Theoretical and Mathematical Physics, **219**(1): 688–703 (2024)

SIMILARITY OF COSMOLOGICAL MODELS AND ITS APPLICATION TO THE ANALYSIS OF COSMOLOGICAL EVOLUTION

Yu. G. Ignat'ev*

Scale transformations of cosmological models based on a statistical system of degenerate fermions with a scalar Higgs interaction are studied. The similarity properties of cosmological models under scale transformations of their fundamental parameters are revealed. The transformation laws for the coordinates of singular points and eigenvalues of the characteristic matrix of the dynamical system of the cosmological model under its scale transformations are established. With the help of the transformation to new variables, the previously studied dynamical system of scalar-charged fermions is modified to a dynamical system with a nondegenerate characteristic matrix; for its nondegenerate branch, the singular points and eigenvalues of the characteristic matrix are found, which coincide with the corresponding values for the vacuum field model. Examples of numerical simulation of such cosmological models are given.

Keywords: scalar-charged plasma, cosmological model, Higgs scalar field, similarity transformation, qualitative analysis

DOI: 10.1134/S0040577924040123

1. Introduction

Methods of similarity theory and the dimensional analysis of dynamical systems [1] have long been successfully used in mechanics, hydro- and gasdynamics [2], as well as in astrophysics and cosmology [3]. These methods allow extending the results of research to other models and are especially valuable in the study of complex, essentially nonlinear dynamical systems, when the use of numerical modeling methods becomes mandatory. Revealing the laws of similarity of such dynamical systems allows extending the results of numerical integration to models with other parameters, thereby providing the possibility of a comprehensive numerical–analytical study of a class of models.

The methods of similarity theory and dimensional analysis become especially effective in the study of cosmological models, based, in turn, on various field-theory models, often containing fundamental constants and parameters not determined at the time of research. In [4], on the basis of microscopic dynamics, a macroscopic model of the Universe was formulated based on statistical systems of fermions with scalar charges, classical and phantom, with the Higgs interaction potential.[1] Subsequently, various versions of this model were constructed and studied (see, e.g., [6]), and were also studied for stability with respect to longitudinal plane-wave gravitational disturbances (see, e.g., [7]). In these works, a short-wave scalar-gravitational instability of a homogeneous cosmological model was identified and studied, which is fundamentally different from the previously studied hydrodynamic and gaseous gravitational instability. The same studies

*Institute of Physics, Kazan Federal University, Kazan, Russia, e-mail: yurii.ignatev.1947@yandex.ru.

[1] We note that apparently the first study in which, at least formally, scalar interaction was included in the general relativistic statistics, was apparently the 1961 work by Tauber and Wuinberg [5].

Translated from *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 219, No. 1, pp. 181–198, April, 2024. Received July 22, 2023. Revised September 13, 2023. Accepted September 19, 2023.

demonstrated the fundamental possibility of supermassive black holes in the early Universe using the mechanism of scalar-gravitational instability. In [8], the evolution of spherical gravitational perturbations in a medium of scalar-charged fermions with the Higgs interaction was studied, including the evolution of localized perturbations without a wavelength limitation.

Studies on the scalar-gravitational stability of the cosmological medium of degenerate scalar-charged fermions, in particular, revealed a close connection between the singular points of the vacuum-field cosmological model and the appearance of unstable phases in the model with charged fermions. This revealed connection makes it necessary to carry out a more detailed qualitative study of the dynamical systems of cosmological models based on statistical systems of scalar-charged fermions. To study the general regularities for these models, the methods of similarity theory are also necessary. This paper is devoted to these questions. We note that the general similarity property for self-gravitating systems with scalar Higgs interaction was formulated in [8].

We also note that from the late 1990s to the present, many researchers (B. Saha, G. N. Shikin, M. O. Ribas, F. P. Devecchi, G. M. Kremer, L. Fabbri, J. Wang, S.-W. Cui, C.-M. Zhang, Y. P. Rybakov, K. A. Bronnikov, T. Boyadzhiev, and others) studied cosmological models based on scalar and nonlinear fermionic (spinor) fields (see, e.g., [9]–[16] and references therein).

2. Cosmological system of fermions with scalar interaction

We briefly state the main provisions of the macroscopic theory[2] for a cosmological model based on a one-component degenerate statistical system of scalar-charged fermions and a scalar Higgs field Φ.

2.1. General model equations. The Lagrange function L_s of the scalar Higgs field is[3]

$$L_s = \frac{1}{16\pi}(g^{ik}\Phi_{,i}\Phi_{,k} - 2V(\Phi)), \tag{1}$$

where

$$V(\Phi) = -\frac{\alpha}{4}\left(\Phi^2 - \frac{m_s^2}{\alpha}\right)^2 \tag{2}$$

is the potential energy of the scalar field, α is the self-action constant, and m_s is the mass of the scalar field quanta. The energy–momentum tensors of scalar fields relative to the Lagrange function (1) and *equilibrium* statistical system are

$$S^i_k = \frac{1}{16\pi}\bigl(2\Phi^{,i}\Phi_{,k} - \delta^i_k\Phi_{,j}\Phi^{,j} + 2V(\Phi)\delta^i_k\bigr), \tag{3}$$

$$T^i_k = (\varepsilon_p + p_p)u^i u_k - \delta^i_k p_p, \tag{4}$$

where u^i is the velocity vector of the statistical system, and ε_p and p_p are the energy density and pressure of the statistical system.

Einstein's equations for the "scalar field + particles" system are:

$$R^i_k - \frac{1}{2}\,\delta^i_k R = 8\pi(T^i_k + S^i_k) + \delta^i_k\Lambda_0, \tag{5}$$

where Λ_0 is the seed value of the cosmological constant, related to its observed value Λ, obtained by removing the constant term in the potential energy, as

$$\Lambda = \Lambda_0 - \frac{m_s^4}{4\alpha}\,. \tag{6}$$

[2]In [4] it is shown how this theory is obtained from the microscopic dynamics.

[3]Here and hereafter, Latin letters range $\overline{1,4}$, and Greek letter, $\overline{1,3}$. We also use the Planck system of units, where $G = \hbar = c = 1$.

The macroscopic consequences of the kinetic theory are the transfer equations [4], including the conservation law for some vector current corresponding to the microscopic conservation law for some fundamental charge

$$\nabla_i q n^i = 0, \tag{7}$$

as well as the energy–momentum conservation laws of the statistical system,

$$\nabla_k T_p^{ik} - \sigma \nabla^i \Phi = 0, \tag{8}$$

where σ is the density of scalar charges with respect to the field Φ [4]. Equations (8) are equivalent to the equations of ideal hydrodynamics

$$(\varepsilon_p + p_p) u^i_{,k} u^k = (g^{ik} - u^i u^k)(p_{p,k} + \sigma \Phi_{,k}), \tag{9}$$

$$\nabla_k [(\varepsilon_p + p_p) u^k] = u^k (p_{p,k} + \sigma \Phi_{,k}), \tag{10}$$

and the fundamental charge conservation laws (7)

$$\nabla_k \rho u^k = 0, \tag{11}$$

where $\rho \equiv qn$ is the *kinematic density of the scalar charge*.

