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The local moduli of Sasaki-Einstein rational homology 7-spheres and invertible polynomials

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Abstract

We study the local moduli space of Sasaki-Einstein metrics on links of invertible polynomials defining rational homology 7-spheres. All these polynomials are either of cycle type or are given as Thom Sebastiani sums of a cycle block and another atomic block. We found that for polynomials of cycle type, the local moduli spaces of Sasaki-Einstein metrics are zero dimensional. For the Thom-Sebastiani sums of an atomic block and a cycle polynomial, the dimensions of the local moduli spaces of Sasaki-Einstein metrics are positive in general. Since all the links under study in this article remain Sasaki-Einstein rational homology 7-spheres under the Berglund-Hübsch rule from classical mirror symmetry (Berglund and Hübsch, Nucl Phys B 393:377–391 (1993), Cuadros et al., Commun Math Phys 405:199 (2024)), we are able to find solutions for the problem associated to the moduli for the Berglund-Hübsch transpose duals of this type of links. For the purpose of doing this, we give specific description of the moduli spaces of complex structures on the weighted quasismooth hypersurfaces cut out by the corresponding invertible polynomials and, in particular, from this description, we can produce families of quasismooth weighted hypersurfaces that degenerate to non-quasismooth with at worst klt singularities.

Keywords Moduli · Sasaki-Einstein metrics · Rational homology spheres · Berglund-Hübsch · Calabi-Yau cones

Mathematics Subject Classification 53C25 · 32G07 · 32G13

Contents

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1 Introduction

For Sasakian manifolds, which are roughly speaking the odd dimensional analogue of Kähler manifolds, the moduli problem has been addressed and an appropriate notion of moduli space has been designed, however little is known about this space, see [3] and references therein. In particular, finding the number of components of this moduli has been studied in order to obtain lower bounds for the dimension of the moduli space of Sasakian structures for links of Brieskorn-Pham polynomials and Smale manifolds [7, 11, 28]. For Sasaki-Einstein structures, the local study on its moduli has given interesting results via deformations of the transverse complex structure in a more general setting, see [40, 48].

In this article, we describe the local moduli of Sasaki-Einstein metrics from the study of invertible polynomials, that is polynomials of the form $f = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}$, where $A = \left(a_{ij}\right)_{i,j=1}^{n}$ is a non-negative integer-valued matrix which is invertible over $\mathbb{Q}$ and where f is quasihomogeneous, i.e., there exist positive integers w_j such that $d := \sum_{j=1}^{n} w_j a_{ij}$ is constant for all i, and f is quasismooth, i.e., $f : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$ has exactly one critical point at the origin. Due to the Kreuzer-Skarke classification of invertible polynomials [32] we know that any invertible polynomial, up to permutation of variables, can be written as a Thom-Sebastiani sum of three types of polynomials usually called atoms:

1. Fermat type: $w = x^a$,
2. Chain type: $w = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \ldots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, and
3. Loop or cycle type: $w = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \ldots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$.

These atoms have remarkable properties, for instance in [29] cyclic polynomials have been studied in order to obtain information on rational surfaces with quotient singularities. Invertible polynomials, thought as defining subsets of weighted projective spaces, produce smooth links with some interesting features both at the topological/differential and the Riemannian level. As a matter of fact, in [4] Boyer and Galicki generalized a method introduced by Kobayashi in [26] to produce Sasaki-Einstein metrics on links of hypersurface singularities. From there, many examples of highly connected Sasaki-Einstein manifolds were found. For such examples the existence of orbifold Kähler-Einstein metrics on various Fano orbifolds is verified using different incarnations of the α-invariant method developed by Tian, Nadel, Demailly, Kollár and others [18, 19, 39, 46]. In [6, 7, 12, 28] special interest was given to 5-dimensional manifolds and homotopy spheres realized as links of Brieskorn-Pham polynomials which are an important block in the classification of invertible polynomials. In these articles the local moduli space of Sasaki-Einstein structures for this kind of links were also studied. For instance, in [7] it was found that for homotopy spheres given as links, the dimension of their moduli grew double exponentially with the dimension and they also showed that all the 28 oriented diffeomorphism classes on S^7 admitted inequivalent families of Sasakian-Einstein structures. In [8], again using links of Brieskorn-Pham polynomials viewed as branched covers over Calabi-Yau hypersurfaces, it is shown that there exist continuous parameter families of Sasakian-Einstein metrics on infinitely many simply connected rational homology spheres in every odd dimension greater than 3. In particular, for dimension 7 they found 38 effective real parameters. Later, in [13] it is shown that the moduli space of positive Sasaki classes with vanishing first Chern class for manifolds of the form $S^{2n} \times S^{2n+1}$ or for odd dimensional homotopy spheres among others, has a countably infinite number of components of dimension greater than one and these contain no extremal Sasaki metrics at all. Again this information is produced through links of Brieskorn-Pham polynomials.

In [5], based on the complete list of Johnson and Kollár of anticanonically embedded Fano 3-folds [24], many rational homology 7-spheres admitting Sasaki-Einstein metrics were constructed using the Kobayashi-Boyer-Galicki method. The ideas in [5] were extended in [14] and [16] and moreover it was shown that all the invertible polynomials producing rational homology 7-spheres admitting Sasaki-Einstein metrics were polynomials of cycle type, or iterated Thom-Sebastiani sums of a cycle type involving three variables and another atomic type involving two variables and they cannot be written as Brieskorn-Pham polynomials. In this article we investigate the space of deformations of Sasaki-Einstein metrics for this sort of links. We found that only the latter case admits non-equivalent Sasaki-Einstein metrics. Actually we show that

- For polynomials of cycle type producing rational homology 7-spheres, the corresponding local moduli space of Sasaki-Einstein metrics has dimension zero, that is, links of this sort do not admit inequivalent families of Sasaki-Einstein structures, see Subsection 3.1.
- For Thom-Sebastiani sums of a cycle type and another atomic type, the dimension of local moduli space of Sasaki-Einstein metrics are given in terms of the rational weights, introduced by Milnor and Orlik in [38], for the corresponding weighted hypersurface and, in general, this dimension is positive, see Subsection 3.2.

Since all the links under study in this article remain Sasaki-Einstein rational homology 7-spheres under the Berglund-Hübsch rule from classical mirror symmetry [16], we found that rational homology 7-spheres given as links coming from polynomials of this type do not admit inequivalent families of Sasaki-Einstein structures with the exception of five elements, see Subsection 3.3. In particular our results give information on the connected components of the moduli of rational homology 7-spheres with specific differential structure which can be expressed as link of isolated hypersurface singularities coming from the Johnson and Kollár list of Fano 3-folds of index 1. Our findings can be interpreted in two different settings:

- Seifert S^1-bundles are rational homology spheres if and only if the corresponding orbifolds are rational homology complex projective spaces [30], so our results describe some components of the moduli space of rational homology complex projective 3-spaces with quotient singularities.
- Sasaki-Einstein structures on the manifold determine Ricci-flat Kähler cone metrics on the corresponding affine cone [9], so our results give information on the moduli of Calabi-Yau cones.

In order to obtain these results, we find the generators of the infinitesimal deformations of the complex structures for orbifolds arising from families of polynomials of the types mentioned above, and then determine the moduli problem for Kähler-Einstein metrics on the corresponding hypersurfaces embedded in weighted projective spaces. In fact, arithmetic conditions on the invertible polynomials are imposed so the corresponding links are rational homology 7-spheres admitting Sasaki-Einstein metrics. Then we we use these arithmetic constraints to obtain all possible monomials

$$z_0^{x_0} z_1^{x_1} z_2^{x_2} z_3^{x_3} z_4^{x_4},$$

of certain degree d.

Due to our interest in smooth Sasaki-Einstein links, our results are based on the information obtained on the subset all quasismooth elements determined by the monomials in $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$. Nonetheless, the explicit description of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ leads to the study of the non-quasismooth polynomial generated by these monomials where, as suggested in [34] (see also [41]), the ones with at worst klt singularities are the candidates to give rise to non-smooth links whose metric cones can be considered as some sort of degenerating Calabi-Yau cones. Actually, non-quasismooth klt hypersurfaces with α-invariant greater than one describe points in the boundary of a compactification of the moduli of quasismooth Kähler-Einstein weighted hypersurface and on the K-moduli of Fano cones as described in [42] for instance. We explore these ideas in the last section of this article and explain how to produce families of quasismooth weighted hipersurfaces that denegerate to non-quasismooth klt varieties through examples.

The paper is organized as follows. Section 2 reviews the background material relevant for this paper. In Section 3 we relate the problem of finding the moduli number of Sasaki-Einstein rational homology 7-spheres to solving the Diophantine equations subject to certain arithmetic conditions extracted from the building blocks of the invertible polynomial associated to this problem. Finally, we find the dimension of the corresponding local moduli space of Sasaki-Einstein metrics and prove our main results. In the last section, we discuss the role of non-quasismooth hypersurfaces in this setting.

2 Preliminaries

2.1 Sasakian structures on smooth links and invertible polynomials

Below we give a very brief review of the main ingredients we need to establish our results. The canonical references here are [9] and [45]. After that, we give arithmetic conditions that allow us to manufacture rational homology 7-spheres from links of invertible polynomials. Finally, we explain how the Berglund-Hübsch transpose rule [1] can be used in this framework.

Sasakian Geometry: Sasakian geometry is a special type of contact metric structure on a $(2n+1)$-dimensional manifold M described by the tensors (Φ, ξ, η, g) such that η is a contact 1-form, Φ is an endomorphism of the tangent bundle, $g = d\eta \circ (\mathbb{I} \times \Phi) + \eta \otimes \eta$ is a Riemannian metric and ξ is the Reeb vector field which is Killing. Moreover, the underlying CR-structure $(\mathcal{D}, \Phi|_{\mathcal{D}})$ is integrable, where $\mathcal{D} = \ker \eta$ denotes the contact structure.

A Sasaki manifold has a transverse Kählerian structure $(\nu(\mathcal{F}_\xi), \bar{J})$ determined by the normal bundle $\nu(\mathcal{F}_\xi)$ of the characteristic foliation $\mathcal{F}_\xi$ determined by ξ and the natural complex structure $\bar{J}$ in $\nu(\mathcal{F}_\xi)$ induced by $\Phi|_{\mathcal{D}}$. In fact, when all the orbits of ξ are closed, the Reeb vector field ξ generates a locally free circle action whose quotient is a Kähler orbifold. In this case the Sasakian structure is called quasiregular. When the action is free it is called regular and the quotient is a Kähler manifold. In the irregular case when the orbits of the Reeb vector field are not closed the local quotients are Kähler.

Alternatively, one can understand a Sasakian structure on the manifold M in terms of the metric cone $C(M) = M \times \mathbb{R}^+$ with symplectic form $\omega = d\left(r^2\eta\right)$ where r is the radial coordinate. Indeed, using the Liouville vector field $\Psi = r\partial_r$ we define a natural complex structure I on $C(M)$ by

$$IX = \Phi X + \eta(X)\Psi, \quad I\Psi = -\xi,$$

where X is a vector field on M, and ξ is understood to be lifted to $C(M)$. By adding the apex of the cone we obtain an affine variety $C(M) \cup \{0\}$ which has been intensely studied to obtain K-stability theorems in the manner of the work of Chen-Donaldson-Sun in the context of Sasakian manifolds [17]. We know that (M, g) is Sasakian if the cone $(C(M), \omega, I, \bar{g})$ is Kähler with Kähler metric given by $\bar{g} = dr^2 + r^2 g$. Moreover, the Sasakian structure corresponds to a polarized affine variety $(C(M) \cup \{0\}, I, \xi)$ polarized by the Reeb vector field ξ and this affine variety admits a Ricci-flat Kähler cone metric if and only if (M, g) admits a Sasaki-Einstein metric.

Let us denote by $\mathcal{S}(M)$ the space of all Sasakian structures on M with the C^∞ Fréchet topology as sections of vector bundles, and let $\mathcal{S}(M, \xi, \bar{J})$ be the subset of Sasakian structures with Reeb vector field ξ and transverse holomorphic structure $\bar{J}$, which is endowed with the subspace topology. Consider $\mathcal{SE}(M)$ the subspace of Sasaki-Einstein metrics in $\mathcal{S}(M, \xi, \bar{J})$ and let $\mathfrak{Aut}(\bar{J})$ be the group of complex automorphisms of $(C(M), I)$ that commute with $\Psi - i\xi$. This group descends to an action on $(M, \mathcal{S})$ commuting with ξ. Then one defines the local moduli space of Sasaki-Einstein metrics [3] by

$$\mathfrak{M}^{SE}(M) = \mathcal{SE}(M)/\mathfrak{Aut}(\bar{J})_0,$$

where $\mathfrak{Aut}(\bar{J})_0$ denotes the connected component of $\mathfrak{Aut}(\bar{J})$.

As we will see in Subsection 2.2, this space has a very concrete description for links of weighted homogeneous hypersurfaces.

Weighted homogeneous hypersurfaces: Consider the weighted $\mathbb{C}^*$-action on the affine space $\mathbb{C}^{n+1}$, defined by

$$(z_0, \ldots, z_n) \mapsto \left(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n\right)$$

where $\mathbf{w} = (w_0, \ldots, w_n)$ is a sequence of positive integers. Then we obtain the weighted projective space with a canonical orbifold structure, defined as the quotient space

$$\mathbb{P}(\mathbf{w}) = \left(\mathbb{C}^{n+1} - \{\mathbf{0}\}\right) / \mathbb{C}^*.$$

The weighted projective space $\mathbb{P}(\mathbf{w})$ is said to be well-formed if the weighted $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ has trivial stabilizers in codimension 1, that is, when $\gcd(w_0, \ldots, \widehat{w}_i, \ldots, w_n) = 1$ for each i. Here the hat symbol means delete that corresponding element. As an algebraic variety, a weighted projective space can be defined as

$$\mathbb{P}(w_0, \ldots, w_n) = \operatorname{Proj}(S(\mathbf{w})),$$

where $S(\mathbf{w}) = \mathbb{C}[x_0, \ldots, x_n]$ is the graded polynomial ring such that the weight of each x_i equals w_i.

