 1} \subseteq \mathbb{R}^N$ satisfies $w_n(x) = u_n(x + y_n)$ and $\{\epsilon_n y_n\} \to y_0 \in \Lambda$. For $L > 0$ and $\gamma > 1$, define

$$g(w_n) = w_n w_{L,n}^{N(\gamma-1)} \text{ and } v_{L,n} = w_n w_{L,n}^{\gamma-1} \tag{4.18}$$

where $w_{L,n} = \min\{w_n, L\}$. On using $g(w_n)$ as a test function in equation (3.40) and a simple change of variable leads to

$$\sum_{s\in\{p,N\}} \int_{\mathbb{R}^N} \Big(|\nabla w_n|^{s-2}\nabla w_n \cdot \nabla g(w_n) + V(\epsilon_n x + a_n)|w_n|^{s-2} w_n g(w_n)\Big)\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho_n(x + y_n) g(w_n)\mathrm{d}x,$$

then

$$\begin{aligned}\sum_{s\in\{p,N\}} \int_{\mathbb{R}^N} (|\nabla w_n|^s w_{L,n} + N(\gamma-1)|\nabla w_n|^{s-2}\nabla w_n \cdot \nabla w_{L,n} w_n w_{L,n}^{N(\gamma-1)-1}\mathrm{d}x \\ + V(\epsilon_n x + a_n)|w_n|^{s-2} w_n g(w_n))\mathrm{d}x \\ = \int_{\mathbb{R}^N} \rho_n(x + y_n) g(w_n)\mathrm{d}x.\end{aligned}$$

Therefore, we have

$$\sum_{s\in\{p,N\}}\int_{\mathbb{R}^N}(|\nabla w_n|^s w_{L,n})\mathrm{d}x + N(\gamma-1)\int_{w_n\le L} w_n w_{L,n}^{N(\gamma-1)-1}|\nabla w_{L,n}|^s\mathrm{d}x$$
$$+\int_{\mathbb{R}^N} V(\epsilon_n x + a_n)|w_n|^{s-2}w_n g(w_n)\mathrm{d}x = \int_{\mathbb{R}^N}\rho_n(x+y_n)g(w_n)\mathrm{d}x. \tag{4.19}$$

Set

$$\Xi(t) = \int_0^t (g'(\tau))^{\frac{1}{N}}\mathrm{d}\tau \text{ and } \theta(t) = \frac{|t|^N}{N}. \tag{4.20}$$

since g is an increasing function, therefore $(s-t)(g(s)-g(t)) \ge 0$ for all $s, t \in \mathbb{R}$.
Further, applying Jensen's inequality, one has

$$|\theta(s)-\theta(t)| \le \Xi'(s-t)(g(s)-g(t)),\ \forall\, s, t \in \mathbb{R}. \tag{4.21}$$

From Eq. (4.20), it follows that

$$\frac{1}{\gamma}w_{L,n}^{\gamma-1}w_n \le \Xi(w_n) \le w_{L,n}^{\gamma-1}w_n. \tag{4.22}$$

Due to equivalent norm and continuous embedding $W_{V_0}^{1,N}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for $s > N$, i.e., there is a best constant $C_* > 0$ such that

$$\frac{1}{\gamma^N}C_*^N\|w_n w_{L,n}^{\gamma-1}\|_{L^s(\mathbb{R}^N)}^N \le C_*^N\|\Xi(w_n)\|_{L^s(\mathbb{R}^N)}^N \le \|\Xi(w_n)\|_{W_{V_0}^{1,N}(\mathbb{R}^N)}^N. \tag{4.23}$$

Then, (4.19) and assumptions **(f1)**, **(f2)** (Remark 1.1) lead us to the following

$$\|\Xi(w_n)\|_{W_{V_0}^{1,N}(\mathbb{R}^N)}^N \le \left(\tau\int_{\mathbb{R}^N}|w_n|^N w_{L,n}^{N(\gamma-1)}\mathrm{d}x + \int_{\mathbb{R}^N}C(\tau)|w_n|^{\nu-1}w_n w_{L,n}^{N(\gamma-1)}\Phi_{\alpha_0,N-2}(w_n)\mathrm{d}x\right). \tag{4.24}$$