Macroscopic scalars for a one-component statistical system of degenerate fermions take the form

$$n = \frac{1}{\pi^2}\,\pi_{\mathrm{F}}^3, \qquad p_p = \frac{e^4\Phi^4}{24\pi^2}(F_2(\psi) - 4F_1(\psi)), \tag{12}$$

$$\sigma = \frac{e^4\Phi^3}{2\pi^2}\,F_1(\psi), \qquad \varepsilon_p = \frac{e^4\Phi^4}{8\pi^2}\,F_2(\psi), \tag{13}$$

where π_{F} is the Fermi momentum, σ is the density of scalar charges e, and

$$\psi = \frac{\pi_{\mathrm{F}}}{|e\Phi|} \tag{14}$$

and the functions $F_1(\psi)$ and $F_2(\psi)$ are introduced as

$$F_1(\psi) = \psi\,\sqrt{1+\psi^2} - \ln\big(\psi + \sqrt{1+\psi^2}\,\big),$$
$$F_2(\psi) = \psi\,\sqrt{1+\psi^2}(1+2\psi^2) - \ln\big(\psi + \sqrt{1+\psi^2}\,\big).$$

The scalar field equation for a system of scalar-charged degenerate fermions can be derives as a corollary of the transport equations

$$\Box\Phi + m_s^2\Phi - \alpha\Phi^3 = -8\pi\sigma \equiv -\frac{4e^4\Phi^3}{\pi}\,F_1(\psi). \tag{15}$$

Thus, the complete system of equations for the mathematical model $\mathbf{M}$ of a system of scalar-charged fermions consists of Einstein's equations (5), hydrodynamic equations (8), and scalar field equation (15) together with definitions of the sources: scalar field energy–momentum tensors (3), fermionic component (4), and scalar charge density (13), as well as fermion energy density (12) and the corresponding pressure (13). As can be seen from the equations of this system and the definition of its coefficients, the solution of the Cauchy problem for this system of equations for given *fundamental parameters*

$$\mathbf{p} = [\,[\alpha, m_s, e], \Lambda] \tag{16}$$

are completely determined by the corresponding initial conditions for the metric functions $g_{ik}(x^j)$, the potential $\Phi(x^j)$, the velocity vector $u^i(x^j)$, and the Fermi momentum $\pi_{\mathrm{F}}(x^j)$. This complete system of equations, together with the definitions of the functions contained in them, given by the initial conditions on a Cauchy hypersurface and a given set of fundamental parameters $\mathbf{p}$, is referred to as the mathematical model $\mathbf{M}$ of a self-gravitating statistical system of degenerate scalar-charged fermions with the Higgs interaction.

2.2. The similarity property of the mathematical model. In [8], the following similarity property of the considered dynamical system was proved.

Assertion 1. *The complete system of equations of the mathematical model* $\mathbf{M}$ *is invariant under simultaneous scaling transformations of fundamental parameters* $\mathbf{P}$

$$\mathcal{S}_k(\mathbf{M})\colon \quad \alpha = k^2\tilde{\alpha}, \qquad m_s = k\tilde{m}_s, \qquad e = \sqrt{k}\,\tilde{e}, \qquad \Lambda = k^2\tilde{\Lambda}, \tag{17}$$

and the coordinates and Fermi momentum

$$x^i = k^{-1}\tilde{x}^i, \qquad \pi_{\mathrm{F}} = \sqrt{k}\,\tilde{\pi}_{\mathrm{F}}, \qquad k = \mathrm{const} > 0 \tag{18}$$

of the mathematical model; in other words, under scaling transformations (17) *and* (18) *and the corresponding transformation of the initial conditions, the solutions of the equations of the original model* $\mathbf{M}$ *and the scaling-transformed model* $\widetilde{\mathbf{M}}$ *coincide*:

$$\Phi(x) = \tilde{\Phi}(\tilde{x}), \qquad g_{ik}(x) = \tilde{g}_{ik}(\tilde{x}), \qquad u^i(x) = \tilde{u}^i(\tilde{x}). \tag{19}$$

The similarity property of the mathematical model allows extending a solution with a given set of fundamental parameters to other values of fundamental parameters. This is practically important in the numerical integration of the model equations in the case of very small or very large values of the parameters and over large intervals of coordinate values.

Under scaling transformations (17) and (18), both parts of Eqs. (5), (8), and (15) are multiplied by k^2, and the scalars and tensors introduced above change in accordance with the laws

$$\begin{aligned} &\psi = \tilde{\psi}, \qquad \sigma = k^2\tilde{\sigma}, \qquad V(\Phi) = k^2\tilde{V}(\tilde{\Phi}),\\ &p_p = k^2\tilde{p}_p, \qquad \varepsilon_p = k^2\tilde{\varepsilon}_p, \qquad S^i_k = k^2\tilde{S}^i_k, \qquad T^i_k = k^2\tilde{T}^i_k. \end{aligned} \tag{20}$$

2.3. Cosmological model equations. In the case of a spatially flat Friedmann metric

$$ds_0^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{21}$$

and a homogeneous isotropic distribution of matter $\Phi = \Phi(t)$, $\pi_{\mathrm{F}} = \pi_{\mathrm{F}}(t)$, $u^i = \delta^i_4$, the energy–momentum tensor of a scalar field takes the form of the energy–momentum tensor of an ideal isotropic fluid,

$$S^{ik} = (\varepsilon_s + p_s)u^i u^k - p_s g^{ik}, \tag{22}$$

where

$$\varepsilon_s = \frac{1}{8\pi}\biggl(\frac{\dot{\Phi}^2}{2} + V(\Phi)\biggr), \qquad p_s = \frac{1}{8\pi}\biggl(\frac{\dot{\Phi}^2}{2} - V(\Phi)\biggr). \tag{23}$$

In this case, material equations (8) and (9) can be integrated exactly [4]:

$$a\pi_{\mathrm{F}} = \mathrm{const}. \tag{24}$$

As a result, the function ψ in (14) is defined in terms of the functions $a(t)$ and $\Phi(t)$,

$$\psi = \frac{\pi_0}{|e\Phi|}\,\mathrm{e}^{-\xi}, \qquad \pi_0 = \pi_{\mathrm{F}}(0), \tag{25}$$

where we put

$$\xi = \ln a, \qquad \xi_0 \equiv \xi(0) = 0. \tag{26}$$

Thus, system of equations (5), (7), (8), and (15) reduces to autonomous dynamical system [4] ($H(t)$ is the Hubble parameter)

$$\dot{\xi} = H \ (\equiv F_1), \qquad \dot{\Phi} = Z \ (\equiv F_3), \tag{27}$$

$$\dot{H} = -\frac{Z^2}{2} - \frac{4}{3\pi}\, e_z^4 \Phi^4 \psi^3 \sqrt{1+\psi^2} \ (\equiv F_2), \tag{28}$$

$$\dot{Z} = -3HZ - m_s^2 \Phi + \Phi^3 \left(\alpha - \frac{4e^4}{\pi}\, F_1(\psi)\right) \ (\equiv F_4), \tag{29}$$

and the Einstein equation for the $^4_4$ component becomes the first integral of this system:

$$3H^2 - \Lambda - \frac{Z^2}{2} - \frac{m_s^2 \Phi^2}{2} + \frac{\alpha \Phi^4}{4} - \frac{e^4 \Phi^4}{\pi}\, F_2(\psi) = 0. \tag{30}$$