Let us recall that a polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ is said to be a weighted homogeneous polynomial of degree d and weight vector $\mathbf{w} = (w_0, \ldots, w_n)$, if for any $\lambda \in \mathbb{C}^*$

$$f\left(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n\right) = \lambda^d f(z_0, \ldots, z_n).$$

We assume that f is chosen so that the affine algebraic variety

$$V_f = \{f = 0\} \subset \mathbb{C}^{n+1}$$

is smooth everywhere except at the origin in $\mathbb{C}^{n+1}$ which is equivalent to say that the weighted variety

$$Z_f = (V_f - \{\mathbf{0}\})/\mathbb{C}^* \subset \mathbb{P}(\mathbf{w})$$

is quasismooth of dimension n. There are well-known conditions that determine when a specific polynomial determines a quasismooth weighted hypersurface [21, 24]:

Lemma 2.1.1 *A weighted hypersurface of degree d in $\mathbb{P}(w_0,\ldots,w_4)$, where $d > w_i$, is quasismooth if and only if the following hold:*

1. *For each $i = 0,\cdots,4$ there is a j and a monomial $z_i^{m_i}z_j$ of degree d. Here $j = i$ is possible.*
2. *For all distinct i, j either there is a monomial $z_i^{b_i}z_j^{b_j}$ of degree d or there exist monomials $z_i^{n_1}z_j^{m_1}z_k, z_i^{n_2}z_j^{m_2}z_l$ of degree d with $\{k,l\} \neq \{i,j\}$ and $k \neq l$.*
3. *For every i, j there exists a monomial of degree d that does not involve either z_i or z_j.*

We have the following important result whose proof can be found in [9], Corollary 5.4.8. First, recall that the index I of a weighted hypersurface is given by the difference $I = |\mathbf{w}| - d$, where $|\mathbf{w}| = \sum_{i=0}^{n} w_i$ denotes the norm of the weight vector $\mathbf{w}$. The weighted hypersurface is said to be Fano if $I > 0$.

Theorem 2.1.1 *Let $Z_f \subset \mathbb{P}(w_0,\ldots w_n)$ be a quasismooth weighted homogeneous Fano hypersurface of degree d. Then Z_f admits a Kähler-Einstein orbifold metric if the following estimate holds:*

$$dI < \frac{n}{(n-1)}\min_{i,j}\left\{w_i w_j\right\}.$$

The weighted hypersurface is well-formed if $\mathbb{P}(\mathbf{w})$ is well-formed and $Z_f \cap \mathrm{sing}(\mathbb{P}(\mathbf{w}))$ has codimension at least 2 in Z_f. When Z_f is well-formed, the canonical divisor satisfies the adjunction formula $K_{Z_f} = \mathcal{O}_{Z_f}(d - w_0 - \cdots - w_n)$. In [21] a criterion to determine the well-formedness of the weighted variety is given: a hypersurface defined by the weighted homogeneous polynomial f of degree d is well-formed in the well-formed weighted projective space $\mathbb{P}(w_0,\ldots,w_n)$ if $\gcd\left(w_0,\ldots,\hat{w}_i,\ldots,\hat{w}_j,\ldots,w_n\right) \mid d$ for distinct $i, j = 0,\ldots,n$.

Links of hypersurface singularities and Sasakian structures. Sasakian structures can be manufactured on links of hypersurface singularities of weighted homogeneous polynomials and we explain how to do this below.

A link $L_f(\mathbf{w},d)$, or L_f for short, is defined as the intersection $V_f \cap S^{2n+1}$, where S^{2n+1} is the $(2n+1)$-sphere in Euclidean space. By the Milnor fibration Theorem [37], $L_f(\mathbf{w},d)$ is a closed $(n-2)$-connected manifold that bounds a parallelizable manifold with the homotopy type of a bouquet of n-spheres. Furthermore, $L_f(\mathbf{w},d)$ admits a quasiregular Sasaki structure $\mathcal{S} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which is the restriction of the weighted Sasaki structure on the sphere S^{2n+1} with Reeb vector field $\xi_{\mathrm{w}} = \sum_{k=0}^{n} w_k\left(y_k\partial_{x_k} - x_k\partial_{y_k}\right)$ and contact form $\eta_{\mathbf{w}} = \frac{\eta}{\sum_{i=0}^{n} w_i\left((x_i)^2+(y_i)^2\right)}$, where η denotes the standard contact 1-form on the sphere S^{2n+1}. If one considers the locally free S^1-action induced by the weighted $\mathbb{C}^*$ action on V_f, the quotient space of the link $L_f(\mathbf{w},d)$ by this action is the weighted hypersurface Z_f, a Kähler orbifold. We have the following commutative diagram [9]

$$\begin{array}{ccc} L_f(\mathbf{w},d) & \longrightarrow & S_{\mathbf{w}}^{2n+1} \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow \\ Z_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where $S_{\mathbf{w}}^{2n+1}$ denotes the unit sphere with a weighted Sasakian structure, $\mathbb{P}(\mathbf{w})$ is a weighted projective space coming from the quotient of $S_{\mathbf{w}}^{2n+1}$ by a weighted circle action generated from the weighted Sasakian structure. The top horizontal arrow is a Sasakian embedding, the bottom arrow is a Kählerian embedding and the vertical arrows are orbifold Riemannian submersions.

It follows from the orbifold adjunction formula that the link L_f admits a positive Ricci curvature if the quotient orbifold Z_f by the natural S^1-action is Fano.

In [26], Kobayashi showed that the link of a cone over a smooth projective variety $Z \subset \mathbb{P}^n$ carries a natural Einstein metric if and only if Z is Fano and Z carries a Kähler-Einstein metric. In [4], the authors generalized this result to weighted cones and furthermore gave an algorithm, the Kobayashi-Boyer-Galicki method, to obtain $(n-1)$-connected Sasaki-Einstein $(2n+1)$-manifolds from the existence of orbifold Fano Kähler-Einstein hypersurfaces Z_f in weighted projective $2n$-space $\mathbb{P}(\mathbf{w})$.

Important topological information of the link can be obtained via the Alexander polynomial. Recall that the Alexander polynomial $\Delta_f(t)$ [37] associated to a link L_f of dimension $(2n-1)$ is the characteristic polynomial of the monodromy map $h_* : H_n(F, \mathbb{Z}) \to H_n(F, \mathbb{Z})$ which is induced by the circle action on the Milnor fibre F. Then $\Delta_f(t) = \det(t\mathbb{I} - h_*)$. Now both F and its closure $\bar{F}$ are homotopy equivalent to a bouquet of n-spheres $S^n \vee \cdots \vee S^n$, and the boundary of $\bar{F}$ is the link L_f, which is $(n-2)$-connected. The Betti numbers $b_{n-1}\left(L_f\right) = b_n\left(L_f\right)$ equal the number of factors of $(t-1)$ in $\Delta_f(t)$. From the Wang sequence of the Milnor fibration (see [44])

$$0 \longrightarrow H_n\left(L_f, \mathbb{Z}\right) \longrightarrow H_n(F, \mathbb{Z}) \xrightarrow{\mathbb{I}-h_*} H_n(F, \mathbb{Z}) \longrightarrow H_{n-1}\left(L_f, \mathbb{Z}\right) \longrightarrow 0$$

one obtains that L_f is a $\mathbb{Q}$-homology sphere if and only if $\Delta_f(1) \neq 0$ and the order of $H_{n-1}\left(L_f, \mathbb{Z}\right)$ equals $|\Delta_f(1)|$. In the case that f is a weighted homogeneous polynomial there is an algorithm due to Milnor and Orlik [38] to compute the Alexander polynomial in terms of the degree and the weights: associate to any monic polynomial f with roots $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^*$ its divisor

$$\operatorname{div} f = \langle \alpha_1 \rangle + \cdots + \langle \alpha_k \rangle$$

as an element of the integral ring $\mathbb{Z}\left[\mathbb{C}^*\right]$. Let $\Lambda_n = \operatorname{div}(t^n - 1)$. Then the divisor of $\Delta_f(t)$ is given by

$$\operatorname{div} \Delta_f = \prod_{i=0}^{n} \left(\frac{\Lambda_{u_i}}{v_i} - \Lambda_1\right),$$

where the $u_i's$ and $v_i's$ are given terms of the degree d of f and the weight vector $\mathbf{w} = (w_0, \ldots w_n)$ by the equations

$$u_i = \frac{d}{\gcd(d, w_i)}, \quad v_i = \frac{w_i}{\gcd(d, w_i)}.$$

Invertible polynomials and rational homology spheres: In [14, 16] we used the Kobayashi-Boyer-Galicki method to establish the existence of Sasaki-Einstein metrics on links of invertible polynomials. From [5] and [14] one notices that all the invertible polynomials taken from the list of Johnson and Kollár of anticanonically embedded Fano 3-folds [24] producing Sasaki-Einstein rational homology 7-spheres were polynomials of cycle type, chain type, or iterated Thom-Sebastiani sums of these types. These polynomials can be described in terms of the following two sets of conditions

- The weights and the degree satisfy $\gcd(d, w_i) = 1$ for all $i = 0, \ldots 4$, which leads to singularities of cycle type.
- The weights are subject to $(w_0, w_1, w_2, w_3, w_4) = (m_3 v_0, m_3 v_1, m_2 v_2, m_2 v_3, m_2 v_4)$ with $\gcd(m_2, m_3) = 1$ and $m_2 m_3 = d$, which leads to singularities that can described

as iterated Thom-Sebastiani sums of chain, cycle type and Fermat singularities. More precisely, the types of singularities obtained have the following form:

$$\text{Type I (Fermat-Cycle): } z_0^{a_0} + z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$
$$\text{Type II (Chain-Cycle): } z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$
$$\text{Type III (Cycle-Cycle): } z_1 z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}.$$

Polynomials with this set of conditions satisfy the following statements whose proofs can be found in [5] and/or in the proof of Theorem 3.2 in [14]:

1. Consider links $L(\mathbf{w}, d)$ of weighted homogeneous polynomials f of the first kind, that is, with $\gcd(d, w_i) = 1$ for all $i = 0, \dots 4$, then the Milnor number $m(L_f)$ for L_f is given by
$$m(L_f) + 1 = d(b_{n-1} + 1) \text{ and } H_3\left(L_f, \mathbb{Z}\right)_{tor} = \mathbb{Z}_d.$$
In particular, if f is given as a polynomial of cycle type
$$f = z_4 z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$
of degree d in the projective space $\mathbb{P}(\mathbf{w})$, where $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ then from the equations
$$a_0 w_0 + w_4 = d, \quad a_1 w_1 + w_0 = d,$$
$$a_2 w_2 + w_1 = d, \quad a_3 w_3 + w_2 = d, \quad a_4 w_4 + w_3 = d$$
we obtain
$$a_0 a_1 a_2 a_3 a_4 = \left(\frac{d - w_4}{w_0}\right)\left(\frac{d - w_0}{w_1}\right)\left(\frac{d - w_1}{w_2}\right)\left(\frac{d - w_2}{w_3}\right)\left(\frac{d - w_3}{w_4}\right) = m\left(L_f\right).$$
So in the case the link is a rational homology 7-sphere we obtain
$$a_0 a_1 a_2 a_3 a_4 = d - 1. \tag{2.1.1}$$
2. Consider links $L(\mathbf{w}, d)$ of the second kind, that it, such that the weight vectors $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ satisfy $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4) = (m_3 v_0, m_3 v_1, m_2 v_2, m_2 v_3, m_2 v_4)$, $\gcd(m_2, m_3) = 1$ and $m_2 m_3 = d$. One obtains the equality
$$\operatorname{div} \Delta_f = \alpha(\mathbf{w})\beta(\mathbf{w})\Lambda_d + \beta(\mathbf{w})\Lambda_{m_3} - \alpha(\mathbf{w})\Lambda_{m_2} - \Lambda_1,$$
with the two positive integers $\alpha(\mathbf{w})$ and $\beta(\mathbf{w})$ depending on the weights:
$$\alpha(\mathbf{w}) = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1} \tag{2.1.2}$$
and
$$\beta(\mathbf{w}) = \left(\frac{m_3}{v_2 v_3} - \frac{1}{v_3} - \frac{1}{v_2}\right)\left(\frac{m_3}{v_4} - 1\right) + \frac{1}{v_4}. \tag{2.1.3}$$
It is known that if the link is a rational homology sphere, then $\beta(\mathbf{w}) = 1$. Furthermore, if f has a cycle block of the form
$$z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4},$$
then
$$a_2 w_2 + w_4 = d, \quad a_3 w_3 + w_2 = d \text{ and } a_4 w_4 + w_3 = d.$$

From the assumptions on the weight vector $\mathbf{w}$ we have

$$\begin{aligned} a_2a_3a_4 &= \left(\frac{d-w_4}{w_2}\right)\left(\frac{d-w_2}{w_3}\right)\left(\frac{d-w_3}{w_4}\right) \\ &= \left(\frac{m_3-v_4}{v_2}\right)\left(\frac{m_3-v_2}{v_3}\right)\left(\frac{m_3-v_3}{v_4}\right) \\ &= \frac{m_3^3-(v_2+v_3+v_4)\,m_3^2+(v_2v_3+v_2v_4+v_3v_4)\,m_3-v_2v_3v_4}{v_2v_3v_4} \\ &= m_3\left(\frac{m_3^2-(v_2+v_3+v_4)\,m_3+v_2v_3+v_2v_4+v_3v_4}{v_2v_3v_4}\right)-1. \end{aligned}$$

Since the corresponding link is a $\mathbb{Q}$-homology sphere, it follows that

$$\beta(\mathbf{w}) = \frac{m_3^2-(v_2+v_3+v_4)\,m_3+v_2v_3+v_2v_4+v_3v_4}{v_2v_3v_4} = 1.$$

Substituting this equality in the previous equation, we obtain

$$a_2a_3a_4 + 1 = m_3. \tag{2.1.4}$$

The Berglund-Hübsch transpose rule: Recall that the Berglund-Hübsch transpose rule considers an invertible polynomial

$$f = \sum_{i=0}^{n}\prod_{j=0}^{n} x_j^{a_{ij}}$$

cutting out an orbifold of degree d in $\mathbb{P}(\mathbf{w})$ and defines the transpose polynomial f^T by transposing the exponential matrix $A = \left(a_{ij}\right)$ of the original polynomial, that is,

$$f^T = \sum_{i=0}^{n}\prod_{j=0}^{n} x_j^{a_{ji}},$$

again an invertible polynomial that cuts out an orbifold of degree $\tilde{d}$ in $\mathbb{P}(\tilde{\mathbf{w}})$. Then one considers the links $L_f(\mathbf{w}, d)$ and $L_{f^T}(\tilde{\mathbf{w}}, \tilde{d})$ associated to each of these polynomials. We will sometimes say that these two links are *Berglund-Hübsch duals* to one another. The following diagram succinctly summarizes the procedure described above, where BH denotes the Berglund-Hübsch transpose rule:

$$\begin{array}{ccc} f=0 & \xrightarrow{BH} & f^T=0 \\ \downarrow & & \downarrow \\ L_f(\mathbf{w},d) & \xrightarrow{BH} & L_{f^T}(\tilde{\mathbf{w}},\tilde{d}). \end{array}$$

In [16] the Berglund-Hübsch transpose rule is used to produce Sasaki-Einstein links with the $\mathbb{Q}$-homology of a 7-sphere. In particular, we found that this rule only produces *twins* for singularities of cycle type or of type I and type III. Recall [5, 16] that two links L_f and L_g of an isolated hypersurface singularity are called twins if both are $\mathbb{Q}$-homology $(2n+1)$-spheres and they satisfy $m\left(L_f\right) = m\left(L_g\right)$, $d_g = d_f$, and $H_n\left(L_f, \mathbb{Z}\right) = H_n\left(L_g, \mathbb{Z}\right)$. However, for polynomials of type II, the Berglund-Hübsch transpose rule does not preserve neither torsion nor Milnor number. Their dual links will receive special attention is Subsection 3.3.