From equation (4.23), we obtain

$$\|\Xi(w_n)\|_{W_{V_0}^{1,N}(\mathbb{R}^N)}^N \le \tau\gamma^N C_*^N\|\Xi(w_n)\|_{W_{V_0}^{1,N}(\mathbb{R}^N)}^N$$
$$+\int_{\mathbb{R}^N}C(\tau)|w_n|^{\nu-1}w_n w_{L,n}^{N(\gamma-1)}\Phi_{\alpha_0,N-2}(w_n)\mathrm{d}x. \tag{4.25}$$

Choose $\tau > 0$ such that $0 < \tau < \gamma^{-N}C_*^{-N}$ and using generalized Hölder's inequality, boundedness of w_n, Lemmas 3.5 and 2.1, one leads to

$$\|w_n w_{n,L}^{\gamma-1}\|_{L^s(\mathbb{R}^N)}^N \le \tilde{C}\gamma^N\|w_n w_{n,L}^{\gamma-1}\|_{L^{N\mu}(\mathbb{R}^N)}^{N\mu} \tag{4.26}$$

for some $\mu > 1$. Since $w_{n,L} \leq w_n$ and as $L \to +\infty$, Fatou's lemma implies that

$$\|w_n\|_{L^{\gamma s}(\mathbb{R}^N)} \leq (\tilde{C})^{\frac{1}{N\gamma}} \gamma^{\frac{1}{\gamma}} \|w_n\|^{\mu}_{L^{N\mu\gamma}(\mathbb{R}^N)}. \tag{4.27}$$

Choose $\gamma = \frac{s}{N\mu}$, then $\gamma^2 N\mu = \gamma s$. Replace γ by γ^2 and using equation(4.27) we obtained that

$$\|w_n\|_{L^{\gamma^2 s}(\mathbb{R}^N)} \leq (\tilde{C})^{\frac{1}{N\gamma^2}} \gamma^{\frac{1}{\gamma^2}} \|w_n\|^{\mu}_{L^{\gamma s}(\mathbb{R}^N)} \leq (\tilde{C})^{\frac{1}{N}(\frac{1}{\gamma}+\frac{1}{\gamma^2})} \gamma^{(\frac{1}{\gamma}+\frac{1}{\gamma^2})} \|w_n\|^{\mu}_{L^{N\mu\gamma}(\mathbb{R}^N)}.$$

On repeating the same arguments k times, we obtain

$$\|w_n\|_{L^{\gamma^k s}(\mathbb{R}^N)} \leq (\tilde{C})^{\frac{1}{N}\sum_{j=1}^{k}(\frac{1}{\gamma^j})} \gamma^{\sum_{j=1}^{k}(\frac{1}{\gamma^j})} \|w_n\|^{\mu}_{L^{N\mu\gamma}(\mathbb{R}^N)}.$$

As $k \to \infty$ in above equation, for any $n \in \mathbb{N}$, we get $\|w_n\|_{L^\infty(\mathbb{R}^N)} \leq K$.
for some constant $K > 0$. Now we will show that

$$\lim_{|x| \to +\infty} w_n(x) = 0, \text{ uniformly in } n \in \mathbb{N}.$$

Due to embedding $W^{1,p}_{V_\epsilon}(\mathbb{R}^N) \cap W^{1,N}_{V_\epsilon}(\mathbb{R}^N) \hookrightarrow W^{1,p}_{V_0}(\mathbb{R}^N) \cap W^{1,N}_{V_0}(\mathbb{R}^N)$, we deduce that $\{w_n\}_{n\geq 1}$ is bounded in $\boldsymbol{Y}$. Moreover, $w_n \to w$ in $L^t(\mathbb{R}^N)$, $\forall\, t \in [p, p*] \cup [N, +\infty)$ and $w_n \to w$ a.e. in $\mathbb{R}^N$. It can be observed that w_n solves