Equation (30) defines some three-dimensional hypersurface $\mathbb{S}_3$ in the four-dimensional arithmetic phase space of the dynamical system

$$\mathbb{S}_3 \subset \mathbb{R}_4 = \{\xi, H, \Phi, Z\} \equiv \{x_1, x_2, x_3, x_4\}, \tag{31}$$

on which all phase trajectories of a dynamical system lie, i.e., specific cosmological models. In what follows, we call $\mathbb{S}_3$ the *Einstein–Higgs hypersurface*. Equation (30) determines the initial value of the Hubble parameter $H(0) \equiv H_0$ given the initial values of the remaining dynamical variables. Two symmetric solutions for the initial value of the Hubble parameter $H_0^{\pm} = \pm H_0$ correspond to starting from an expanding state $(+)$ or from a contracting state $(-)$. Autonomous system (27)–(29) is invariant under time translations $t \to t + t_0$, which allows us to choose (26) ($\xi_0 = 0$) as the initial condition. Thus, with a fixed sign of the initial value of the Hubble parameter, only two initial values remain free, Φ_0 and Z_0, which we arrange into an ordered list

$$\mathbf{I} = [\Phi_0, Z_0], \qquad \Phi_0 = \Phi(0), \qquad Z_0 = Z(0). \tag{32}$$

Taking the exact integral (24) into account, we also assume that the initial value of the Fermi momentum π_0 is a fundamental parameter of the cosmological model; in what follows, we arrange the fundamental parameters of the model $\mathbf{M}$ into an ordered list [4]

$$\mathbf{P} = [\,[\alpha, m_s, e, \pi_0], \Lambda]. \tag{33}$$

2.4. Similarity of cosmological models. We consider two cosmological models: $\mathbf{M}$ with the fundamental parameters $\mathbf{P}$ and the initial conditions $\mathbf{I}$, and a similar model $\widetilde{\mathbf{M}}$ with the fundamental parameters $\widetilde{\mathbf{P}}$ and the initial conditions $\tilde{\mathbf{I}}$:

$$\tilde{\mathbf{I}} = \left[\Phi_0, \frac{1}{k}\, Z_0\right], \tag{34}$$

$$\widetilde{\mathbf{P}} = \left[\left[\frac{\alpha}{k^2}, \frac{m_s}{k}, \frac{e}{\sqrt{k}}, \frac{\pi_0}{\sqrt{k}}\right], \frac{\Lambda}{k^2}\right]. \tag{35}$$

Functions $f(t) = \tilde{f}(\tilde{t})$ that are invariant under similarity transformation (17), (18) are transformed according to the rules

$$\tilde{f}(\tilde{t}) = f\left(\frac{\tilde{t}}{k}\right). \tag{36}$$

Let the solutions of dynamical system (27)–(29), (30) for the model $\mathbf{M}$ in (32) and (33) be

$$\mathbf{S}(t) = [\xi(t), H(t), \Phi(t), Z(t)].$$

Then the solutions of the corresponding equations for such a model $\widetilde{\mathbf{M}}$ in (34) and (35) are

$$\widetilde{\mathbf{S}}(t) = [\tilde{\xi}(t), \widetilde{H}(t), \widetilde{\Phi}(t), \widetilde{Z}(t)] \equiv \left[\xi\left(\frac{t}{k}\right), \frac{1}{k}\, H\left(\frac{t}{k}\right), \Phi\left(\frac{t}{k}\right), \frac{1}{k}\, Z\left(\frac{t}{k}\right)\right]. \tag{37}$$

2.5. Transformation of the dynamical system matrix and its eigenvalues. We find how the eigenvalues of the dynamical system matrix transform under scaling transformations (17), (18). The characteristic matrix $\mathbf{A}$ of the dynamical system $\mathbf{M}$ in (27)–(29) at a point M (see, e.g., [17]), according to phase coordinates (31), is

$$\mathbf{A}(M) = \|A_i^k\| \equiv \left\| \frac{\partial F_i}{\partial x_k} \right\| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \dfrac{\partial F_2}{\partial \xi} & 0 & \dfrac{\partial F_2}{\partial \Phi} & -Z \\ 0 & 0 & 0 & 1 \\ \dfrac{\partial F_4}{\partial \xi} & -3Z & \dfrac{\partial F_4}{\partial \Phi} & -3H \end{pmatrix}. \tag{38}$$

According to the law of transformation of fundamental parameters (17), the coordinates, and the Fermi momentum, the right-hand sides of dynamical system (27)–(29) that is similar to the dynamical system $\widetilde{\mathbf{M}}$ are obtained according to the rules

$$\widetilde{F}_1 = \frac{1}{k}\, F_1, \qquad \widetilde{F}_2 = \frac{1}{k^2}\, F_2, \qquad \widetilde{F}_3 = \frac{1}{k}\, F_3, \qquad \widetilde{F}_4 = \frac{1}{k^2}\, F_4. \tag{39}$$

Thus, the characteristic matrix $\tilde{\mathbf{A}}$ of the image $\widetilde{\mathbf{M}}$ of the dynamical system is

$$\tilde{\mathbf{A}}(\widetilde{M}) = \|\tilde{A}_i^k\| \equiv \left\| \frac{\partial \widetilde{F}_i}{\partial \tilde{x}_k} \right\| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \dfrac{1}{k^2}\,\dfrac{\partial F_2}{\partial \xi} & 0 & \dfrac{1}{k^2}\,\dfrac{\partial F_2}{\partial \Phi} & -\dfrac{1}{k}\, Z \\ 0 & 0 & 0 & 1 \\ \dfrac{1}{k^2}\,\dfrac{\partial F_4}{\partial \xi} & -\dfrac{3}{k}\, Z & \dfrac{1}{k^2}\,\dfrac{\partial F_4}{\partial \Phi} & -\dfrac{3}{k}\, H \end{pmatrix}. \tag{40}$$

The comparison of (38) and (40) shows that the characteristic matrix $\tilde{\mathbf{A}}$ of the image $\widetilde{\mathbf{M}}$ of the dynamical system $\mathbf{M}$ is not similar to the matrix $\mathbf{A}$, which may lead us to a wrong conclusion. It must be borne in mind that this matrix is not an independent object, but is connected precisely with the characteristic equation of the qualitative theory of dynamical systems,

$$\mathbf{A} \cdot \mathbf{X} = \lambda \mathbf{X}, \qquad \mathbf{X} = \begin{pmatrix} \xi \\ H \\ \Phi \\ Z \end{pmatrix}, \tag{41}$$

where λ are the eigenvalues and X is a column matrix of phase coordinates of a point in the dynamical system. The corresponding equations for the image $\widetilde{\mathbf{M}}$ are

$$\tilde{\mathbf{A}} \cdot \widetilde{\mathbf{X}} = \tilde{\lambda} \widetilde{\mathbf{X}}, \qquad \widetilde{\mathbf{X}} = \begin{pmatrix} \xi \\ \dfrac{H}{k} \\ \Phi \\ \dfrac{Z}{k} \end{pmatrix}. \tag{42}$$

Multiplying $\mathbf{A}$ in (38) and $\mathbf{X}$ in accordance with (41) and the matrices $\tilde{\mathbf{A}}$ in (40) and $\widetilde{\mathbf{X}}$ in accordance with (42), it is easy to see that the resulting systems of linear homogeneous algebraic equations with respect to phase coordinates become equivalent if and only if the eigenvalues λ and $\tilde{\lambda}$ are related as $\tilde{\lambda} = \lambda/k$.

Thus, the following assertion is true.