2.2 Deformations of transverse holomorphic Sasakian structures

Locally the moduli space of Sasaki isotopy classes is determined by the deformation theory of the transverse holomorphic structure of the foliation $\mathcal{F}_\xi$. So the usual thing to do is to fix the contact structure and deform the transverse holomorphic structure via Kodaira-Spencer theory.

A germ of a deformation of a transverse holomorphic foliation $\mathcal{F}_\xi$ on M with base space $(B, 0)$ is given by an open cover $\{U_\alpha\}$ of M and a family of local submersions $f_{\alpha,t} : U_\alpha \to \mathbb{C}^n$ parametrized by $(B, 0)$ that are holomorphic in $t \in B$ for each $x \in U_\alpha$. For $\Theta_{\mathcal{F}_\xi}$ denoting the sheaf of transversely holomorphic vector fields on M, we have a Kodaira-Spencer map $\rho : T_0B \to H^1\left(M, \Theta_{\mathcal{F}_\xi}\right)$ that sends $\frac{\partial}{\partial t}$ to a certain class in $H^1\left(M, \Theta_{\mathcal{F}_\xi}\right)$ defined by a section $\theta_{\alpha,\beta}$ of the sheaf $\Theta_{\mathcal{F}_\xi} \mid U_\alpha \cap U_\beta$. One can consider the full cohomology ring $H^*\left(M, \Theta_{\mathcal{F}_\xi}\right)$, these were proven to be finite dimensional. In [20] it is shown that there is a versal Kuranishi space of deformations given by the map $\Phi : U \to H^2\left(M, \Theta_{\mathcal{F}_\xi}\right)$, for U open set in $H^1\left(M, \Theta_{\mathcal{F}_\xi}\right)$, here, as in the complex case, the base of parametrizations is given by $\Phi^{-1}(0)$. We have that if $H^2\left(M, \Theta_{\mathcal{F}_\xi}\right) = 0$, then the Kuranishi family of deformations of $\mathcal{F}_\xi$ is isomorphic to an open set in $H^1\left(M, \Theta_{\mathcal{F}_\xi}\right)$. Otherwise, the Kuranishi space may be singular. For a quasiregular Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with quotient orbifold Z one obtains the following sequence [9]

$$0 \longrightarrow H^1(Z, \Theta_Z) \longrightarrow H^1\left(M, \Theta_{\mathcal{F}_\xi}\right) \longrightarrow H^0(Z, \Theta_Z) \longrightarrow H^2(Z, \Theta_Z)$$

where Θ_Z denotes the sheaf of germs of holomorphic vector fields on the complex orbifold Z and $H^0(Z, \Theta_Z)$ can be considered as the Lie algebra of the group of holomorphic automorphisms of Z. Thus the deformation of the transverse holomorphic structures of the foliation can be understood in terms of the deformation of the complex structure of the orbifold, which are described by $H^1(Z, \Theta_Z)$ and in terms of the deformations of the Reeb vector in $H^0(Z, \Theta_Z)$ which are described by the Sasaki cone [10]. Of course, if $H^0(Z, \Theta_Z) = 0$ we obtain an isomorphisms between $H^1\left(M, \Theta_{\mathcal{F}_\xi}\right)$ and $H^1(Z, \Theta_Z)$. Deformations of the transversely holomorphic structure may not remain Sasakian unless the $(0, 2)$ component of the basic Euler class $\left[d\eta^{0,2}\right] \in H_B^{(0,2)}(M, \mathcal{F}_\xi)$ is zero, where $H_B^{0,2}(M, \mathcal{F}_\xi)$ is the basic Dolbeault cohomology for the transversal complex structure. However, in case the Sasakian structure is positive, which is the case we are interested in this article, it has been proven in [40] that $H_B^{0,q}(M, \mathcal{F}_\xi) = 0$ for $q > 0$.

We pay particular attention to the local moduli of quasiregular Sasakian structures on links of isolated singularities, so we focus on orbifolds that are quasismooth weighted hypersurfaces in certain weighted projective spaces. Moreover, all the orbifolds under discussion in this article have finite automorphism group, so we will assume $H^0(Z, \Theta_Z) = 0$. The proof of the next two theorems can be found in [9]. (See also [7].)

Theorem 2.2.1 *Let Z_f be a quasismooth weighted hypersurface in $\mathbb{P}(\mathbf{w})$ corresponding to the weighted homogenous polynomial f of degree d and weight vector $\mathbf{w} = (w_0, \ldots, w_n)$ with $H^0\left(Z_f, \Theta_{Z_f}\right) = 0$. Assume also that $n \geqslant 3$. Then the complex orbifolds Z_f form a continuous family of complex dimension*

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(d)) - \sum_i \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)). \tag{2.2.1}$$

Furthermore, if the index $I = |\mathbf{w}| - d > 0$ and Z_f admits a Kähler-Einstein metric for a generic f then it admits a $2\left[h^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i h^0\left(\mathbb{P}(\mathbf{w}), \mathcal{O}\left(w_i\right)\right)\right]$ dimensional family of Kähler-Einstein metrics up to homothety.

For quasiregular Sasaki-Einstein metrics we have the following:

Theorem 2.2.2 *Let M be a smooth compact manifold and let $\mathcal{S} = (\xi, \eta, \Phi_t, g_t)$ be a family of quasiregular Sasaki-Einstein structures on M induced by a continuous family of inequivalent complex orbifolds $\mathcal{Z}_t$ with Kähler-Einstein metrics. Then the metrics g_t are inequivalent as Sasaki-Einstein metrics.*

From these two theorems it is clear that one can compute the dimension of the local moduli of Sasaki-Einstein metrics through the information given by the moduli of Kähler-Einstein metrics on Fano orbifolds. We will do this for rational homology 7-spheres that are obtained as links of invertible polynomials.

The procedure to determine the moduli of Kähler-Einstein metrics on Fano orbifolds goes as follows

(a) Consider $X_d \subset \mathbb{P}(w_0, \ldots, w_n)$ a well-formed and quasismooth weighted projective hypersurface of degree d with finite automorphism group, cut out by an invertible polynomial f of certain defined type.
(b) Determine the linear system

$$\mathcal{X}_d = \left|\mathcal{O}_{\mathbb{P}(\mathbf{w})}(d)\right| = \mathbb{P}H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)),$$

that is, the parameter space for all hypersurfaces of degree d in $\mathbb{P}(\mathbf{w})$.
(c) Determine the automorphism group $\mathcal{G}(\mathbf{w})$ of $\mathbb{P}(\mathbf{w})$.
(d) Since $\mathcal{G}(\mathbf{w})$ acts on $\left|\mathcal{O}_{\mathbb{P}(\mathbf{w})}(d)\right|$ and there is an inclusion of the set $\mathcal{X}_d^{QS}$ of quasismooth hypersurfaces of degree d in $\mathbb{P}(\mathbf{w})$, as an open set in $\mathcal{X}_d$, one obtains the quotient

$$\mathcal{X}_d^{QS}/\mathcal{G}(\mathbf{w}),$$

which is a coarse moduli space, see [25] Corollary 1.2. Actually, since

$$\mathcal{X}_d^{QS}/\mathcal{G}(\mathbf{w}) \subset \mathbb{P}H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))/\mathbb{P}G(\mathbf{w}) = H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))/G(\mathbf{w}),$$

where $G(\mathbf{w})$ is the group of automorphisms of the graded ring $S(\mathbf{w})$, we will give a precise description of the former quotient.
(e) Additionally, we will assume that the weighted hypersurface $X_d \subset \mathbb{P}(\mathbf{w})$ satisfies the estimate $dI < \frac{n}{(n-1)} \min_{i,j}\left\{w_i w_j\right\}$ in Theorem 2.1.1, which implies the existence of Kähler-Einstein metrics in all the elements in $\mathcal{X}_d^{QS}$. It follows from Theorems 2.2.1 and 2.2.2 that the number of parameters of the space of inequivalent Sasaki-Einstein metrics on the corresponding link $L_f = V_f \cap S^{2n+1}$ is given by the dimension of the moduli $\mathcal{X}_d^{QS}/\mathcal{G}(\mathbf{w})$.

Notice that well-formedness is required, otherwise we only obtain upper bounds in the dimension of the moduli space. For instance consider $Z_9 \subset \mathbb{P}(3, 3, 3, 3, 3)$ a hypersurface of degree 9 which is isomorphic as a variety to $Z_3 \subset \mathbb{P}(1, 1, 1, 1, 1)$ a hypersurface of degree 3 with the same equation. In Subsection 3.3, where we study the moduli of the Berglund-Hübsch duals, we deal with non well-formed weighted hypersurface where the procedure descried above suffices to determine the precise dimension of the moduli in most cases. In Section 4 we also discuss the role of the boundary points of $\overline{\mathcal{X}_d^{QS}/\mathcal{G}(\mathbf{w})}$, that is, the non-quasismooth weighted varieties, in the construction of Sasaki-Einstein metrics on non-smooth links.

3 Local moduli of Sasaki-Einstein metrics on smooth rational homology 7-spheres

In this section we will study the space of deformations of polynomials of cycle type, type I, type II and type III. We focus only on these types of singularities for the reasons that were argued in Section 2.1: if a link L_f is a smooth rational homology 7-sphere and admits Sasaki-Einstein metric then f has to be one of the aforementioned types. In the process we find the monomials generating $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ and the group of automorphisms of the weighted projective space $\mathbb{P}(\mathbf{w})$ that contains the corresponding orbifold.

3.1 Rational homology 7-spheres: the cycle type

Let f be an invertible polynomial of the form

$$f = z_4 z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4} \tag{3.1.1}$$

of degree d and associated weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ with $\gcd(d, w_i) = 1$, which in particular implies that the weighted hypersurface $Z_f \subset \mathbb{P}(\mathbf{w})$ is well-formed. From (3.1.1), we have the following relation between the weights w_i's and the degree:

$$a_0 w_0 + w_4 = d, \quad a_1 w_1 + w_0 = d, \quad a_2 w_2 + w_1 = d, \quad a_3 w_3 + w_2 = d, \quad a_4 w_4 + w_3 = d. \tag{3.1.2}$$

Moreover, if the link L_f is a rational homology sphere, from the equality (2.1.1) $d = 1 + a_0 a_1 a_2 a_3 a_4$, we can express each weight w_i as:

$$w_i = 1 - a_{i+1} + a_{i+1} a_{i+2} - a_{i+1} a_{i+2} a_{i+3} + a_{i+1} a_{i+2} a_{i+3} a_{i+4}, \tag{3.1.3}$$

where the subscript is taken mod 5. Furthermore, considering the relations given in (3.1.2) and the fact that $\gcd(d, w_i) = 1$, we conclude that two consecutive weights w_i and w_{i+1} are always co-prime.

Remark 3.1.1 Notice that the assumption (e) below Theorem 2.2.2 and the relations in (3.1.2) imply that $a_i > 1$ for $i = 0, \ldots, 4$. This fact will be used throughout the proof of Lemma 3.1.2. Indeed, without loosing generality, let us assume that $a_2 = 1$. Then the third equation in (3.1.2) gives $w_1 + w_2 = d$ and the estimate $dI < \frac{4}{3} \min_{i,j} \{w_i w_j\}$ can be written as

$$(w_1 + w_2)(w_0 + w_3 + w_4) < \frac{4}{3} \min_{i,j} \{w_i w_j\}.$$

Since $6 \min_{i,j} \{w_i w_j\} \leqslant (w_0 + w_3 + w_4)(w_1 + w_2)$ we obtain a contradiction.

Now, we find the generators of the space of deformations of the orbifold Z_f. In [7], it was proven that the automorphism group of any orbifold Z_f is finite as long as $w_i < \frac{1}{2} d$ for all but one of the $w_i's$ for f quasismooth. Since $\gcd(d, w_i) = 1$, the weight vector $\mathbf{w}$ does not admit polynomials that contain blocks of the form $z_i^2 + z_j^2$, thus $H^0(Z_f, \Theta_{Z_f}) = 0$. So the complex dimension of the moduli of Z_f is given by

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)). \tag{3.1.4}$$

Let us compute $\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$. Here we consider all possible monomials $z_0^{x_0} z_1^{x_1} z_2^{x_2} z_3^{x_3} z_4^{x_4}$ of degree d. Notice that it is equivalent to solving the following Diophantine equation

$$w_0x_0 + w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 = d \tag{3.1.5}$$

with variables $x_i \in \mathbb{Z}_0^+$ and where at least one of them is nonzero.

First recall a well-known result for Diophantine equations that will be used in the arguments that follow.

Lemma 3.1.1 *We consider the Diophantine equation in variables x, y:*

$$ax + by = c$$

If $\gcd(a, b) = 1$ and (x_0, y_0) is a particular solution, then all the solutions of the Diophantine equation are given by

$$(x, y) = (x_0 - bk, y_0 + ak), \quad \textit{where } k \in \mathbb{Z}.$$

The next lemma determines all solutions of Equation (3.1.5).

Lemma 3.1.2 *Let $\mathbf{w}$ be defined as in (3.1.2). Then the Diophantine equation (3.1.5) has exactly five solutions. These solutions are*

$$(a_0, 0, 0, 0, 1), (1, a_1, 0, 0, 0), (0, 1, a_2, 0, 0), (0, 0, 1, a_3, 0) \textit{ and } (0, 0, 0, 1, a_4).$$

Thus, the set of generators of the space $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by

$$\{z_0^{a_0}z_4, z_0z_1^{a_1}, z_1z_2^{a_2}, z_2z_3^{a_3}, z_3z_4^{a_4}\}.$$

Proof Since the polynomial f associated to $\mathbf{w}$ is cycle, we can assume, without loss of generality, that $w_4 = \min_i w_i$.