$$-\Delta_p w_n - \Delta_N w_n + V_0(w_n^{p-1} + w_n^{N-1}) \leq \tau w_n^{N-1} + C(\tau) w_n^{\nu-1} \Phi_{\alpha_0, N-2}(w_n) \tag{4.28}$$

in the weak sense. Using the arguments used in Liu and Van Rooij (1969) it can be proved that

$$\tau w_n^{N-1} + C(\tau) w_n^{\nu-1} \Phi_{\alpha_0, N-2}(w_n) \in L^{\frac{N}{N-1}}(\mathbb{R}^N) \subset Y^*$$

where $\boldsymbol{Y}^*$ denotes the dual space of $\boldsymbol{Y}$. Define the operator $T : \boldsymbol{Y} \to \boldsymbol{Y}^*$

$$\langle T(u), v \rangle = \sum_{s \in \{p,N\}} \int_{\mathbb{R}^N} |\nabla u|^{s-2} \nabla u \cdot \nabla v + V_0 |u|^{s-2} u v \mathrm{d}x \tag{4.29}$$

for all $u, v \in \boldsymbol{Y}$. By following the arguments used in Mahanta et al. (2025b), we can show that T is surjective. Due to which we have $z_n \in \boldsymbol{Y}$ such that $T(z_n) = \tau w_n^{N-1} + C(\tau) w_n^{\nu-1} \Phi_{\alpha_0, N-2}(w_n)$. This means for all $v \in \boldsymbol{Y}$

$$\sum_{s \in \{p,N\}} \int_{\mathbb{R}^N} |\nabla z_n|^{s-2} \nabla z_n \cdot \nabla v + V_0 |z_n|^{s-2} z_n v \mathrm{d}x = \int_{\mathbb{R}^N} \left(\tau w_n^{N-1} + C(\tau) w_n^{\nu-1} \Phi_{\alpha_0, N-2}(w_n) \right) v \mathrm{d}x. \tag{4.30}$$

Assume $v = z_n^-$ as a test function in equation (4.30), then we obtain $z_n^- = 0$ a.e. in $\mathbb{R}^N$. Hence $z_n \geq 0$ a.e. in $\mathbb{R}^N$. On using equations (4.28), (4.30) and (Brasco et al., 2022, Theorem 4.1), we conclude that

$$0 \leq w_n \leq z_n \text{ a.e. in } \mathbb{R}^N. \tag{4.31}$$

By using the Young's inequality (i.e. for any $\vartheta > 0$ and $a, b \geq 0$ with $\frac{1}{s} + \frac{1}{s'} = 1$,

$$ab \leq \vartheta a^s + C(\vartheta) b^{s'},$$

where C is a constant depends only on ϑ. Hölder's inequality and Lemma 2.1 we can obtain the boundedness of $\{z_n\}_{n\geq 1}$ in $\boldsymbol{Y}$. Consequently, up to a subsequence still denoted by itself

$$z_n \rightharpoonup z \text{ in } \boldsymbol{Y} \text{ and } z_n \to z \text{ a.e. in } \mathbb{R}^N.$$

So,

$$\|z\|^p_{W^{1,p}_{V_0}(\mathbb{R}^N)} + \|z\|^N_{W^{1,N}_{V_0}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \Big(\tau w^{N-1} + C(\tau) w^{\nu-1} \Phi_{\alpha_0, N-2}(w)\Big) z \mathrm{d}x. \tag{4.32}$$

Next we will show that $\lim_{n\to+\infty} \int_{\mathbb{R}^N} w_n^{N-1} z_n \mathrm{d}x = \int_{\mathbb{R}^N} w^{N-1} z \mathrm{d}x$.