Assertion 2. *Under scaling transformations* (17) *and* (18)*, the eigenvalues of the characteristic matrix of dynamical system* (27)–(29) *transform in accordance with the law*

$$\tilde{\lambda} = \frac{\lambda}{k}\,. \tag{43}$$

The coordinates of singular points transform the same as arbitrary coordinates of a phase trajectory of the dynamical system, i.e., in accordance with the law (42) (*or, equivalently,* (37)).

Due to the proportionality of the eigenvalues of such models, the nature of singular points is an invariant property of similarity.

2.6. Vacuum-field cosmological model. The transition to the vacuum-field cosmological model in which there is no scalar-charged matter is carried out by substituting $e = 0$ in system of equations (28)–(30). As a result, we obtain the system

$$\dot{\xi} = H\ (\equiv F_1), \qquad \dot{\Phi} = Z\ (\equiv F_3), \tag{44}$$

$$\dot{H} = -\frac{Z^2}{2}\ (\equiv F_2), \tag{45}$$

$$\dot{Z} = -3HZ - m_s^2\Phi + \alpha\Phi^3\ (\equiv F_4), \tag{46}$$

$$3H^2 - \Lambda - \frac{Z^2}{2} - \frac{m_s^2\Phi^2}{2} + \frac{\alpha\Phi^4}{4} = 0. \tag{47}$$

This dynamical system is a special case of the general system in (27)–(29), and it hence inherits all the similarity properties discussed above. On the other hand, as studies [4], [6], [7] showed, the cosmological system of scalar-charged particles inherits the behavior of vacuum-field cosmological models, and therefore the importance of studying their global properties persists. However, this dynamical system is also fundamentally different from dynamical system (27)–(29) considered above: all functions F_i of this system are independent of the scaling function $\xi(t)$. As a result, dynamical system (44)–(46) reduces to an autonomous subsystem in the three-dimensional phase space $R_3 = \{H, \Phi, Z\}$. Reducing the dimension of the phase space, in turn, leads to the removal of conditions on the Hubble parameter by the subsystem of dynamical equations; as a result, the Hubble parameter at singular points is determined from Einstein equation (47).

We demonstrate the above assertions by analyzing singular points of the single-field model in Eqs. (44)–(47). The singular points of the model are determined by the vanishing of the right-hand sides of the dynamical equations, which implies that their phase coordinates are determined by the system of equations

$$F_i(\mathbf{X}) = 0, \qquad i = \overline{1,4}. \tag{48}$$

For system (44)–(46), Eqs. (48) and their solutions take the form

$$H = 0, \qquad Z = 0, \tag{49}$$

$$-\,m_s^2\Phi + \alpha\Phi^3 = 0 \quad \Rightarrow \quad \Phi_0 = 0, \qquad \Phi_\pm = \pm\frac{m}{\sqrt{\alpha}}. \tag{50}$$

Substituting these solutions in the first integral in (47) leads to conditions on the value of the cosmological constant,

$$\begin{aligned} \Phi &= \Phi_0 \quad \Rightarrow \quad \Lambda = 0, \\ \Phi &= \Phi_\pm \quad \Rightarrow \quad \Lambda_0 = 0, \end{aligned} \tag{51}$$

where Λ_0 is the seed value of the cosmological constant and Λ is its observed value (see (6)). Further, because $H = 0$, we conclude that $\xi = \mathrm{const} = 0$, and therefore the Universe at the singular point is Euclidean and the total energy density is equal to zero.

However, if we use the autonomous subsystem of dynamical system (44)–(46), dropping the first equation (44) from it, this subsystem does not imply any condition on the Hubble parameter that we find from (47) by substituting the values $Z = 0$ from (49) and Φ from (50) in this equation. Thus, we obtain correct results for the characteristics of singular points (see, e.g., [6], [18]). The above example indicates that not any dynamical function can be good enough for a qualitative analysis of a dynamical system; in some cases, poor choice can lead to system degeneration.

3. Modified system of dynamical equations

3.1. Transformation of the dynamical system to a nondegenerate form. To eliminate the disadvantage discussed above, we transform dynamical system (28)–(30) to new variables. For this, we note that the right-hand sides of these equations depend on $\xi(t)$ only through the function $\psi(t)$, and we can express the scale-invariant function $\xi(t)$ in relation (25) in terms of a couple of other scale-invariant functions:

$$\xi = -\ln\left|\frac{e\psi\Phi}{\pi_0}\right|. \tag{52}$$

Thus, instead of (28)–(30), we obtain a new system of equations (only the first equation of the system formally changes)

$$\dot{\psi} = \psi\left(H - \frac{Z}{\Phi}\right) \ (\equiv G_1), \qquad \dot{\Phi} = Z \ (\equiv G_3), \tag{53}$$

$$\dot{H} = -\frac{Z^2}{2} - \frac{4}{3\pi} e_z^4 \Phi^4 \psi^3 \sqrt{1+\psi^2} \ (\equiv G_2), \tag{54}$$

$$\dot{Z} = -3HZ - m_s^2 \Phi + \Phi^3\left(\alpha - \frac{4e^4}{\pi} F_1(\psi)\right) \ (\equiv G_4). \tag{55}$$

The first integral of system (30) does not change.

3.2. Singular points of the modified system. We now find the singular point coordinates. The equation $G_1 = 0$ has two solutions: $Z = H\Phi$ and $\psi = 0$. It is easy to see that the first solution eventually brings us back to the previous situation ($H = 0$). We therefore turn to the second solution. With $Z = 0$, this solution turns the equation $G_2 = 0$ into an identity. Because $F_1(0) = 0$ in accordance with (15), we can use (55) to obtain solutions for the coordinate of a singular point, coincident with the solutions for the one-field vacuum model (50). Thus, dynamical equations (53)–(55) do not impose any restrictions on $H(t)$. As in the case of the vacuum field model, we obtain this quantity from the first integral (30), taking into account that $F_2(0) = 0$:

$$\begin{aligned} H_0^{\pm} &= \pm\sqrt{\frac{\Lambda}{3}}, \qquad \Phi = \Phi_0, \\ H_{\pm} &= \pm\sqrt{\frac{\Lambda_0}{3}}, \qquad \Phi = \Phi_\pm. \end{aligned} \tag{56}$$

We write the coordinates of the six singular points of the system (with the signs taking independent values)

$$M_0^{\pm} = \left[0, \pm\sqrt{\frac{\Lambda}{3}}, 0, 0\right], \tag{57}$$

$$M_{\pm}^{\pm} = \left[0, \pm\sqrt{\frac{\Lambda_0}{3}}, \pm\frac{m_s}{\sqrt{\alpha}}, 0\right]. \tag{58}$$

We note, first, that the coordinates of the singular points $[H, \Phi, Z]$ for dynamical system (53)–(55) with integral condition (30) coincide with the coordinates of the singular points of the vacuum-field model for the scalar singlet (see [18]), which explains the previously noted features of the behavior of the cosmological model with charged fermions near singular points of the vacuum field model. Second, these points correspond to the vanishing of the function $\psi(t)$, $(\xi \to +\infty)$, i.e., to the late stages of cosmological evolution, when the role of matter is negligible. Third, we note that all similarity laws, together with the scaling transformation laws for eigenvalues, are also preserved for the upgraded dynamical system.