Now, in Equation (3.1.5), as $\gcd(w_3, w_4) = 1$, we can define a new variable $y_3 = w_3x_3 + w_4x_4 \geqslant 0$. Replacing this in Equation (3.1.5), we obtain a new Diophantine equation

$$w_0x_0 + w_1x_1 + w_2x_2 + y_3 = d.$$

Here, we define the variable $y_2 = w_2x_2 + y_3 \geqslant 0$. Putting this in the equation above, we get

$$w_0x_0 + w_1x_1 + y_2 = d.$$

Repeating the previous process, we define the new variable $y_1 = w_1x_1 + y_2 \geqslant 0$. Again, replacing this above, we obtain

$$w_0x_0 + y_1 = d. \tag{3.1.6}$$

From (3.1.2), we obtain a particular solution (a_0, w_4) of Equation (3.1.6). Thus, the general solution is given by

$$x_0 = a_0 - t \quad \text{and} \quad y_1 = w_4 + tw_0, \quad \text{where } t \in \mathbb{Z}.$$

Now, we solve the Diophantine equation

$$w_1x_1 + y_2 = y_1. \tag{3.1.7}$$

Since $y_1 = w_4 + tw_0$ and $w_0 = d - a_1w_1$, we obtain

$$w_1x_1 + y_2 = y_1 = w_4 + t(d - a_1w_1) = -ta_1w_1 + w_4 + td.$$

Then we have a particular solution $(-ta_1, w_4 + td)$. Therefore, the general solution of (3.1.7) is given by

$$x_1 = -ta_1 - s \quad \text{and} \quad y_2 = w_4 + td + sw_1, \quad \text{where } t, s \in \mathbb{Z}$$

On the other hand, to solve the Diophantine equation

$$w_2x_2 + y_3 = y_2 \tag{3.1.8}$$

we take into account that $y_2 = w_4 + td + sw_1$ and $w_1 = d - a_2w_2$. Thus, we obtain

$$w_2x_2 + y_3 = y_2 = w_4 + td + sw_1 = -sa_2w_2 + w_4 + (t + s)d$$

Then, a particular solution of equation above is $(-sa_2, w_4+(t+s)d)$. So the general equation is

$$x_2 = -sa_2 - r \quad \text{and} \quad y_3 = w_4 + (t + s)d + rw_2 \quad \text{where } r, s, t \in \mathbb{Z}$$

Finally, we will solve the Diophantine equation

$$w_3x_3 + w_4x_4 = y_3 \tag{3.1.9}$$

Since $y_3 = w_4 + (t + s)d + rw_2$ and $w_2 = d - a_3w_3$, we have

$$w_3x_3 + w_4x_4 = y_3 = w_4 + (t + s)d + rw_2 = (t + s + r - ra_3)w_3 + (1 + a_4(t + s + r))w_4$$

This implies that a particular solution of (3.1.9) is $(t + s + r - ra_3,\ 1 + a_4(t + s + r))$. Then, the general solution is given by

$$x_3 = t + s + r - ra_3 - qw_4 \quad \text{and} \quad x_4 = 1 + (t + s + r)a_4 + qw_3 \quad \text{where } q, r, s, t \in \mathbb{Z}$$

Considering all of the above, we have the general solution for (3.1.5):

$$(x_0, x_1, x_2, x_3, x_4) = (a_0-t, -ta_1-s, -sa_2-r, t+s+r-ra_3-qw_4, 1+(t+s+r)a_4+qw_3) \tag{3.1.10}$$

where $q, r, s, t \in \mathbb{Z}$. Notice that in order to solve (3.1.5), we require $x_i \geqslant 0$ for all i. This implies that

$$t \leqslant a_0, \quad s \leqslant -ta_1, \quad r \leqslant -sa_2, \quad ra_3+qw_4 \leqslant t+s+r \quad \text{and} \quad -qw_3-1 \leqslant (t+s+r)a_4. \tag{3.1.11}$$

On the other hand, for the variables y_i's, we have

$$(y_1, y_2, y_3) = (w_4+tw_0, w_4+td+sw_1, w_4+td+sd+rw_2), \quad \text{where } r, s, t \in \mathbb{Z} \tag{3.1.12}$$

As $x_i \geqslant 0$, then we have $y_i \geqslant 0$. So we obtain the following inequalities

$$tw_0 + w_4 \geqslant 0, \quad w_4 + td + sw_1 \geqslant 0 \quad \text{and} \quad w_4 + td + sd + rw_2 \geqslant 0. \tag{3.1.13}$$

<u>Claim 1</u>: $q \in \{-1, 0\}$. First, we prove that $q \leqslant 0$. Indeed, from (3.1.11) we have

$$x_3 = t + s + r - ra_3 - qw_4 \geqslant 0 \Rightarrow t(1 - a_3) + s(1 - a_3) + r(1 - a_3) \geqslant qw_4 - ta_3 - sa_3$$
$$\Rightarrow t + s + r \leqslant \frac{ta_3 + sa_3 - qw_4}{a_3 - 1}.$$

Also, from (3.1.11), we obtain

$$t + s + r \geqslant \frac{-qw_3 - 1}{a_4}.$$

From the two inequalities above, we get

$$\frac{-qw_3 - 1}{a_4} \leqslant \frac{ta_3 + sa_3 - qw_4}{a_3 - 1} \Rightarrow -qa_3w_3 - a_3 + qw_3 + 1 \leqslant ta_3a_4 + sa_3a_4 - qa_4w_4.$$

Using (3.1.2), we have $a_3w_3 = d - w_2$ and $a_4w_4 = d - w_3$. Replacing these in the previous inequality and simplifying, we obtain

$$qw_2 \leqslant a_3 - 1 + ta_3a_4 + sa_3a_4. \tag{3.1.14}$$

Considering the equality (3.1.3) for w_2, we have

$$\begin{aligned} qw_2 &\leqslant a_3 - 1 + a_0a_3a_4 - ta_1a_3a_4 \\ &= -1 + a_3 - a_3a_4 + a_0a_3a_4 - a_0a_1a_3a_4 + (a_0 - t)a_1a_3a_4 + a_3a_4 \\ &= -w_2 + (a_0 - t)a_1a_3a_4 + a_3a_4. \end{aligned}$$

On the other hand, since $\gcd(w_0, w_4) = 1$ and w_4 is the minimum of all w_i's, we have $w_4 < w_0$. Moreover, using (3.1.13), we obtain $t \geqslant -\dfrac{w_4}{w_0} > -1$. Thus, we have $t \geqslant 0$. Thus

$$qw_2 \leqslant -w_2 + (a_0 - t)a_1a_3a_4 + a_3a_4 \leqslant -w_2 + a_0a_1a_3a_4 + a_3a_4$$

which implies

$$q \leqslant -1 + \frac{a_0a_1a_3a_4 + a_3a_4}{w_2}. \tag{3.1.15}$$

Now, we will show that $2w_2 > a_0a_1a_3a_4 + a_3a_4$. Indeed, we have

$$\begin{aligned} 2w_2 > a_0a_1a_3a_4 + a_3a_4 &\Longleftrightarrow 2(1 - a_3 + a_3a_4 - a_0a_3a_4 + a_0a_1a_3a_4) > a_0a_1a_3a_4 + a_3a_4 \\ &\Longleftrightarrow a_0a_3a_4(a_1 - 2) + a_3(a_4 - 2) + 2 > 0. \end{aligned}$$

From Inequality (3.1.15) we have

$$q \leqslant -1 + \frac{a_0a_1a_3a_4 + a_3a_4}{w_2} < -1 + 2 = 1.$$

Therefore, we have $q \leqslant 0$.

Next, we will show that $q \geqslant -1$. First, we will prove that $d < 2a_1w_1$. We remember that $d = 1 + a_0a_1a_2a_3a_4$ and using (3.1.3) we write $w_1 = 1 - a_2 + a_2a_3 - a_2a_3a_4 + a_0a_2a_3a_4$. Then

$$\begin{aligned} d < 2a_1w_1 &\Longleftrightarrow 1 + a_0a_1a_2a_3a_4 < 2a_1(1 - a_2 + a_2a_3 - a_2a_3a_4 + a_0a_2a_3a_4) \\ &\Longleftrightarrow 1 < 2a_1 - 2a_1a_2 + 2a_1a_2a_3 - 2a_1a_2a_3a_4 + a_0a_1a_2a_3a_4 \\ &\Longleftrightarrow 1 < 2a_1 + 2a_1a_2(a_3 - 1) + a_1a_2a_3a_4(a_0 - 2). \end{aligned}$$

Since $a_3 \geqslant 2$ and $a_0 \geqslant 2$, we conclude that $d < 2a_1w_1$. Now, from (3.1.11) and (3.1.13), we have

$$sa_2w_2 \leqslant -rw_2 \leqslant w_4 + td + sd.$$

Replacing $a_2w_2 = d - w_1$ in the above inequality, we obtain $-sw_1 \leqslant w_4 + td$. Moreover, as $d < 2a_1w_1$ and $w_4 = \min_i w_i \leqslant w_1$, we get

$$-sw_1 \leqslant w_1 + td < w_1 + 2ta_1w_1 \Rightarrow -s < 1 + 2ta_1 \Rightarrow -s \leqslant 2ta_1.$$

By (3.1.11), $a_0 \geqslant t$, then we have

$$-s \leqslant 2ta_1 \leqslant 2a_0a_1. \tag{3.1.16}$$

Also, from (3.1.11) we have $1 + (t + s + r)a_4 + qw_3 \geqslant 0$, $-sa_2 \geqslant r$ and $t \leqslant a_0$. Then

$$-qw_3 \leqslant 1 + a_4(t + s + r) \leqslant 1 + a_4(t - s(a_2 - 1)) \leqslant 1 + a_4(a_0 - s(a_2 - 1)).$$

Replacing (3.1.16) in the above inequality, we obtain

$$-qw_3 \leqslant 1 + a_4(a_0 + 2a_0a_1(a_2 - 1)) = 1 + a_0a_4 - 2a_0a_1a_4 + 2a_0a_1a_2a_4.$$

Adding $1 + a_0a_4 - 2a_4 = 1 + a_4(a_0 - 2) > 0$ to the right of the last inequality, we have

$$-qw_3 < 2(1 - a_4 + a_4a_0 - a_4a_0a_1 + a_4a_0a_1a_2) = 2w_3 \Rightarrow q > -2 \Rightarrow q \geqslant -1.$$

So $q \in \{-1, 0\}$.

Next, we will determine the number of solutions for the two values of q.

a) If $q = 0$, we have the unique solution $(a_0, 0, 0, 0, 1)$. Indeed from (3.1.11) we have

$$t \leqslant a_0, \quad s \leqslant -ta_1, \quad r \leqslant -sa_2, \quad ra_3 \leqslant t+s+r \quad \text{and} \quad -1 \leqslant (t+s+r)a_4. \tag{3.1.17}$$

Since $t + s + r - ra_3 \geqslant 0$, we obtain $(t + s + r)(1 - a_3) \geqslant -sa_3 - ta_3$ which implies

$$t + s + r \leqslant \frac{a_3(s + t)}{a_3 - 1}. \tag{3.1.18}$$

On the other hand, from the last inequality in (3.1.17), we have $t + s + r \geqslant -\dfrac{1}{a_4} > -1$ which means $t + s + r \geqslant 0$. So from Inequality (3.1.18) we have $s + t \geqslant 0$.
Now from $\gcd(w_4, w_0) = 1$ and $w_4 = \min_i w_i$, we have $w_4 < w_0$. Then $-1 < -\dfrac{w_4}{w_0} \leqslant t \leqslant a_0$. So we obtain $0 \leqslant t \leqslant a_0$. In addition, from (3.1.17) we have $s \leqslant -ta_1 \leqslant 0$. As $-t \leqslant s$, we have

$$s \leqslant -ta_1 \leqslant sa_1 \Rightarrow 0 \leqslant s(a_1 - 1) \Rightarrow s \geqslant 0.$$

Since $s \leqslant 0$, we obtain $s = 0$ so $t = 0$. Finally, from (3.1.17) we have $r \leqslant -sa_2 = 0$ and since $t = s = 0$, we have $r = t + s + r \geqslant 0$. Thus, $r = 0$. Then, we have a solution for (3.1.5): $(a_0, 0, 0, 0, 1)$.

b) If $q = -1$, we have to analyze for cases: In (3.1.11) we have

$$t \leqslant a_0, \quad s \leqslant -ta_1, \quad r \leqslant -sa_2, \quad ra_3 - w_4 \leqslant t+s+r \quad \text{and} \quad w_3 - 1 \leqslant (t+s+r)a_4. \tag{3.1.19}$$

<u>Claim 2</u>: $t \in \{a_0 - 1, a_0\}$. Indeed, from (3.1.3) we write $w_3 = 1 - a_4 + a_4a_0 - a_4a_0a_1 + a_4a_0a_1a_2$. Replacing this in the last inequality in (3.1.19) we obtain

$$t+s+r \geqslant \frac{w_3 - 1}{a_4} = \frac{-a_4 + a_4a_0 - a_4a_0a_1 + a_4a_0a_1a_2}{a_4} = -1 + a_0 - a_0a_1 + a_0a_1a_2. \tag{3.1.20}$$

From (3.1.19), using $a_0 - t \geqslant 0$ and $-sa_2 \geqslant r$ in (3.1.20), we obtain

$$-sa_2 + s \geqslant r + s \geqslant -1 + a_0 - t - a_0a_1 + a_0a_1a_2 \geqslant -1 - a_0a_1 + a_0a_1a_2. \tag{3.1.21}$$

Since $1 - a_2 < 0$, then

$$s \leqslant \frac{-1 - a_0a_1(1 - a_2)}{1 - a_2} = \frac{1}{a_2 - 1} - a_0a_1 \leqslant 1 - a_0a_1.$$

Actually $s < 1 - a_0a_1$. Indeed if $s = 1 - a_0a_1$ (which would force $a_2 = 2$) from (3.1.21) we can write

$$-s \geqslant r + s \geqslant -1 + a_0 - t + a_0a_1 \geqslant -1 + a_0a_1 = -s.$$

In this inequality, we have that $t = a_0$. Thus, in the second inequality of (3.1.19) we obtain $s \leqslant -a_0a_1$, which contradicts the assumption $s = 1 - a_0a_1$. So we have $s < 1 - a_0a_1$ so we say

$$s \leqslant -a_0a_1. \tag{3.1.22}$$

Replacing (3.1.22) in (3.1.20), we obtain

$$r \geqslant -1 + (a_0 - t) + (-s - a_0a_1) + a_0a_1a_2 \geqslant -1 + a_0a_1a_2. \tag{3.1.23}$$

On the other hand, from (3.1.13) and (3.1.22) we obtain

$$w_4 + td \geqslant -sw_1 \geqslant a_0a_1w_1.$$

Since $a_1w_1 = d - w_0$ and $w_4 = d - a_0w_0$ the previous inequality can be rewritten as

$$a_0(d - w_0) \leqslant d - a_0w_0 + td \Rightarrow a_0 - 1 \leqslant t.$$

Also, from (3.1.19) we have $t \leqslant a_0$. Thus $t \in \{a_0 - 1, a_0\}$.