Since $w^{N-1} \in L^{\frac{N}{N-1}}(\mathbb{R}^N)$ and using the weak convergence of z_n in $L^N(\mathbb{R}^N)$, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} w^{N-1}(z_n - z) \mathrm{d}x = 0. \tag{4.33}$$

On the other side, due to $w_n \to w$ in $L^N(\mathbb{R}^N)$ and Lebesgue dominated convergence theorem, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |w_n^{N-1} - w^{N-1}|^{\frac{N}{N-1}} \mathrm{d}x = 0.$$

Now

$$\begin{aligned}\int_{\mathbb{R}^N} (w_n^{N-1} z_n - w^{N-1} z) \mathrm{d}x &= \int_{\mathbb{R}^N} (w_n^{N-1} z_n - w^{N-1} z_n + w^{N-1} z_n - w^{N-1} z) \mathrm{d}x \\ &= \int_{\mathbb{R}^N} (w_n^{N-1} z_n - w^{N-1} z_n) \mathrm{d}x + \int_{\mathbb{R}^N} (w^{N-1} z_n - w^{N-1} z) \mathrm{d}x.\end{aligned}$$

Using Hölder's inequality and boundedness of $\{z_n\}$ in $\boldsymbol{Y}$, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} (w_n^{N-1} z_n - w^{N-1} z_n) \mathrm{d}x = 0. \tag{4.34}$$

From Eqs. (4.33) and (4.34) it follows that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} w_n^{N-1} z_n \mathrm{d}x = \int_{\mathbb{R}^N} w^{N-1} z \mathrm{d}x. \tag{4.35}$$

Similarly, it follows that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} w_n^{N-1} z_n \Phi_{\alpha_0,N-2}(w_n)\mathrm{d}x = \int_{\mathbb{R}^N} w^{N-1} z \Phi_{\alpha_0,N-2}(w)\mathrm{d}x. \tag{4.36}$$

By virtue of equations (4.35) and (4.36)

$$\|z_n\|^p_{W^{1,p}_{V_0}(\mathbb{R}^N)} + \|z_n\|^N_{W^{1,N}_{V_0}(\mathbb{R}^N)} = \|z\|^p_{W^{1,p}_{V_0}(\mathbb{R}^N)} + \|z\|^N_{W^{1,N}_{V_0}(\mathbb{R}^N)} + o_n(1)$$

as $n \to +\infty$. From Brézis-Lieb lemma (Brézis and Lieb 1983), it follows that $z_n \to z$ in $\boldsymbol{Y}$. Since $0 \le w_n \le z_n$ a.e. in $\mathbb{R}^N$, again using Moser iteration argument as in the first part of this lemma, we have

$$\|z_n\|_{L^\infty(\mathbb{R}^N)} \le C$$

for some constant $C > 0$. Hence from Tong et al. (2025) and (4.31), it follows that

$$\lim_{|x|\to+\infty} w_n(x) = 0, \text{ uniformly in } n \in \mathbb{N}.$$

This completes the proof. □

5 Proof of The Main Theorem

Now we are empowered enough to conclude the proof of the main Theorem of this article.

Proof of Theorem 1.4 We will prove that there exists $\tilde{\epsilon}$, $\beta > 0$ such that for every $\epsilon \in (0, \tilde{\epsilon})$, $\beta \in (0, \hat{\beta})$ and every solution $u_{\epsilon,\beta}$ of problem $(\mathscr{P}^a_{\epsilon,\beta})$ satisfies

$$\|u_{\epsilon,\beta}\|_{L^\infty(\Lambda^c_\epsilon)} < a. \tag{5.1}$$

On contrary suppose that (5.1) does not hold, i.e., for some subsequence $\epsilon_n, \beta_n \to 0$ we have $\{u_n\}(:= \{u_{\epsilon_n,\beta_n}\})$ such that $I_{\epsilon_n,\beta_n}(u_n) = c_{\epsilon_n,\beta_n}$, $0 \in \partial I_{\epsilon_n,\beta_n}(u_n)$ and

$$\|u_{\epsilon,\beta}\|_{L^\infty(\Lambda^c_\epsilon)} \ge a. \tag{5.2}$$

From Lemma 4.3, we have $(y_n) \subset \mathbb{R}^N$ such that $w_n(x) = u_n(x + y_n) \to w$ in $\boldsymbol{Y}$ and $\epsilon_n y_n \to y_0$ with $V(y_0) = V_0$. Set $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$ and hence

$$B_{\frac{r}{\epsilon_n}}\left(\frac{y_0}{\epsilon_n}\right) \subset \Lambda_{\epsilon_n}.$$