3.3. Characteristic matrix and eigenvalues of a nondegenerate dynamical system. Introducing an ordered set of new phase coordinates $[\psi, H, \Phi, Z]$, we write the matrix of dynamical system (53)–(55) at singular points as

$$M_0 = [0, H, \Phi, 0] \quad \Rightarrow \quad \mathbf{Y} = \begin{pmatrix} 0 \\ H \\ \Phi \\ 0 \end{pmatrix},$$

$$\mathbf{B}(M_0) = \|B_i^k\| \equiv \left\|\frac{\partial G_i}{\partial y_k}\right\| = \begin{pmatrix} H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -m_s^2 + 3\alpha\Phi^2 & -3H \end{pmatrix}, \tag{59}$$

where we must substitute Φ and H from (50) and (56).

The eigenvalues of matrix (59) are

$$\lambda_1 = 0, \qquad \lambda_2 = H, \qquad \lambda_{3,4} = -\frac{3}{2}H \pm \frac{3}{2}\sqrt{H^2 + \frac{4}{9}(3\alpha\Phi^2 - m_s^2)}\,.$$

Substituting Φ and H from (50) and (56), we finally find the eigenvalues of matrix (59) at singular points:

$$\mathbf{M}_0^{\pm}\colon \begin{cases} \lambda_2 = \pm\sqrt{\dfrac{\Lambda}{3}}, \\ \lambda_{3,4} = \mp\dfrac{1}{2}\sqrt{3\Lambda} \pm \dfrac{1}{2}\sqrt{3\Lambda - 4m_s^2}, \end{cases} \tag{60}$$

$$\mathbf{M}_{\pm}^{\pm}\colon \begin{cases} \lambda_2 = \pm\sqrt{\dfrac{\Lambda_0}{3}}, \\ \lambda_{3,4} = \mp\dfrac{1}{2}\sqrt{3\Lambda_0} \pm \dfrac{1}{2}\sqrt{3\Lambda_0 + 8m_s^2}\,. \end{cases} \tag{61}$$

We note that the expressions for eigenvalues (60), (61), as well as the expressions for singular points coordinates (50) and (56), in accordance with the transformation law for fundamental parameters (17), once again confirm the validity of Assertion 2.

Assertion 3. *The coordinates of the eigenpoints of the dynamical system of the cosmological model* **M** *at $H \not\equiv 0$ in the subspace $\mathbb{R}_3 \equiv \{H, \Phi, Z\} \subset \mathbb{R}_4$, as well as the eigenvalues of the characteristic matrix coincide with the coordinates of the eigenpoints and the eigenvalues of the characteristic matrix of the vacuum-field cosmological model.*

We note that papers [4], [6] cited above did not reveal these singular points due to the choice of dynamical variables $[\xi, H, \Phi, Z]$ that gave rise to a degenerate characteristic matrix of the dynamical system. In these papers, the singular points of only one branch of the solutions of the equation $G_1 = 0$ in Sec. 3.2, corresponding to the infinite future of the Universe ($\xi \to +\infty$, $H \to 0$), were identified and studied. We also note that the solution branch found in this paper corresponds to $\psi \to 0$, i.e., formally also corresponds to the $\xi \to +\infty$ case, but the model remains in the inflationary mode—that of the scalar field dominance over particles.

4. Examples of numerical simulation of the dynamical system

According to Assertions 1 and 3, the coordinates of singular points (57) and (58) of the dynamical system under study transform under the $\mathcal{S}_k$ similarity transformation as

$$\widetilde{M}_0^{\pm} = \left[0, \pm\frac{1}{k}\sqrt{\frac{\Lambda}{3}}, 0, 0\right], \tag{62}$$

$$\widetilde{M}_{\pm}^{\pm} = \left[0, \pm\frac{1}{k}\sqrt{\frac{\Lambda_0}{3}}, \pm\frac{m_s}{\sqrt{\alpha}}, 0\right]. \tag{63}$$

4.1. Standard example. We consider the example of a numerical simulation of two similar systems **M** and $\widetilde{\mathbf{M}} = \mathcal{S}_k(\mathbf{M})$ with the similarity coefficient $k = 10^4$:

$$\mathbf{M} : \mathbf{P} = [\,[1, 1, 1, 0.1], 3 \cdot 10^{-3}], \qquad \mathbf{I} = [1, 0, 1], \tag{64}$$

$$\widetilde{\mathbf{M}} : \widetilde{\mathbf{P}} = [\,[10^{-8}, 10^{-4}, 10^{-2}, 10^{-3}], 3 \cdot 10^{-11}], \qquad \tilde{\mathbf{I}} = [1, 0, 1]. \tag{65}$$

The eigenpoints of these models in the subspaces $\mathbb{S}_3 = \{H, \Phi, Z\}$ and $\widetilde{\mathbb{S}}_3 = \{\widetilde{H}, \Phi, \widetilde{Z}\}$ have the following coordinates (quoting approximate values for simplicity):

$$\mathbf{M}\colon \begin{cases} M_0 = [3.16 \cdot 10^{-2}, 0, 0], \\ M_\pm = [2.90 \cdot 10^{-1}, 1, 0], \end{cases} \tag{66}$$

$$\widetilde{\mathbf{M}}\colon \begin{cases} \widetilde{M}_0 = [3.16 \cdot 10^{-6}, 0, 0], \\ \widetilde{M}_\pm = [2.90 \cdot 10^{-5}, 1, 0]. \end{cases} \tag{67}$$

As is easy to see, the coordinates of singular points transform exactly in accordance with similarity laws (37) with the similarity coefficient $k = 10^4$.

It is easy to calculate the eigenvalues of the characteristic matrix at these points (the signs take independent values)

$$\mathbf{M}\colon \begin{cases} k_0 = \mp 0.0474 \mp i, \\ k_\pm = \mp 1.915 \pm 1.044, \end{cases} \tag{68}$$

$$\widetilde{\mathbf{M}}\colon \begin{cases} \tilde{k}_0 = \mp 4.74 \cdot 10^{-6} \mp i \cdot 10^{-4}, \\ \tilde{k}_\pm = \mp 1.915 \cdot 10^{-4} \pm 1.044 \cdot 10^{-4}. \end{cases} \tag{69}$$

The eigenvalues of the characteristic matrices of two similar dynamical cosmological systems also transform in exact correspondence with similarity law (43) with the similarity coefficient $k = 10^4$. Thus, in the considered example, the points M_0 and $\widetilde{M}_0$ are attracting foci, and the points $M_\pm$ and $\widetilde{M}_\pm$ are saddle or nodal foci.

In Fig. 1, we show projections of the Einstein–Higgs surface of two models onto the hyperplane $\xi = \tilde{\xi} = 0.1$, $S_3 = [H, \Phi, Z,]$, and in Fig. 2, projections on the hyperplanes $Z = 0.1$ and $\widetilde{Z} = 10^{-5}$ of the dynamical system phase space. As is easy to see from these figures, the graphs of the Einstein surfaces of two similar models are also similar in all projections: the scales along the OH and OZ axes are contracted by 10^4 times for the similar model, while the scales along the $O\xi$ and $O\Phi$ axes are preserved. We also note that according the equation for the secant hyperplane $Z = \text{const}$ is also transformed in accordance with similarity laws (37).

In Fig. 3, we show graphs of the evolution of the scale functions $\xi(t)$ and $H(t)$ in the $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ models, and in Fig. 4, phase diagrams of the corresponding models. It can be seen that all corresponding pairs of graphs are similar with the similarity coefficient $k = 10^4$ in accordance with similarity laws (37). In Fig. 3, in particular, we can observe a rigorous time scaling $\tilde{t} = 10^4 t$.