Next, we will determine the solutions of the Diophantine equation (3.1.5) for each value of t.

i) If $t = a_0 - 1$, from (3.1.13) we obtain

$$w_4 + (a_0 - 1)d + sd + rw_2 \geqslant 0 \Rightarrow w_4 - d + a_0d + sd \geqslant -rw_2.$$

From (3.1.19) we have $r \leqslant -sa_2$. In addition, the weights verify $d = w_4 + a_0w_0 = w_1 + a_2w_2$. Thus, the inequality above can be written

$$-a_0w_0 + a_0d + sd \geqslant -rw_2 \geqslant sa_2w_2 = sd - sw_1.$$

As $a_1w_1 = d - w_0$, then $-a_0a_1 \leqslant s$. Also, by (3.1.22) we know that $s \leqslant -a_0a_1$. Hence $s = -a_0a_1$. On the other hand, replacing $t = a_0 - 1$ and $s = -a_0a_1$ in (3.1.20), we obtain $r \geqslant a_0a_1a_2$. From (3.1.19) we have

$$ra_3 - w_4 \leqslant t + s + r \Rightarrow r \leqslant \frac{w_4 + t + s}{a_3 - 1} = \frac{w_4 + (a_0 - 1) + (-a_0a_1)}{a_3 - 1}.$$

As $w_4 = 1 - a_0 + a_0a_1 - a_0a_1a_2 + a_0a_1a_2a_3$, we obtain $r \leqslant a_0a_1a_2$. Thus, $r = a_0a_1a_2$. In this case, from (3.1.10) it follows that a solution for (3.1.5) is $(1, a_1, 0, 0, 0)$.

ii) If $t = a_0$. Following a similar process that in i) and using (3.1.19) and (3.1.22) we have

$$r \leqslant \frac{t + s + w_4}{a_3 - 1} = \frac{a_0 + s + w_4}{a_3 - 1} \leqslant \frac{a_0 - a_0a_1 + w_4}{a_3 - 1}. \tag{3.1.24}$$

Moreover, since $w_4 = 1 - a_0 + a_0a_1 - a_0a_1a_2 + a_0a_1a_2a_3$, we obtain

$$r \leqslant \frac{a_0 - a_0a_1 + w_4}{a_3 - 1} \leqslant \frac{1 - a_0a_1a_2 + a_0a_1a_2a_3}{a_3 - 1} = \frac{1}{a_3 - 1} + a_0a_1a_2.$$

When $a_3 = 2$, it is possible that $r = 1 + a_0a_1a_2$. Let us see that this situation cannot happen. Indeed, if we assume that $r = 1 + a_0a_1a_2$, then $a_3 = 2$. It implies that $w_4 = 1 - a_0 + a_0a_1 + a_0a_1a_2$. Replacing in (3.1.24), we obtain

$$r \leqslant 1 + a_0a_1 + s + a_0a_1a_2 \leqslant 1 + a_0a_1a_2 = r.$$

Thus, we get $s = -a_0a_1$. Putting this in the third inequality of (3.1.19), we have $r \leqslant a_0a_1a_2$, which results in a contradiction. Therefore, we have $r \leqslant a_0a_1a_2$. Also, from (3.1.23), it verifies $r \geqslant -1 + a_0a_1a_2$. Hence, we have $r \in \{-1 + a_0a_1a_2, a_0a_1a_2\}$. Next, we detail each case.

- If $r = -1 + a_0a_1a_2$, we have in the last inequality of (3.1.19):

$$a_0 + s - 1 + a_0a_1a_2 = t + s + r \geqslant \frac{w_3 - 1}{a_4} = -1 + a_0 - a_0a_1 + a_0a_1a_2,$$

  which implies that $s \geqslant -a_0a_1$. Using (3.1.22), we have $s = -a_0a_1$. Thus, in this case from (3.1.10) it follows that the solution of (3.1.5) is $(0, 0, 1, a_3, 0)$.
- If $r = a_0a_1a_2$, in a similar way as we have worked above, in the last inequality of (3.1.19) we have

$$a_0 + s + a_0a_1a_2 = t + s + r \geqslant \frac{w_3 - 1}{a_4} = -1 + a_0 - a_0a_1 + a_0a_1a_2,$$

  which implies that $s \geqslant -1 - a_0a_1$. From (3.1.22), $s \leqslant -a_0a_1$, we obtain $s = -1 - a_0a_1$ or $s = -a_0a_1$. If $s = -1 - a_0a_1$, the solution for (3.1.5) is $(0, 1, a_2, 0, 0)$. On the other hand, if $s = -a_0a_1$, from (3.1.10) it follows that $(0, 0, 0, 1, a_4)$ is the solution for the equation (3.1.5).

Therefore, the equation (3.1.5) has exactly five solutions:

$$(a_0, 0, 0, 0, 1), (1, a_1, 0, 0, 0), (0, 1, a_2, 0, 0), (0, 0, 1, a_3, 0) \text{ and } (0, 0, 0, 1, a_4).$$

□

Next, we compute the generators of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i))$. For this, we need to find all monomials $z_0^{a_0}z_1^{a_1}z_2^{a_2}z_3^{a_3}z_4^{a_4}$ of degree w_i. It is equivalent to solving the Diophantine equation

$$w_0x_0 + w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 = w_i \tag{3.1.25}$$

with variables $x_i \in \mathbb{Z}_0^+$ and where at least one of them is nonzero.

Lemma 3.1.3 *The equation (3.1.25) has a unique solution for each w_i and the set of generators of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i))$ is given by $\{z_i\}$ for $i = 0 \ldots, 4$.*

Proof Since f is a cycle polynomial, we can work without loss of generality with $w_i = w_1$. Thus, the equation (3.1.25) results in

$$w_0x_0 + w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 = w_1. \tag{3.1.26}$$

From (3.1.2), we have $a_2w_2 + w_1 = d$. Then, if we add a_2w_2 to both sides of (3.1.26), we arrive to a new Diophantine equation

$$w_0x_0 + w_1x_1 + w_2\tilde{x}_2 + w_3x_3 + w_4x_4 = d, \tag{3.1.27}$$

where the new variable $\tilde{x}_2 = x_2 + a_2 \geqslant a_2$. Now, we remember that Equation (3.1.27) has five solutions, which were obtained in the lemma above:

$$(a_0, 0, 0, 0, 1), (1, a_1, 0, 0, 0), (0, 1, a_2, 0, 0), (0, 0, 1, a_3, 0) \text{ and } (0, 0, 0, 1, a_4).$$

Since $\tilde{x}_2 \geqslant a_2$, we have a unique option that solve (3.1.27): $(0, 1, a_2, 0, 0)$. As a consequence, the equation (3.1.26) has a unique solution. Thus returning to Equation (3.1.26), after subtracting the vector $(0, 0, a_2, 0, 0)$ we obtain the solution $(0, 1, 0, 0, 0)$ that gives as generator the monomial z_1. Similarly we obtain the other solutions which are exactly the solutions given by Equation (3.1.27). It follows that the generator for $H^0(\mathbb{P}(\mathbf{w}), w_i)$ is given by the monomial z_i for $i = 0 \ldots 4$. □

Following [6], we collect the information given by the previous lemmas

Proposition 3.1.1 *Let us choose the definition of the weighted projective space* $\mathbb{P}(\mathbf{w})$ *as a scheme* $\mathrm{Proj}(S(\mathbf{w}))$*, where*

$$S(\mathbf{w}) = \bigoplus_d S^d(\mathbf{w}) = \mathbb{C}\left[z_0, z_1, z_2, z_3, z_4\right].$$

The ring of polynomials $\mathbb{C}\left[z_0, z_1, z_2, z_3, z_4\right]$ *is graded with grading defined by the weights* $\mathbf{w} = (w_1, w_1, w_2, w_3, w_4)$*. From Lemma 3.1.3, the group* $G(\mathbf{w})$ *of automorphisms of the graded ring* $S(\mathbf{w})$ *can be defined on generators by*

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

where $\alpha_i \in \mathbb{C}^*$*. The group* $\mathcal{G}(\mathbf{w})$ *of complex automorphisms of* $\mathbb{P}(\mathbf{w})$ *is the projectivization of* $G(\mathbf{w})$ *which in this case is given by* $\mathcal{G}(\mathbf{w}) = (\mathbb{C}^*)^4$ *. Actually since the generating set of* $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(d))$ *is given by the monomials* $z_0^{a_0} z_4, z_0 z_1^{a_1}, z_1 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}$*. Thus, the moduli of the orbifold* Z_f *is included in the space*

$$Span\{z_0^{a_0} z_4, z_0 z_1^{a_1}, z_1 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}\}/G(\mathbf{w})$$

which determines a 0-dimensional quotient.

□

From the two previous lemmas, we obtain the following outcome in the context of Sasaki-Einstein structures for rational homology 7-spheres.

Proposition 3.1.2 *Let* f *be a cycle polynomial as in (3.1.1) of degree* d *with associated weight vector* $\boldsymbol{w} = (w_0, w_1, w_2, w_3, w_4)$ *such that* $\gcd(d, w_i) = 1$*. Then the complex dimension of the moduli of the orbifold* $Z_f = (f = 0)/\mathbb{C}^*(\boldsymbol{w})$*, is equal to* 0*. Moreover, the generators of the spaces* $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(d))$ *and* $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i))$ *for all* $i = 0, \ldots, 4$ *are given in Proposition 3.1.1. Additionally, if* f *belongs to one of the 236 rational homology spheres admitting Sasaki-Einstein metrics found in [5] and [14], then the dimension of the local moduli of Sasaki-Einstein metrics of* $\mathbb{Q}$*-homology 7-spheres at* L_f *is zero dimensional. Thus rational homology 7-spheres given as links coming from polynomials* f *as above do not admit inequivalent families of Sasaki-Einstein structures.*

□

Example 3.1.1 Consider the following cycle polynomial $f = z_4 z_0^2 + z_0 z_1^8 + z_1 z_2^4 + z_2 z_3^{30} + z_3 z_4^3$ which can be found in the Johnson and Kollár list of anticanonically embedded Fano Kähler-Einsten 3-folds. It follows that the corresponding weight vector is $\mathbf{w} = (1945, 477, 1321, 148, 1871)$ and the degree is $d = 5761$ so $\gcd(d, w_i) = 1$. As shown in [6] the corresponding link L_f is a Sasaki-Einstein rational homology 7-sphere. By Proposition 3.1.1, the generating set of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by the set of monomials $\{z_4 z_0^2, z_0 z_1^8, z_1 z_2^4, z_2 z_3^{30}, z_3 z_4^3\}$. Furthermore, the moduli of the orbifold Z_f is included in

$$Span\{z_4 z_0^2, z_0 z_1^8, z_1 z_2^4, z_2 z_3^{30}, z_3 z_4^3\}/G(\mathbf{w})$$

where $G(\mathbf{w})$ is defined on generators by

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4. \end{pmatrix}$$

In this case L_f does not admit inequivalent Sasaki-Einstein metrics.

3.2 Rational homology 7-spheres: Thom-Sebastiani sums of invertible polynomials

In this subsection we consider the following types of invertible polynomials:

$$\begin{aligned} \text{Type I (Fermat-Cycle): } f &= z_0^{a_0} + z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4} \\ \text{Type II (Chain-Cycle): } f &= z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4} \\ \text{Type III (Cycle-Cycle): } f &= z_1 z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}. \end{aligned}$$

Let us assumed that the weight vectors associated to these families of polynomials have the form:

$$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4) = (m_3 v_0, m_3 v_1, m_2 v_2, m_2 v_3, m_2 v_4) \qquad (3.2.1)$$

where $\gcd(m_2, m_3) = 1$ and degree $d = m_2 m_3$. We will impose the conditions $\gcd(v_0, v_1) = 1$ and $\gcd(v_i, v_j) = 1$ for $i \neq j$ with $i, j \in \{2, 3, 4\}$, which in particular implies that the weighted hypersurface $Z_f \subset \mathbb{P}(\mathbf{w})$ is well-formed.

Before we find the generators of the space of deformations of the orbifold Z_f, we will prove some technical lemmas on the different types of polynomials described above.

Lemma 3.2.1 *Let f be an invertible polynomial of type I, II or III with associated weight vector $\mathbf{w}$ described in (3.2.1) and with degree $d = m_2 m_3$. Then we have*

a) If f is an invertible polynomial of type I, then $v_0 = v_1 = 1$ in $\mathbf{w}$ and hence $w_0 = w_1$.
b) If f is an invertible polynomial of type II, then $v_0 = 1$ and if the pair $(\mathbf{w}, \mathbf{d})$ does not admit a polynomial of type I, then $v_1 \neq 1$.
c) If f is an invertible polynomial of type III, such that $(\mathbf{w}, \mathbf{d})$ does not admit a polynomial of type II, then $v_0 \neq 1$ and $v_1 \neq 1$.

Proof a) For a polynomial f of type I:

$$f = z_0^{a_0} + z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$

with associated weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ and degree $d = m_2 m_3$, we have

$$\begin{cases} a_0 w_0 = d \Rightarrow a_0 m_3 v_0 = m_2 m_3 \\ a_1 w_1 = d \Rightarrow a_1 m_3 v_1 = m_2 m_3. \end{cases}$$

The equalities above imply $v_0 \mid m_2$ and $v_1 \mid m_2$. Since $w_i = m_2 v_i$ for $i = 2, 3, 4$, we have $v_0 \mid \gcd(w_0, w_2, w_3, w_4)$. Finally, since $\mathbf{w}$ is well-formed, we have $v_0 = 1$. A similar argument shows that $v_1 = 1$.

b) For a polynomial f of type II

$$f = z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$

with associated weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ and degree $d = m_2 m_3$, we have

$$a_0 w_0 = d \Rightarrow a_0 m_3 v_0 = m_2 m_3.$$

Thus $v_0 \mid m_2$. As $w_i = m_2 v_i$ for $i = 2, 3, 4$, we also obtain $v_0 \mid \gcd(w_0, w_2, w_3, w_4)$. Since $\mathbf{w}$ is well-formed, we conclude that $v_0 = 1$. Now, if we suppose that $v_1 = 1$, then we can choose the polynomial

$$\tilde{f} = z_0^{m_2} + z_1^{m_2} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$

which is of type I for the weight vector $\mathbf{w}$, but this contradicts the hypothesis. Thus, we have $v_1 \neq 1$.

c) We consider an invertible polynomial of type III:

$$f = z_1 z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4},$$

which has associated the weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ and degree $d = m_2 m_3$. If we suppose that $v_0 = 1$, then $\mathbf{w}$ admits a polynomial $\tilde{f}$ of type II:

$$\tilde{f} = z_0^{m_2} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$

which is a contradiction. Thus, we have $v_0 \neq 1$. A similar process as the one given above leads to $v_1 \neq 1$.

□

In the next lemma, we show that the existence of certain type of polynomial for a given weight vector $\mathbf{w}$ implies the presence of a different type of invertible polynomial associated to the same weight vector $\mathbf{w}$.

Lemma 3.2.2 *Let f be an invertible polynomial of type I, II or III, where its associated weight vector $\mathbf{w}$ is described as in (3.2.1) and its degree satisfies $d = m_2 m_3$. Then the following hold:*

a) If f is a polynomial of type II, then its associated weight vector $\mathbf{w}$ also admits a polynomial of type III.

b) If f is a polynomial of type I, then its associated weight vector $\mathbf{w}$ also admits polynomials of type II and III.