So, for all $y \in B_{\frac{r}{\epsilon_n}}(y_n)$,

$$\left|y - \frac{y_0}{\epsilon_n}\right| \leq |y - y_n| + \left|y_n - \frac{y_0}{\epsilon_n}\right| < \frac{2r}{\epsilon_n}$$

for large n and in that case, there holds

$$\Lambda^c_{\epsilon_n} \subset B^c_{\frac{r}{\epsilon_n}}(y_n). \tag{5.3}$$

By Lemma 4.4, we have

$$w_n(x) \to 0 \text{ as } |x| \to +\infty \text{ uniformly in } n.$$

Hence there is $\mathcal{R} > 0$ such that

$$|w_n(x)| < a, \ \forall\, |x| \geq \mathcal{R}, \ n \in \mathbb{N}.$$

So $u_n(x) < a$ for $x \in B^c_{\mathcal{R}}(y_n)$ and from (5.3) there exist $n_0 \in \mathbb{N}$ such that

$$\Lambda^c_{\epsilon_n} \subset B^c_{\frac{r}{\epsilon_n}}(y_n) \subset B^c_{\mathcal{R}}(y_n), \ \forall\, n \geq n_0.$$

This means $u_n(x) < a$ for all $x \in \Lambda^c_{\epsilon_n}$ and $n \geq n_0$, which contradicts (5.2). Hence, our claim holds. Since $u_{\epsilon,\beta}$ is solution for problem $(\mathscr{P}^a_{\epsilon,\beta})$ and satisfies

$$\|u_{\epsilon,\beta}\|_{L^\infty(\Lambda^c_\epsilon)} < a. \tag{5.4}$$

From Remark 3.2, it follows that $u_{\epsilon,\beta}$ is also a solution for main problem $(\mathscr{P}_{\epsilon,\beta})$ for all $\epsilon \in (0, \tilde{\epsilon})$, and $\beta \in (0, \hat{\beta})$. Next, we will explore the behaviour of the maximum points of $u_{\epsilon,\beta}$ for small enough ϵ and β. From assumption **(g3)**, one can conclude that there is $\tau \in (0, a)$ such that

$$|g(\epsilon x, s)s| \leq \frac{V_0}{2} s^N, \ \forall\, x \in \mathbb{R}^N \text{ and } s \leq \tau. \tag{5.5}$$

From (5.4) we have

$$\|u_n\|_{L^\infty(B^c_{\mathcal{R}}(y_n))} < \tau.$$

Also, we can assume that

$$\|u_n\|_{L^\infty(B_{\mathcal{R}}(y_n))} \geq \tau. \tag{5.6}$$

Suppose (5.6) does not hold. Then, using the fact that $0 \in \partial I_{\epsilon_n,\beta_n}(u_n)$ and from (5.5), we have

$$0 < \int\limits_{\mathbb{R}^N} \left(|\nabla u_n|^p + V_0|u_n|^p\right) \mathrm{d}x + \int\limits_{\mathbb{R}^N} \left(|\nabla u_n|^N + V_0|u_n|^N\right) \mathrm{d}x \leq \frac{V_0}{2} \int\limits_{\mathbb{R}^N} |u_n|^N \mathrm{d}x$$

which leads to $u_n \to 0$ as $n \to \infty$. Consequently $I_{\epsilon_n,\beta_n}(u_n) \to 0$; which is not possible. Hence (5.6) holds. Let x_n be the global maximum point of u_n. This means $x_n \in B_{\mathcal{R}}(y_n)$. Set $x_n = y_n + z_n$ for some $z_n \in B_{\mathcal{R}}$. So $\epsilon_n x_n = \epsilon_n y_n + \epsilon_n z_n \to y_0$ as $n \to +\infty$. Now, using the continuity of the function V, we have $\lim\limits_{n\to+\infty} V(\epsilon_n x_n) = V_0$, which concludes the proof. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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