Thus, the numerical simulation results rigorously and clearly confirm all the similarity properties formulated above (Assertions 1–3) for similar models $\mathbf{M}$ in (64) and $\widetilde{\mathbf{M}}$ in (65).

4.2. Possible violation of the similarity symmetry near saddle singular points. Although the numerical simulation results given above for the standard example reveal the rigorous fulfillment of the formulated similarity properties (Assertions 1–3), violations of the similarity symmetry of cosmological models are possible in building numerical models. These violations can occur at very small values of the scalar charge e if in this case the phase trajectory of the model passes through an unstable singular point of the dynamical system. We consider the example

$$\mathbf{M_1} : \mathbf{P} = [\,[1, 1, 10^{-7}, 0.1], 3 \cdot 10^{-3}], \qquad \mathbf{I} = [1, 0, 1], \tag{70}$$

$$\widetilde{\mathbf{M_1}} : \widetilde{\mathbf{P}} = [\,[10^{-8}, 10^{-4}, 10^{-9}, 10^{-3}], 3 \cdot 10^{-11}], \qquad \tilde{\mathbf{I}} = [1, 0, 1]. \tag{71}$$

We note that in this case, the singular point coordinates coincide with the coordinates of the singular points of models (66) and (67), because the parameters e and π_0 do not affect either these coordinates or the nature of the points. We also note that the initial conditions Φ_0 and Z_0 in the above example, as in the case under considaration, coincide with the coordinates of the unstable singular points $M_\pm$ and $\widetilde{M}_\pm$. But in the current case, we reduced the scalar charge by 10^7 times. In Fig. 5, the evolution of the scale functions $\xi(t)$ and $H(t)$ in the $\mathbf{M_1}$ and $\widetilde{\mathbf{M_1}}$ models is plotted.

These figures, despite being superficially similar, demonstrate a violation of the similarity symmetry. Indeed, the similarity coefficient for these models is equal to $k = 10^4$. Therefore, according to the plot in Fig. 5a, the value $\xi = 70$ should be reached in the model $\widetilde{\mathbf{M_1}}$ at the time $t \approx 2.4 \cdot 10^6$, but we see from the plot in Fig. 5b that this value is reached at $t \approx 2 \cdot 10^7$, i.e., an order of magnitude later. The value of the Hubble parameter according to the plot in Fig. 5a in the $\widetilde{\mathbf{M_1}}$ model must be about $H \approx 3 \cdot 10^{-5}$, but from the plot in Fig. 5b, we find the value $H \approx 3 \cdot 10^{-6}$, i.e., an order of magnitude smaller. To resolve this discrepancy, we present the phase diagrams of the models in the $\{\Phi, Z\}$ plane (Fig. 6).

Regarding the plot in Fig. 6a, we note that taking the accuracy of the calculations into account, this plot represents one point on the plane $\{\Phi, Z\}$: $\Phi = 1$, $Z = 0$, i.e., describes the “sticking” of the trajectory at a singular point.

To demonstrate the effect of a singular point on phase trajectories, we consider an example with initial conditions close to this point: we replace the initial value $\Phi_0 = 1$ with $\Phi_0 = 0.999$ close to it:

$$\mathbf{M_{1a}} : \mathbf{P} = [\,[1, 1, 10^{-7}, 0.1], 3 \cdot 10^{-3}], \qquad \mathbf{I} = [0.999, 0, 1], \tag{72}$$

$$\widetilde{\mathbf{M_{1a}}} : \widetilde{\mathbf{P}} = [\,[10^{-8}, 10^{-4}, 10^{-9}, 10^{-3}], 3 \cdot 10^{-11}], \qquad \tilde{\mathbf{I}} = [0.999, 0, 1]. \tag{73}$$

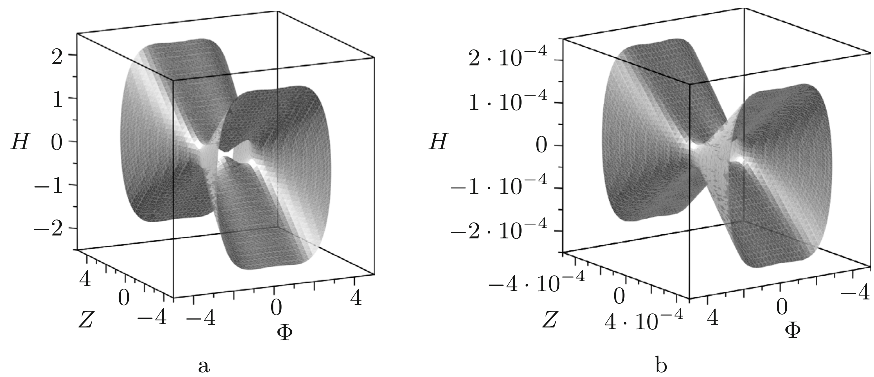

Fig. 1. Projection of the Einstein–Higgs hypersurface of the models $\mathbf{M}$ (64) (a) and $\widetilde{\mathbf{M}}$ (65) (b) onto the hyperplane $\xi = 0.1$.

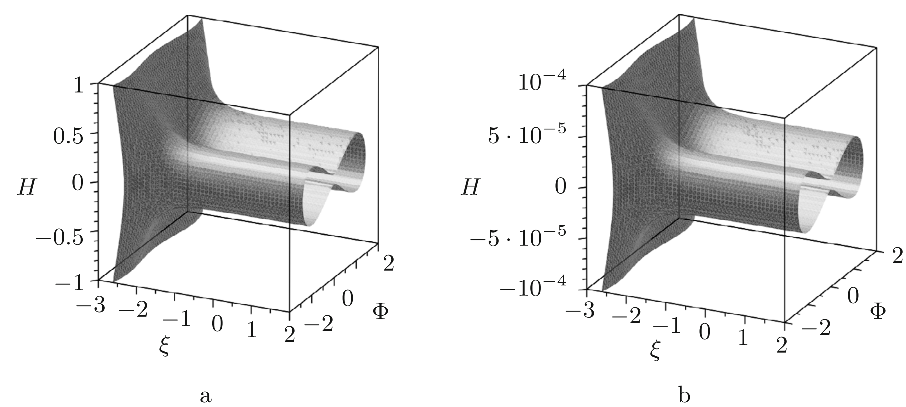

Fig. 2. Projection of the Einstein–Higgs hypersurface of the model $\mathbf{M}$ (64) onto the hyperplane $Z = 0.1$ (a) and the model $\widetilde{\mathbf{M}}$ (65) onto the hyperplane $\widetilde{Z} = 10^{-5}$ (b).

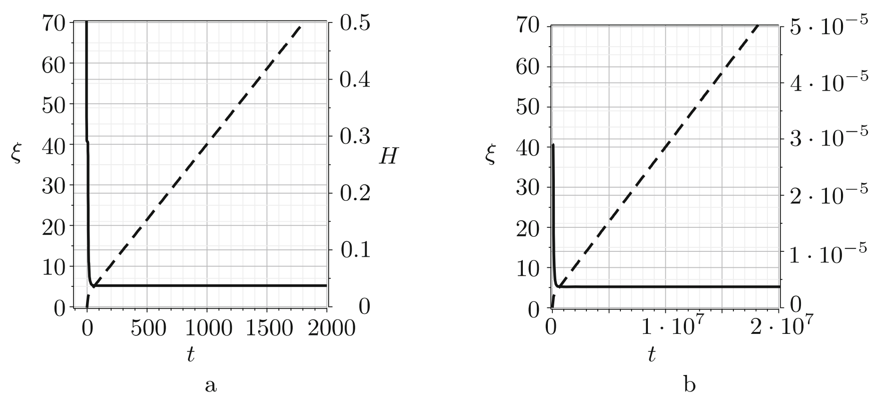

Fig. 3. Evolution of the scale functions $\xi(t)$ (dashed lines) and $H(t)$ (solid lines) in the models $\mathbf{M}$ (64) (a) and $\widetilde{\mathbf{M}}$ (65) (b).