Proof (a) Let f be a polynomial of type II:

$$f = z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}.$$

By Lemma 3.2.1, we know $w_0 = m_3$. As $d = m_2 m_3$, we have $d - w_1 = m_3(m_2 - v_1) > m_3$. This implies that $m_3 \mid d - w_1$ and there exists an integer $\tilde{a}_0 > 1$ such that

$$\tilde{a}_0 w_0 = \tilde{a}_0 m_3 = d - w_1.$$

Thus, the weights vector $\mathbf{w}$ admits a polynomial $\tilde{f}$ of type III:

$$\tilde{f} = z_1 z_0^{\tilde{a}_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}.$$

(b) If $\mathbf{w}$ is the associated weight vector to the invertible polynomial f of type I:

$$f = z_0^{a_0} + z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4},$$

then by Lemma 3.2.1, we have $w_0 = w_1 = m_3$. As $d = a_1 m_3$ and $d > w_0 + w_1 = 2m_3$, these imply that $a_1 \geqslant 3$. Now, if we take $\tilde{a}_1 = a_1 - 1 \geqslant 2$, we can verify that $w_0 + \tilde{a}_1 w_1 = d$. Therefore, **w** also admits a polynomial $\tilde{f}$ of type II:

$$\tilde{f} = z_0^{a_0} + z_0 z_1^{\tilde{a}_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}.$$

Finally, from (a) of this lemma, we have that **w** also admits a polynomial of type III.

□

For any invertible polynomial f of type I, II or III, whose associated weight vector **w** is defined as in (3.2.1) and with degree $d = m_2 m_3$, we have that f contains no block of the form $z_i^2 + z_j^2$ (recall $\gcd(m_2, m_3) = 1$). This implies that $\dim \mathfrak{Aut}(Z_f) = 0$, see [7]. Thus, for the hypersurface $\{f = 0\} \subset \mathbb{C}^5$, the complex dimension of the orbifold $Z_f \subset \mathbb{P}(\mathbf{w})$ is given by the formula

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)). \tag{3.2.2}$$

We will begin computing the generators of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$. As before, we look for all different monomials $z_0^{y_0} z_1^{y_1} z_2^{x_2} z_3^{x_3} z_4^{x_4}$ of degree d. Since $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ is the associated weight vector to f, equivalently we can solve the following Diophantine equation:

$$w_0 y_0 + w_1 y_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = d \tag{3.2.3}$$

with variables $y_i, x_i \in \mathbb{Z}_0^+$ and where at least one of them is nonzero. The following lemma allows us to split Equation (3.2.3) in two new Diophantine equations.

Lemma 3.2.3 *In the Diophantine equation (3.2.3), if some y_i is a positive integer, then $x_j = 0$, for all $j = 2, 3, 4$. The converse is also true.*

Proof On the contrary, let us assume that there exist some positive integers y_i and x_j. In this case, we can suppose without loss of generality that $y_0 > 0$ and $x_2 > 0$. Since **w** is defined as in (3.2.1), we can write Equation (3.2.3) as:

$$m_3 v_0 y_0 + m_3 v_1 y_1 + m_2 v_2 x_2 + m_2 v_3 x_3 + m_2 v_4 x_4 = m_2 m_3.$$

Then, we have

$$m_2(v_2 x_2 + v_3 x_3 + v_4 x_4) = m_3(m_2 - v_0 y_0 - v_1 y_1).$$

Since $\gcd(m_2, m_3) = 1$, then $m_2 \mid (m_2 - v_0 y_0 - v_1 y_1)$. Moreover, since $m_2 - v_0 y_0 - v_1 y_1 < m_2$, we obtain $m_2 - v_0 y_0 - v_1 y_1 = 0$. Thus, we have

$$m_2(v_2 x_2 + v_3 x_3 + v_4 x_4) = m_3(m_2 - v_0 y_0 - v_1 y_1) = 0.$$

Finally, as $x_2 > 0$ and $x_3, x_4 \in \mathbb{Z}_0^+$, the equality above implies that $m_2 = 0$, which is not possible. For the converse, the process is similar to the process in the previous argument. □

From this lemma we conclude that the solutions of (3.2.3) can be obtained putting together the solutions of each one of the following Diophantine equations:

$$w_0 y_0 + w_1 y_1 = d \tag{3.2.4}$$

and

$$w_2 x_2 + w_3 x_3 + w_4 x_4 = d. \tag{3.2.5}$$

Next, we solve Equations (3.2.4) and (3.2.5). The next lemma will allow us to find all the solutions of Equation (3.2.4).

Lemma 3.2.4 *Let f be an invertible polynomial of type I, II or III, whose associated weight vector is $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ satisfying (3.2.1) and with degree $d = m_2 m_3$. Then we have:*

a) If f is of type I, then Equation (3.2.4) has $m_2 + 1$ solutions.
b) If f is of type II and its associated weight vector $\mathbf{w}$ does not admit polynomial of type I, then Equation (3.2.4) has $\dfrac{m_2}{v_1} - \dfrac{1}{v_1} + 1$ solutions.
c) If f is of type III and its associated weight vector $\mathbf{w}$ does not admit polynomial of type II, then Equation (3.2.4) has $\dfrac{m_2}{v_0 v_1} - \dfrac{1}{v_0} - \dfrac{1}{v_1} + 1$ solutions.

Moreover, all these quantities which represent the number of solutions of (3.2.4) are equivalent to

$$\left\lfloor \frac{m_2}{v_0 v_1} \right\rfloor + 1.$$

Proof Indeed

a) Since f is of type I, we have $v_0 = v_1 = 1$. These imply that $w_0 = w_1 = m_3$. Replacing in Equation (3.2.4), we obtain

$$m_3 y_0 + m_3 y_1 = d = m_2 m_3.$$

Simplifying, we obtain a new Diophantine equation $y_0 + y_1 = m_2$. Since all solutions are in $\mathbb{Z}_0^+$, these are given by the pairs

$$\{(0, m_2), (1, m_2 - 1), \ldots (m_2, 0)\}.$$

Thus Equation (3.2.4) has $m_2 + 1$ solutions.

b) By Lemma 3.2.1, we have $v_0 = 1$ and $v_1 \neq 1$. As a consequence, we obtain $w_0 = m_3$. Moreover, as $d = m_2 m_3$ and $w_1 = m_3 v_1$, the Equation (3.2.4) can be reduced to

$$y_0 + v_1 y_1 = m_2. \tag{3.2.6}$$

Since $(m_2, 0)$ is a particular solution of (3.2.6), by Lemma 3.1.1 we have that all its solutions are given by the pairs

$$(y_0, y_1) = (m_2 - k v_1, k) \quad \text{where } k \in \mathbb{Z}.$$

Since $y_0, y_1 \in \mathbb{Z}_0^+$, we obtain

$$0 \leqslant k \leqslant \frac{m_2}{v_1}. \tag{3.2.7}$$

Moreover, as $\mathbf{w}$ admits a polynomial of type II, we have $w_0 + a_1 w_1 = d$. By above, it implies that $1 + a_1 v_1 = m_2$. Thus, we conclude that $\dfrac{m_2 - 1}{v_1}$ is an integer. Considering this in (3.2.7), we write

$$0 \leqslant k \leqslant \left(\frac{m_2}{v_1} - \frac{1}{v_1} \right) + \frac{1}{v_1}.$$

Moreover, as $v_1 \neq 1$, we have the number of solutions of (3.2.4):

$$\left\lfloor \frac{m_2}{v_1} \right\rfloor + 1 = \frac{m_2}{v_1} - \frac{1}{v_1} + 1.$$

c) Let f be the polynomial of type III associated to $\mathbf{w}$:

$$f = z_1 z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}.$$

Replacing $w_0 = m_3 v_0$, $w_1 = m_3 v_1$ and $d = m_2 m_3$ in (3.2.4) and then simplifying, we obtain the new equation

$$v_0 y_0 + v_1 y_1 = m_2 \tag{3.2.8}$$

Since $\gcd(v_0, v_1) = 1$ and the pair $(a_0, 1)$ is a solution of (3.2.8), we have that all solutions of the Diophantine equation (3.2.8) are given by the pairs

$$(y_0, y_1) = (a_0 - kv_1, 1 + kv_0)$$

As $y_0, y_1 \in \mathbb{Z}_0^+$, then k is restricted to

$$-\frac{1}{v_0} \leqslant k \leqslant \frac{a_0}{v_1} \tag{3.2.9}$$

Since $\mathbf{w}$ does not admit polynomials of type II, we have $v_0 \neq 1$. Thus, we obtain $k \geqslant 0$. On the other hand, as the polynomial f is associated to $\mathbf{w}$, we have $a_0 w_0 + w_1 = d$. This implies that $a_0 v_0 + v_1 = m_2$. Then

$$\frac{a_0}{v_1} = \frac{m_2 - v_1}{v_0 v_1} = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} = \left(\frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1} \right) + \frac{1}{v_1}.$$

Using again that $\mathbf{w}$ does not admit polynomial of type II, we have $v_1 \neq 1$, which implies that $\frac{1}{v_1} < 1$. In addition, since $\alpha(\mathbf{w}) = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1}$ is a integer, from (3.2.9) we obtain

$$0 \leqslant k \leqslant \left\lfloor \frac{a_0}{v_1} \right\rfloor = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1}$$

Finally, as $\gcd(v_0, v_1) = 1$, then $0 < \frac{1}{v_0} + \frac{1}{v_1}$. From $\gcd(v_0, v_1) = 1$, it follows that the number of solutions of (3.2.4) is

$$\left\lfloor \frac{a_0}{v_1} \right\rfloor + 1 = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1} + 1 = \left\lfloor \frac{m_2}{v_0 v_1} \right\rfloor + 1.$$

□

Remark 3.2.1 We have the following remarks.

a) Notice that if f is a polynomial of type II such that its associated weight vector $\mathbf{w}$ admits a polynomial of type I, then we can work as in a) of the lemma above. Therefore, the equation (3.2.4) has $m_2 + 1$ solutions.
b) On the other hand, if f is a polynomial of type III whose associated weight vector $\mathbf{w}$ admits a polynomial of type II but not type I, then we can use the case b) of the previous lemma.
c) Finally, if f is a polynomial of type III whose associated weight vector admits a polynomial of type I and II, then the number of solutions of (3.2.4) is obtained as in a) of the previous lemma.

It remains to compute the number of solutions of Equation (3.2.5). Due to the fact that the polynomials of type I, II or III have the same cycle block:

$$z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}, \tag{3.2.10}$$

the number of solutions of the Diophantine Equation (3.2.5) is independent of any type. On the other hand, as the polynomial f has degree $d = m_2 m_3$ and $\mathbf{w}$ is written as in (3.2.1), then we obtain from (3.2.10) the following equations for the weights w_i's:

$$a_2 w_2 + w_4 = d \Rightarrow v_4 + a_2 v_2 = m_3 \tag{3.2.11}$$
$$a_3 w_3 + w_2 = d \Rightarrow v_2 + a_3 v_3 = m_3 \tag{3.2.12}$$
$$a_4 w_4 + w_3 = d \Rightarrow v_3 + a_4 v_4 = m_3. \tag{3.2.13}$$

In addition, as the link L_f is a rational homology sphere, it verifies Equality (2.1.4): $m_3 = a_2 a_3 a_4 + 1$. Thus, we can write v_2, v_3 and v_4 as

$$v_2 = a_4 a_3 - a_3 + 1, \quad v_3 = a_2 a_4 - a_4 + 1, \quad \text{and} \;\; v_4 = a_3 a_2 - a_2 + 1. \tag{3.2.14}$$

Next, we will solve Equation (3.2.5):

Lemma 3.2.5 *The Diophantine equation*

$$w_2 x_2 + w_3 x_3 + w_4 x_4 = d,$$

where w_i's are defined as in (3.2.1), has only three solutions.

Proof Since $w_j = m_2 v_j$, for $j = 2, 3, 4$ and $d = m_2 m_3$, we have an equivalent equation to (3.2.5):

$$v_2 x_2 + v_3 x_3 + v_4 x_4 = m_3. \tag{3.2.15}$$

As $\gcd(v_2, v_3) = 1$, we can define a new variable $\tilde{x} = v_2 x_2 + v_3 x_3$. Thus, we can write Equation (3.2.15) as

$$\tilde{x} + v_4 x_4 = m_3. \tag{3.2.16}$$

By (3.2.13), a particular solution of the equation (3.2.16) is given by the pair (v_3, a_4). Then the general solution of (3.2.16) is

$$(\tilde{x}, x_4) = (v_3 + t v_4, a_4 - t), \quad \text{where } t \in \mathbb{Z}. \tag{3.2.17}$$

Now, we consider the Diophantine equation

$$v_2 x_2 + v_3 x_3 = \tilde{x} = v_3 + t v_4 \tag{3.2.18}$$

From (3.2.11) and (3.2.12), we have

$$v_4 = m_3 - a_2 v_2 = (v_2 + a_3 v_3) - a_2 v_2 = a_3 v_3 + (1 - a_2) v_2.$$

Therefore, a particular solution of (3.2.18) is the pair $(t(1 - a_2), 1 + t a_3)$. Then the general solution of (3.2.18) is given by

$$(x_2, x_3) = (t(1 - a_2) + s v_3, 1 + t a_3 - s v_2), \quad \text{where } t, s \in \mathbb{Z}. \tag{3.2.19}$$

From (3.2.17) and (3.2.19), we obtain the general solution of the Diophantine equation (3.2.15):

$$(x_2, x_3, x_4) = (t(1 - a_2) + s v_3, 1 + t a_3 - s v_2, a_4 - t), \quad \text{where } t, s \in \mathbb{Z}. \tag{3.2.20}$$

Claim: $\mathbf{s \in \{0, 1\}}$

Since we require $x_2, x_3 \in \mathbb{Z}_0^+$, we have that $x_2 = t(1 - a_2) + s v_3 \geqslant 0$ and $x_3 = 1 + t a_3 - s v_2 \geqslant 0$. This inequalities imply that

$$\frac{s v_2 - 1}{a_3} \leqslant t \leqslant \frac{s v_3}{a_2 - 1}.$$

From this, we obtain

$$\frac{sv_2 - 1}{a_3} \leqslant \frac{sv_3}{a_2 - 1} \Rightarrow (a_2 - 1)(sv_2 - 1) \leqslant sv_3a_3$$
$$\Rightarrow -(a_2 - 1) \leqslant s(v_3a_3 - v_2(a_2 - 1)).$$

Using (3.2.11) and (3.2.12) we have $v_4 = v_3a_3 - v_2(a_2 - 1)$. Replacing above, we can write

$$-(a_2 - 1) \leqslant sv_4 \Rightarrow -\frac{a_2 - 1}{v_4} \leqslant s.$$

Now, using the expression for v_4 given in (3.2.14) and the fact of that $a_2a_3 \geqslant 2a_2 > 2a_2 - 1$, we have

$$v_4 = a_2a_3 - a_2 + 1 > a_2 \Rightarrow \frac{a_2 - 1}{v_4} < 1.$$

Hence

$$-1 < -\frac{a_2 - 1}{v_4} \leqslant s. \tag{3.2.21}$$

On the other hand, as $x_3 \geqslant 0$ and $x_4 \geqslant 0$, then $1 + ta_3 \geqslant sv_2$ and $a_4 \geqslant t$, respectively. Putting together these two inequalities, we obtain

$$1 + a_4a_3 \geqslant 1 + ta_3 \geqslant sv_2.$$

Also, from the expression given for v_2 given in (3.2.14) and the inequality $a_4a_3 \geqslant 2a_3 > 2a_3 - 1$, we have

$$2v_2 = 2a_4a_3 - 2a_3 + 2 > 1 + a_4a_3.$$

Since $sv_2 \leqslant 1 + a_4a_3$, we conclude that

$$s \leqslant \frac{1 + a_4a_3}{v_2} < 2. \tag{3.2.22}$$

From (3.2.21) and (3.2.22), we conclude that $s \in \{0, 1\}$.

Next, we will exhibit the solutions that are obtained for each s.

For $\mathbf{s = 0}$: In this case, the general solution of (3.2.15) is given by

$$(x_2, x_3, x_4) = (t(1 - a_2), 1 + ta_3, a_4 - t).$$

Since $x_i \geqslant 0$, we have $t(1 - a_2) \geqslant 0$, $1 + ta_3 \geqslant 0$ and $a_4 - t \geqslant 0$. As $a_2 - 1 > 0$, then $t \leqslant 0$. On the other hand, the inequality $1 + ta_3 \geqslant 0$ implies that $t \geqslant -\frac{1}{a_3} > -1$ so $t = 0$. Therefore, $(0, 1, a_4)$ is the unique solution for Equation (3.2.15).

For $\mathbf{s = 1}$: The general solution of (3.2.15) is given by

$$(x_2, x_3, x_4) = (t(1 - a_2) + v_3, 1 + ta_3 - v_2, a_4 - t).$$

Since $x_4 = a_4 - t \geqslant 0$, we have $a_4 \geqslant t$. On the other hand, as $x_3 = 1 + ta_3 - v_2 \geqslant 0$ and $v_2 = a_4a_3 - a_3 + 1$ in (3.2.14), we obtain

$$t \geqslant \frac{v_2 - 1}{a_3} = a_4 - 1$$

Thus, $t \in \{a_4 - 1, a_4\}$. So the two solutions for $s = 1$ are

$$\begin{cases} (1, a_3, 0), & \text{if } t = a_4, \\ (a_2, 0, 1), & \text{if } t = a_4 - 1. \end{cases}$$

□

Thus, for f be an invertible polynomial with associated weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ satisfying (3.2.1) and with degree $d = m_2 m_3$ we have

- If f is a polynomial of type I, then the set of generators of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by the following monomials of degree d:
$$\{z_1^{m_2}, z_0 z_1^{m_2-1}, \dots, z_0^{m_2-1} z_1, z_0^{m_2}, z_4 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}\}.$$
- If f is a polynomial of type II and its associated weight vector $\mathbf{w}$ does not admit polynomial of type I, then the set of generators of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by the following monomials of degree d :
$$\left\{ z_0^{m_2 - kv_1} z_1^k, z_4 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}, \text{ where } 0 \leqslant k \leqslant \left\lfloor \frac{m_2}{v_1} \right\rfloor \right\}.$$
- If f is a polynomial of type III and its associated weight vector $\mathbf{w}$ does not admit polynomial of type II, then the set of generators of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by the following monomials of degree d :
$$\left\{ z_0^{a_0 - kv_1} z_1^{1+kv_0}, z_4 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}, \text{ where } 0 \leqslant k \leqslant \left\lfloor \frac{m_2}{v_0 v_1} \right\rfloor \right\}.$$

Now we will compute $\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i))$ for each w_i, that is, we will find all the solutions of the following Diophantine equation

$$w_0 y_0 + w_1 y_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = w_i, \tag{3.2.23}$$

with $x_j, y_k \in \mathbb{Z}_0^+$, where at least one of them is nonzero. In the next lemma, we will do this for either $w_i = w_0$ or $w_i = w_1$.

Lemma 3.2.6 *Let f be an invertible polynomial of type I, II or III, with associated weight vector $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ satisfying (3.2.1) and with degree $d = m_2 m_3$. We consider the following Diophantine equations*

$$w_0 y_0 + w_1 y_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = w_0 \tag{3.2.24}$$
$$w_0 y_0 + w_1 y_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = w_1. \tag{3.2.25}$$

Then, we have:

a) If f is of type I, then equations (3.2.24) and (3.2.25) both have two solutions. Moreover, the set of generators of $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(w_i))$ is given by $\{z_0, z_1\}$ for $i = 0, 1$.
b) If f is of type II and its associated weight vector $\mathbf{w}$ does not admit polynomial of type I, then equations (3.2.24) and (3.2.25) have one and two solutions, respectively. Moreover, the set of generators of $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(w_0))$ is given by $\{z_0\}$ and the set generators for $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(w_1))$ is $\{z_0^{v_1}, z_1\}$.
c) If f is of type III and its associated weight vector $\mathbf{w}$ does not admit polynomial of type II, then the equations (3.2.24) and (3.2.25) both have one solution. Moreover, the set of generators for $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(w_0))$ is $\{z_0\}$ and the set of generators for $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(w_1))$ is $\{z_1\}$

Proof Let us study each case:

(a) If f is of type I, we can write

$$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4) = (m_3, m_3, m_2 v_2, m_2 v_3, m_2 v_4).$$

Since $w_0 = m_3$ and $d = m_2 m_3$, adding $(m_2-1)w_0$ on both sides of (3.2.24), we obtain

$$w_0\hat{y}_0 + w_1 y_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = d, \tag{3.2.26}$$

where the new variable $\hat{y}_0 = y_0 + (m_2-1) \geqslant m_2 - 1$. From Lemmas 3.2.4 and 3.2.5, it follows that the solutions of Equation (3.2.26) belong to the set

$$\{(t, m_2 - t, 0, 0, 0), (0, 0, a_2, 0, 1), (0, 0, 1, a_3, 0), (0, 0, 0, 1, a_4), \text{ where } t \in \mathbb{Z}_0^+, t \leqslant m_2\}.$$

Since $\hat{y}_0 \geqslant m_2 - 1$, we will find only two possible solutions for (3.2.26): if $\hat{y}_0 = m_2$, we have $y_0 = 1$ and $y_1 = 0$. On the other hand, if $\hat{y}_0 = m_2 - 1$, we obtain $y_0 = 0$ and $y_1 = 1$. Thus, the solutions obtained are

$$(1, 0, 0, 0, 0) \text{ and } (0, 1, 0, 0, 0).$$

Since $w_1 = w_0$, we also obtain two solutions for Equation (3.2.25).

(b) First, we will solve Equation (3.2.24). Since $f = z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$ is a polynomial of type II and its associated weight vector $\mathbf{w}$ does not admit polynomials of type I, we can express $\mathbf{w}$ as

$$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4) = (m_3, m_3 v_1, m_2 v_2, m_2 v_3, m_2 v_4), \quad \text{where } v_1 \neq 1.$$

As $w_0 = m_3$ and $d = m_2 m_3$, then adding $(m_2-1)w_0$ on both sides of the Equation (3.2.24), we obtain

$$w_0\hat{y}_0 + w_1 y_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = d \tag{3.2.27}$$

with new variable $\hat{y}_0 = y_0 + (m_2-1) \geqslant m_2 - 1$. By Lemmas 3.2.4 and 3.2.5, the solutions of (3.2.27) belong to the set

$$\left\{(m_2 - tv_1, t, 0, 0, 0), (0, 0, a_2, 0, 1), (0, 0, 1, a_3, 0), (0, 0, 0, 1, a_4), \text{ where } t \in \mathbb{Z}_0^+, t \leqslant \left\lfloor \frac{m_2}{v_1} \right\rfloor\right\}$$

Notice that this is possible only when $\hat{y}_0 = m_2 - tv_1 \geqslant m_2 - 1$. Since $v_1 \neq 1$ then $t \leqslant \dfrac{1}{v_1} < 1$. Thus, we conclude that $t = 0$. In this case, we obtain $\hat{y}_0 = m_2$, which implies $y_0 = 1$ and $y_1 = 0$. Hence, the unique solution for (3.2.24) is $(1, 0, 0, 0, 0)$.

On the other hand, to solve Equation (3.2.25), consider the term $z_0 z_1^{a_1}$ of f, that leads to $w_0 + a_1 w_1 = d$. If we add $w_0 + (a_1 - 1)w_1$ to both sides of Equation (3.2.25), we obtain the new Diophantine equation

$$w_0\hat{y}_0 + w_1\hat{y}_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = d, \tag{3.2.28}$$

where $\hat{y}_0 = y_0 + 1 \geqslant 1$ and $\hat{y}_1 = y_1 + a_1 - 1 \geqslant a_1 - 1$. By the solutions given above, we can write $\hat{y}_0 = m_2 - tv_1$ and $\hat{y}_1 = t$. Thus, we obtain

$$\frac{m_2 - 1}{v_1} - 1 = a_1 - 1 \leqslant t \leqslant \frac{m_2 - 1}{v_1} = \left\lfloor \frac{m_2}{v_1} \right\rfloor.$$

We have

- If $t = \dfrac{m_2 - 1}{v_1} - 1$, then $\hat{y}_0 = 1 + v_1$ and $\hat{y}_1 = \dfrac{m_2 - 1}{v_1} - 1$. Thus, $y_0 = v_1$ and $y_1 = 0$.
- If $t = \dfrac{m_2 - 1}{v_1}$, then $\hat{y}_0 = 1$ and $\hat{y}_1 = \dfrac{m_2 - 1}{v_1}$. Thus, $y_0 = 0$ and $y_1 = 1$.

Thus, Equation (3.2.25) has two solutions.

(c) We consider f be an invertible polynomial of type III:

$$f = z_1 z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4},$$

whose associated weight vector $\mathbf{w}$ does not admit polynomial of type II. To solve (3.2.24), we add $a_1 w_1$ to both sides of this equation. Since $w_0 + a_1 w_1 = d$, we obtain

$$w_0 y_0 + w_1 \hat{y}_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = d \tag{3.2.29}$$

where $\hat{y}_1 = y_1 + a_1 \geqslant a_1$. By Lemmas 3.2.4 c) and 3.2.5, the solutions of (3.2.29) are in the set

$$\left\{(a_0 - tv_1, 1 + tv_0, 0, 0, 0), (0, 0, a_2, 0, 1), (0, 0, 1, a_3, 0), (0, 0, 0, 1, a_4), \text{ where } t \in \mathbb{Z}_0^+, t \leqslant \left\lfloor \frac{m_2}{v_0 v_1} \right\rfloor\right\}$$

Then $\hat{y}_1 = 1 + tv_0 \geqslant a_1$. Thus $t \geqslant \dfrac{a_1 - 1}{v_0}$. As $w_0 + a_1 w_1 = d$, we have that $v_0 + a_1 v_1 = m_2$. We obtain

$$t \geqslant \frac{a_1 - 1}{v_0} = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1} = \left\lfloor \frac{m_2}{v_0 v_1} \right\rfloor.$$

So $t = \left\lfloor \dfrac{m_2}{v_0 v_1} \right\rfloor$. Thus, Equation (3.2.29) has only one solution given by $(a_0 - tv_1, 1 + tv_0, 0, 0, 0)$. Returning to Equation (3.2.24) (after subtracting the vector $(0, a_1, 0, 0, 0)$) we obtain the vector $(a_0 - tv_1, 1 + tv_0 - a_1, 0, 0, 0)$ as solution. In this case, since $t = \dfrac{m_2}{v_0 v_1} - \dfrac{1}{v_0} - \dfrac{1}{v_1}$, it follows that the generator of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_0))$ is given by $\{z_0\}$. Solutions to Equation (3.2.25) can be found using a similar argument as the one given above. It follows that the generator of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_1))$ is given by $\{z_1\}$.

□

Remark 3.2.2 We have the following comments.

a) Notice that if f is a polynomial of type II such that its associated weight vector $\mathbf{w}$ admits a polynomial of type I, then part a) of the previous lemma above holds. Thus, Equations (3.2.24) and (3.2.25) have 2 solutions each one.
b) If f is a polynomial of type III whose associated weight vector $\mathbf{w}$ admits a polynomial of type II but not type I, then we can use part b) of the previous lemma above. Thus, Equations (3.2.24) and (3.2.25) have 1 and 2 solutions, respectively.
c) Finally, if f is a polynomial of type III whose associated weight vector admits polynomial of type I and II, then the number of solutions of the equations can be obtained as in a) of the previous lemma. That is, Equations (3.2.24) and (3.2.25) both have 2 solutions.

Next, we will solve Equation (3.2.23) for the remaining cases: $i = 2, 3$ or 4.

Lemma 3.2.7 *Let f be a polynomial of type I, II or III and $\mathbf{w}$ the associated weight vector defined as in (3.2.1). Then the equation (3.2.23):*

$$w_0 y_0 + w_1 y_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = w_i$$

has a unique solution for each $i = 2, 3, 4$. It follows that the set of generators of $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(w_i))$ is given by $\{z_i\}$ for $i = 2, 3, 4$.

Proof Since any polynomial f of type I, II or III has the same block of cycle type

$$z_4z_2^{a_2} + z_2z_3^{a_3} + z_3z_4^{a_4},$$

we can work with any of these types. On the other hand, by the cyclic form of this block, it is enough to assume $w_i = w_2$. Thus, Equation (3.2.23) can be written as

$$w_0y_0 + w_1y_1 + w_2x_2 + w_3x_3 + w_4x_4 = w_2 \tag{3.2.30}$$

Adding a_3w_3 to both sides of (3.2.30), we obtain

$$w_0y_0 + w_1y_1 + w_2x_2 + w_3\hat{x}_3 + w_4x_4 = d$$

where $\hat{x}_3 = x_3 + a_3 \geqslant a_3$. Since $\hat{x}_3 > 0$, the solutions of the above equation are in the set

$$\{(0, 0, a_2, 0, 1), (0, 0, 1, a_3, 0), (0, 0, 0, 1, a_4)\}.$$

Since $a_3 > 1$, we have only one solution $(0, 0, 1, a_3, 0)$. From $\hat{x}_3 = a_3$, we obtain $x_3 = 0$. Then the solution of Equation (3.2.30) is $(0, 0, 1, 0, 0)$. Similarly we obtain the solutions $(0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 1)$ for w_3 and w_4 respectively. The last statement of the lemma follows for similar arguments as the ones given in the previous lemmas. □

In the same vein as Proposition 3.1.1 we give the following result.