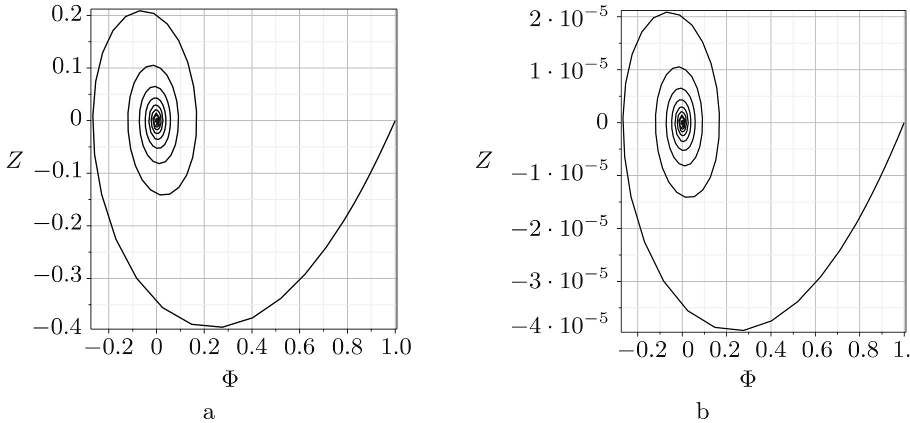

Fig. 4. Phase diagram of the models $\mathbf{M}$ (64) (a) and $\widetilde{\mathbf{M}}$ (65) (b) on the $\{\Phi, Z\}$ plane.

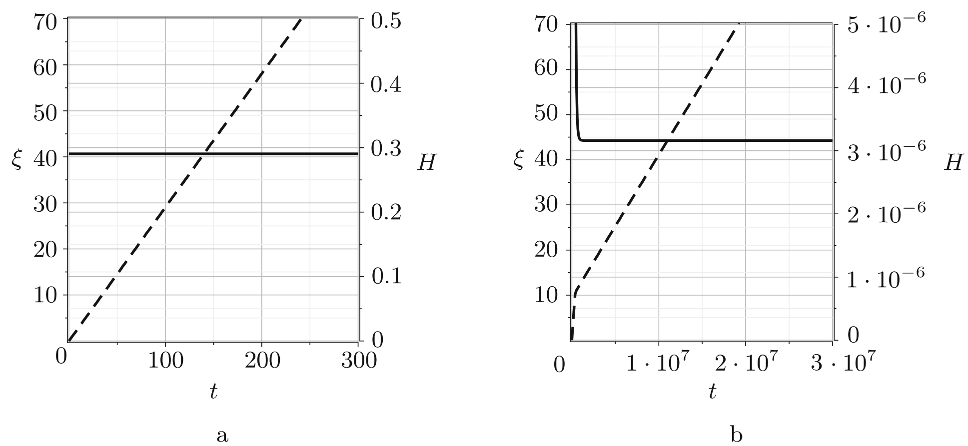

Fig. 5. Evolution of the scaling functions $\xi(t)$ (dashed lines) and $H(t)$ (solid lines) in the models $\mathbf{M_1}$ (70) (a) and $\widetilde{\mathbf{M}}_{\mathbf{1}}$ (71) (b).

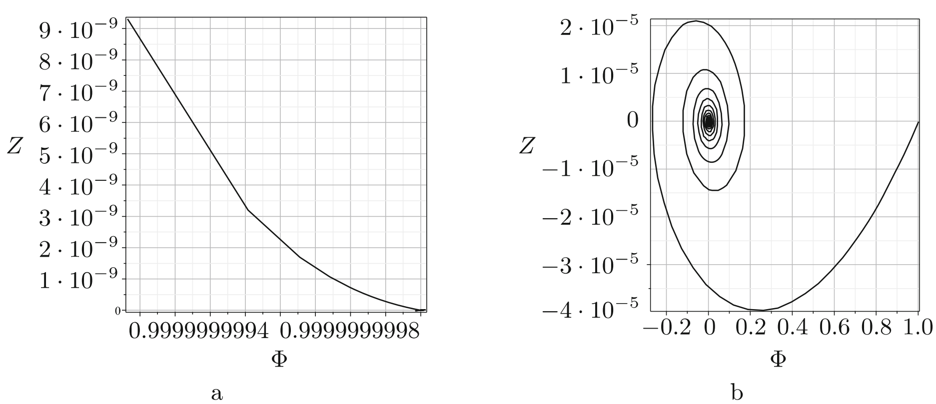

Fig. 6. Phase diagram of the models $\mathbf{M_1}$ (70) (a) and $\widetilde{\mathbf{M}}_{\mathbf{1}}$ (71) (b) in the $\{\Phi, Z\}$ plane.

In Figs. 7 and 8, we plot the evolution of the scale functions $\xi(t)$ and $H(t)$ and phase diagrams in the $\{\Phi, Z\}$ plane of the $\mathbf{M_{1a}}$ model. In this case, the plots of the $\widetilde{\mathbf{M}}_{\mathbf{1a}}$ model do not differ from the corresponding plots in Figs. 5b and 6b. Thus, we can verify that the similarity symmetry is restored with a slight shift of the initial conditions away from the coordinates of the singular point. Nevertheless, the considered case shows that the similarity transformation must be applied with caution near singular points.

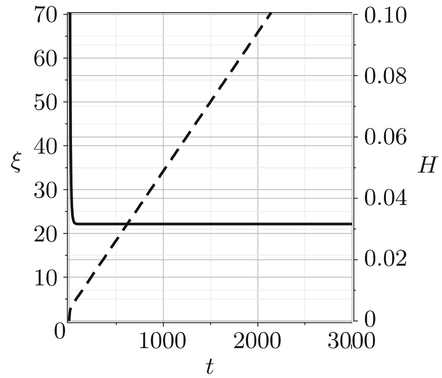

Fig. 7. Evolution of the scaling functions $\xi(t)$ (dashed line) and $H(t)$ (solid line) in the model $\mathbf{M_{1a}}$ (72).

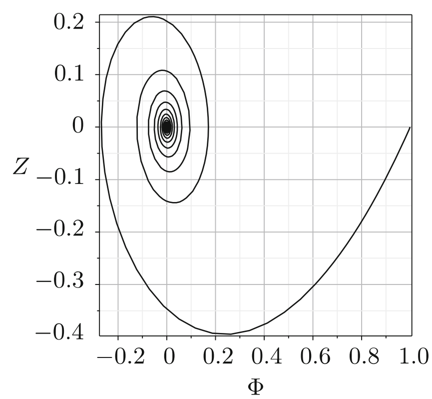

Fig. 8. Phase diagram of the model $\mathbf{M_{1a}}$ (70) in the $\{\Phi, Z\}$ plane.

5. Conclusions

To conclude, we note, first, that *realistic cosmological models* must correspond to physically realizable values of the fundamental parameters corresponding to the scales of field models of the $SU(5)$ type

$$SU(5): \quad \alpha \lesssim 10^{-8}, \qquad m \lesssim 10^{-4}, \qquad e \lesssim 10^{-2}, \tag{74}$$

or the standard SM model

$$SM: \quad \alpha \lesssim 10^{-30}, \qquad m \lesssim 10^{-15}, \qquad e \lesssim 10^{-9}. \tag{75}$$

But with such small values of the fundamental parameters, purely technical difficulties in numerically integrating a nonlinear system of dynamical equations allow extending the calculations only up to time values of the order of $t \lesssim 10^4$ (see, e.g., [4]). However, by performing a scale transformation with a similarity coefficient of the order of $k = 10^4$, we pass to the $\mathbf{M}$ cosmological model with the parameters $\alpha = m = e = 1$, which is already amenable to numerical simulations up to significantly longer times $t \sim 10^8$. Using the results of integrating the $\mathbf{M}$ model by the scaling rules, we thereby extend the results for the original $\widetilde{\mathbf{M}}$ model to times of the order of $t \gtrsim 10^8$. Similarly, in the case of the standard model, we must choose the coefficient $k = 10^{15}$, which gives rise to $t \sim 10^{19}$ (see [19]). We note that the application of the similarity transformation to the theory of the formation of early nuclei of supermassive black holes [19] leads to the formation times of these objects $t \sim 10^{13}$ for the $SU(5)$ model and $t \sim 10^{23}$ for the Standard Model. Both these values fit within the observed permissible time interval with a margin.

Second, a physically important circumstance is that during the transition from the studied cosmological model $\mathbf{M}$ to the similar cosmological model $\widetilde{\mathbf{M}}$ with the similarity coefficient $k \sim 10^4 \div 10^5$, we extend the time interval of cosmological evolution by the same factor, passing to times $kt \gg t_{\mathrm{Pl}}$, at which no quantum field consideration of the cosmological model is required, but its classical description suffices. Making the inverse transition from the classical model $\widetilde{\mathbf{M}}$ to the similar model $\mathbf{M}$ with the similarity coefficient k^{-1}, we obtain a classical model at times comparable to Planck's times. This model, however, in no way claims to have any physical significance. It serves only as a *convenient computational model*, similar to the classical cosmological model under study at late evolution times $t \gg t_{\mathrm{Pl}}$.

Third, the explicit dependence of solutions (37) of the dynamical system of equations describing the cosmological model on the similarity coefficient allows analytically continuing the obtained numerical solutions to similar models and similar time intervals, which significantly expands the possibilities of analyzing numerical models. Simultaneously extendec are the results of a qualitative analysis of the global properties of dynamical models.

Fourth, finally, it is obvious that the established laws of similarity of dynamical systems can be successfully applied to other cosmological models with scalar fields, including, which is significant, the previously considered cosmological models based on scalar multipoles, an asymmetric scalar doublet [18], or a multifield model with an exponential interaction potential [20].

Funding. This research was supported by Kazan Federal University under the Strategic Academic Leadership Program.

Conflict of interest. The author of this work declares that he has no conflicts of interest.

REFERENCES

1. S. J. Kline, *Similitude and Approximation Theory*, Springer, New York (1986).
2. L. I. Sedov, *Similarity and Dimensional Methods in Mechanics*, CRC Press, Boca Raton (1993).
3. E. A. Dibai and S. A. Kaplan, *Dimensions and Similarity of Astrophysical Quantities* [in Russian], Nauka, Moscow (1076).
4. Yu. G. Ignat'ev and D. Yu. Ignatyev, "Cosmological models based on a statistical system of scalar charged degenerate fermions and an asymmetric Higgs scalar doublet," *Theoret. and Math. Phys.*, **209**, 1437–1472 (2021), arXiv: 2111.00492.
5. G. E. Tauber and J. W. Weinberg, "Internal state of a gravitating gas," *Phys. Rev.*, **122**, 1342–1365 (1961).
6. Yu. G. Ignat'ev, A. A. Agathonov, and D. Yu. Ignatyev, "Cosmological evolution of a statistical system of degenerate scalarly charged fermions with an asymmetric scalar doublet. II. One-component system of doubly charged fermions," *Gravit. Cosmol.*, **28**, 10–24 (2022), arXiv: 2203.12766.
7. Yu. G. Ignat'ev, "Two-field model of gravitational-scalar instability and formation of supermassive black holes in the early universe," *Gravit. Cosmol.*, **28**, 163–176 (2023), arXiv: 2305.15456.
8. Yu. G. Ignatev, "Evolution of spherical perturbations in the cosmological environment of degenerate scalar-charged fermions with a scalar," *Theoret. and Math. Phys.*, **215**, 862–892 (2023), arXiv: 2306.17185.

9. B. Saha and G. N. Shikin, “Interacting spinar and scalar fields in Bianchi type I universe filled with perfect fluid: exact self-consistent solutions,” *Gen. Rel. Grav.*, **29**, 1099–1113 (1997).
10. B. Saha, “Spinor field in Bianchi type-I universe: Regular solutions,” *Phys. Rev. D*, **64**, 123501, 15 pp. (2001).
11. L. P. Chimento, F. P. Devecchi, M. Forte, and G. M. Kremer, “Phantom cosmologies and fermions,” *Class. Quantum Grav.*, **25**, 085007, 10 pp. (2008), arXiv: 0707.4455.
12. M. O. Ribas, F. P. Devecchi, and G. M. Kremer, “Cosmological model with non-minimally coupled fermionic field,” *Europhys. Lett.*, **81**, 19001, 6 pp. (2008), arXiv: 0710.5155.
13. J. Wang, S.-W. Cui, and C.-M. Zhang, “Thermodynamics of spinor quintom,” *Phys. Lett. B*, **683**, 101–107 (2010), arXiv: 0806.3890.
14. L. Fabbri, “Conformal gravity with the most general ELKO matter,” *Phys. Rev. D*, **85**, 047502, 5 pp. (2012), arXiv: 1101.2566.
15. B. Saha, “Nonlinear spinor fields in Bianchi type-I spacetime: problems and possibilities,” *Astrophys. Space Sci.*, **357**, 28, 16 pp. (2015), arXiv: 1409.4993.
16. K. A. Bronnikov, Yu. P. Rybakov, and B. Saha, “Spinor fields in spherical symmetry: Einstein–Dirac and other space-times,” *Eur. Phys. J. Plus*, **135**, 124, 10 pp. (2020), arXiv: 1909.04789.
17. Ia. B. Zel’dovich and I. Novikov, *Structure and Evolution of the Universe* [in Russian], Nauka, Moscow (1975).
18. Yu. G. Ignat’ev and I. A. Kokh, “Complete cosmological model based on an asymmetric scalar Higgs doublet,” *Theoret. and Math. Phys.*, **207**, 514–552 (2021), arXiv: 2104.01054.
19. Yu. G. Ignat’ev, “Formation of supermassive nuclei of black holes in the early universe by the mechanism of scalar-gravitational instability. I. Local picture,” *Gravit. Cosmol.*, **29**, 327–344 (2023), arXiv: 2308.03192.
20. G. Leon, A. Paliathanasis, and J. L. Morales, “The past and future dynamics of quintom dark energy models,” *Eur. Phys. J C*, **78**, 753, 22 pp. (2018), arXiv: 1808.05634.