Proposition 3.2.1 *The group $G(\mathbf{w})$ of complex automorphisms of the graded ring $S(\mathbf{w})$ with* $\mathrm{Proj}(S(\mathbf{w}))$ *can be defined on generators for polynomials of type I, II and III thanks to Lemmas 3.2.6 and 3.2.7 and we can describe the moduli of the corresponding orbifold as before. Indeed, let $\alpha_i, \beta_1 \in \mathbb{C}^*$ and $\mathbb{A} \in GL(2, \mathbb{C})$, then we have*

- *If f is a polynomial of type I, then $G(\mathbf{w})$ is given on generators by*

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \mathbb{A}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

  and the moduli of the orbifold Z_f is included in the quotient

$$Span\left\{z_1^{m_2}, z_0z_1^{m_2-1}, \ldots, z_0^{m_2-1}z_1, z_0^{m_2}, z_4z_2^{a_2}, z_2z_3^{a_3}, z_3z_4^{a_4}\right\}/G(\mathbf{w}).$$

- *If f is a polynomial of type II and its associated weight vector $\mathbf{w}$ does not admit polynomial of type I, then $G(\mathbf{w})$ is given on generators by*

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 + \beta_1 z_0^{v_1} \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

  and the moduli of the orbifold Z_f is included in the quotient

$$Span\left\{z_0^{m_2-kv_1}z_1^k, z_4z_2^{a_2}, z_2z_3^{a_3}, z_3z_4^{a_4},\ \text{where } 0 \leqslant k \leqslant \left\lfloor \frac{m_2}{v_1} \right\rfloor\right\}/G(\mathbf{w}).$$

- *Finally, If f is a polynomial of type III and its associated weight vector* $\mathbf{w}$ *does not admit polynomial of type II, then* $G(\mathbf{w})$ *is given on generators by*

$$\varphi_{\mathrm{w}}\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

and the moduli of the orbifold Z_f *is included in the quotient*

$$Span\left\{z_0^{a_0-kv_1}z_1^{1+kv_0}, z_4z_2^{a_2}, z_2z_3^{a_3}, z_3z_4^{a_4},\ where\ 0 \leqslant k \leqslant \left\lfloor \frac{m_2}{v_0v_1} \right\rfloor\right\}/G(\mathbf{w}).$$

□

For the local moduli of Sasaki-Einstein structures for rational homology spheres we can say that this is non-trivial, more precisely we have:

Proposition 3.2.2 *Let f be an invertible polynomial of type I, II or III, whose associated weight vector is $\boldsymbol{w} = (w_0, w_1, w_2, w_3, w_4)$ which is defined as in (3.2.1) with degree $d = m_2m_3$. Then the complex dimension μ of the local moduli of the orbifold $Z_f = \{f = 0\}/\mathbb{C}^*(\boldsymbol{w})$ is given as follows*

a) If f is of type I, then $\mu = m_2 - 3$.
b) If f is of type II and its associated weight vector $\mathbf{w}$ *does not admit a polynomial of type I, then $\mu = \dfrac{m_2}{v_1} - \dfrac{1}{v_1} - 2$.*
c) If f is of type III and its associated weight vector $\mathbf{w}$ *does not admit a polynomial of type II, then $\mu = \dfrac{m_2}{v_0v_1} - \dfrac{1}{v_0} - \dfrac{1}{v_1} - 1$.*

All these quantities are equivalent to

$$\mu = \frac{m_2}{v_0v_1} - \frac{1}{v_0} - \frac{1}{v_1} - 1.$$

Moreover, the generators of the spaces $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(d))$ and $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i))$ for all $i = 0, \ldots, 4$ are given in the previous lemmas of this subsection. Additionally, if f belongs to one of the 236 rational homology spheres admitting Sasaki-Einstein metrics found in [5] and [14], then the real dimension of the local moduli of Sasaki-Einstein metrics of rational homology 7-spheres at L_f equals $2\left\lceil \dfrac{m_2}{v_0v_1} - \dfrac{1}{v_0} - \dfrac{1}{v_1} - 1\right\rceil$.

Proof Let us prove these statements case by case:

a) If f is of type I, from Lemmas 3.2.4 and 3.2.5 we know $\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) = m_2+4$. Also, from Lemmas 3.2.6 and 3.2.7, we have

$$\sum_{i=0}^{4} \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)) = 7.$$

Thus

$$\mu = \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)) = m_2 - 3.$$

b) If f is of type II with associated weight vector $\mathbf{w}$ not admitting a polynomial of type I, from Lemmas 3.2.4 and 3.2.5 we know $\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) = \frac{m_2}{v_1} - \frac{1}{v_1} + 4$. On the other hand, Lemmas 3.2.6 and 3.2.7, establish the equality

$$\sum_{i=0}^{4} \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)) = 6.$$

Thus

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)) = \frac{m_2}{v_1} - \frac{1}{v_1} - 2.$$

c) If f is of type III and its associated weight vector $\mathbf{w}$ does not admit a polynomial of type II, then we obtain from Lemmas 3.2.4 and 3.2.5 that

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), d) = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1} + 4.$$

Moreover, from Lemmas 3.2.6 and 3.2.7, we have

$$\sum_{i=0}^{4} \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)) = 5.$$

Thus,

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)) = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1} - 1.$$

As we notice that for any of the cases mentioned above, μ is equivalent to

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i \dim_{\mathbb{C}} H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i)) = \frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1} - 1.$$

The last statement then follows from Theorem 2.2.2 □

Example 3.2.1 Let us consider the following examples, where all the polynomials are taken from the list of Johnson and Kollár of anticanonically embedded Fano Kähler-Einstein 3-folds.

1. The polynomial $f = z_0^9 + z_1^9 + z_4 z_2^2 + z_2 z_3^2 + z_3 z_4^{19}$ of type I has associated weight vector $\mathbf{w} = (77, 77, 333, 180, 27)$ and degree $d = m_3 m_2 = 693$, where $m_3 = 77$ and $m_2 = 9$. The generating set of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by the set of thirteen monomials

$$\{z_0^9, z_0^8 z_1, \ldots, z_0 z_1^8, z_1^9, z_4 z_2^2, z_2 z_3^2, z_3 z_4^{19}\}.$$

Then the moduli of the orbifold Z_f is included in

$$Span\left\{z_0^9, z_0^8 z_1, \ldots, z_0 z_1^8, z_1^9, z_4 z_2^2, z_2 z_3^2, z_3 z_4^{19}\right\}/G(\mathbf{w}).$$

where $G(\mathbf{w})$ is expressed as above. Thus the complex dimension of this moduli is six and the corresponding link L_f has a local moduli of Sasaki-Einstein metrics of real dimension twelve.

2. The polynomial $f = z_0^{125} + z_0z_1^4 + z_4z_2^2 + z_2z_3^7 + z_3z_4^3$ of type II with degree $d = m_3m_2 = 5375$, where $m_3 = 43$ and $m_2 = 125$, and with associated weight vector is $\mathbf{w} = (43, 1333, 1875, 500, 1625)$. We notice that $\mathbf{w}$ does not admit polynomials of type I. Since $v_1 = 31$, then we have that the generating set of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by $\left(\left\lfloor \frac{125}{31} \right\rfloor + 1\right) + 3 = 8$ monomials. Indeed, the generating set is given by
$$\{z_0^{125}, z_0^{94}z_1, z_0^{63}z_1^2, z_0^{32}z_1^3, z_0z_1^4, z_4z_2^2, z_2z_3^7, z_3z_4^3\}$$
The moduli of the orbifold Z_f is obtained in a similar way as above. Thus the complex dimension of this moduli is two and the corresponding link L_f has a local moduli of Sasaki-Einstein metrics of real dimension four.
3. The polynomial $f = z_1z_0^5 + z_0z_1^{15} + z_4z_2^2 + z_2z_3^4 + z_3z_4^4$ of type III, with associated weight vector $\mathbf{w} = (231, 66, 481, 185, 259)$ and degree $d = m_3m_2 = 1221$, where $m_3 = 33$ and $m_2 = 37$. Here, the weight vector $\mathbf{w}$ does not admit polynomials type I or II. Moreover, as $v_0 = 7$ and $v_1 = 2$, then the generating set of $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is formed by $\left(\left\lfloor \frac{37}{(7)(2)} \right\rfloor + 1\right) + 3 = 6$ given by
$$\{z_0^5z_1, z_0^3z_1^8, z_0z_1^{15}, z_4z_2^2, z_2z_3^4, z_3z_4^4\}.$$
It follows that the the complex dimension of this moduli is one and the corresponding link L_f has a local moduli of Sasaki-Einstein metrics of real dimension two.

3.3 Rational homology sphere: the Berglund-Hübsch dual of chain-cycle polynomials

As mentioned before, in [16] it was proven that the Berglund-Hübsch transpose rule only produces twins for singularities of cycle type, type I and type III. Actually these types are preserved under the Berglund-Hübsch transpose:

- Type I polynomials are sent to type I polynomials.
- Type III polynomials are sent to type III polynomials, and moreover
- m_2, v_0 and v_1 are invariant under the Berglund-Hübsch rule.

So the the moduli of orbifolds Z_{f^T} determined by Berglund-Hübsch transpose dual f^T of f for cycle polynomials and polynomials of type I and type III have moduli described by Propositions 3.2.2 and 3.1.1. From Proposition 3.2.2 the real dimension $\mu_{\mathbb{R}}$ of the local moduli for links arising from polynomials of type I and type III are given by $\mu_{\mathbb{R}} = 2[\frac{m_2}{v_0v_1} - \frac{1}{v_0} - \frac{1}{v_1} - 1]$, and from Proposition 3.1.2 $\mu_{\mathbb{R}} = 0$ in case the singularity is given by a cycle type polynomial given in Subsection 3.1. Thus we only need to study polynomials of type II, that is polynomials of chain-cycle type, where the Berglund-Hübsch transpose rule does not preserve neither torsion nor Milnor number, and hence m_2, v_0 and v_1 vary. We will also assume that the index $I = |\mathbf{w}| - \mathbf{d}$ equals 1, as done in [16]. Since our weighted varieties produce Berglund-Hübsch duals embedded in non well-formed weighted projective spaces, in principal our procedure will give only upper bounds for the dimension of the moduli, however in most cases we found that these bounds are zero.

Let
$$z_0^{a_0} + z_0z_1^{a_1} + z_4z_2^{a_2} + z_2z_3^{a_3} + z_3z_4^{a_4}, \tag{3.3.1}$$

where its associated weight vector $\mathbf{w}$ satisfies condition (3.2.1). The exponential matrix of f is given by

$$A_f = \begin{bmatrix} a_0 & 0 & 0 & 0 & 0 \\ 1 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 1 \\ 0 & 0 & 1 & a_3 & 0 \\ 0 & 0 & 0 & 1 & a_4 \end{bmatrix} \tag{3.3.2}$$

Applying the Berglund-Hübsch transpose rule, one obtains the matrix

$$A_f^T = \begin{bmatrix} a_0 & 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 1 & 0 \\ 0 & 0 & 0 & a_3 & 1 \\ 0 & 0 & 1 & 0 & a_4 \end{bmatrix}$$

which has associated invertible polynomial

$$f^T = z_0^{a_0} z_1 + z_1^{a_1} + z_3 z_2^{a_2} + z_4 z_3^{a_3} + z_2 z_4^{a_4}, \tag{3.3.3}$$

In [16], the associated weight vector $\tilde{\mathbf{w}}$ to f^T was obtained:

$$\tilde{\mathbf{w}} = (m_3 v_1 (a_1 - 1), m_3 m_2 v_1, m_2(m_2 - 1)\tilde{v}_2, m_2(m_2 - 1)\tilde{v}_3, m_2(m_2 - 1)\tilde{v}_4),$$

where the degree is $\tilde{d} = m_3 m_2 (m_2 - 1)$ and

$$\tilde{v}_2 = a_3 a_4 - a_4 + 1, \quad \tilde{v}_3 = a_2 a_4 - a_2 + 1 \quad \text{and} \quad \tilde{v}_4 = a_2 a_3 - a_3 + 1. \tag{3.3.4}$$

Also, since $m_2 - 1 = a_1 v_1$, we can simplify $\tilde{\mathbf{w}}$ and obtain

$$\tilde{\mathbf{w}} = (m_3(a_1 - 1), m_3 m_2, m_2 a_1 \tilde{v}_2, m_2 a_1 \tilde{v}_3, m_2 a_1 \tilde{v}_4), \tag{3.3.5}$$

where the degree is $\tilde{d} = m_3 m_2 a_1$.

The complex dimension of the moduli for Z_{f^T} is bounded from below by

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{d})) - \sum_i \dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i)) \tag{3.3.6}$$

We begin computing $\dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{d}))$. Since this refers to the number of all possible monomials $z_0^{x_0} z_1^{x_1} z_2^{x_2} z_3^{x_3} z_4^{x_4}$ of degree $\tilde{d}$, we calculate this number counting the solutions of the Diophantine equation

$$\tilde{w}_0 x_0 + \tilde{w}_1 x_1 + \tilde{w}_2 x_2 + \tilde{w}_3 x_3 + \tilde{w}_4 x_4 = \tilde{d} \tag{3.3.7}$$

where the unknowns x_i are non-negative integers such that at least one of them is non-zero. We will solve this equation adding certain mild constraints on the exponent a_1 and m_3: either $\gcd(a_1, m_3) = 1$ and $a_1 > 2$ or $a_1 = 2$. As we will see at the end of this section, the remaining cases of interest can be computed case by case.

Lemma 3.3.1 *Let f^T be an invertible polynomial described as in (3.3.3) with an associated weight vector $\tilde{\mathbf{w}}$ given as in (3.3.5). If we add the additional conditions $\gcd(a_1, m_3) = 1$ and $a_1 > 2$, then the equation (3.3.7) has exactly five solutions. Moreover, the set of generators of $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(\tilde{d}))$ is given by the set*

$$\{z_0^{m_2} z_1, z_1^{a_1}, z_3 z_2^{a_2}, z_4 z_3^{a_3}, z_2 z_4^{a_4}\}.$$

Proof Using the expression given in (3.3.5), we can write the Diophantine equation (3.3.7) as

$$m_3(a_1 - 1)x_0 + m_3m_2x_1 + m_2a_1\tilde{v}_2x_2 + m_2a_1\tilde{v}_3x_3 + m_2a_1\tilde{v}_4x_4 = m_3m_2a_1, \quad (3.3.8)$$

which we can rewrite as

$$m_3((a_1 - 1)x_0 + m_2x_1) = a_1m_2(m_3 - \tilde{v}_2x_2 - \tilde{v}_3x_3 - \tilde{v}_4x_4).$$

Since $\gcd(m_2, m_3) = 1$, we obtain $m_2 \mid (a_1 - 1)x_0 + m_2x_1$. This implies that $m_2 \mid (a_1 - 1)x_0$. It is not difficult to obtain $\gcd(m_2, a_1 - 1) = 1$ (see equation (3.4.11) in the proof of Theorem 3.1 in [16]), so $m_2 \mid x_0$. On the other hand, since $m_3(a_1 - 1)x_0 \leqslant \tilde{d} = m_3m_2a_1$, we have

$$x_0 \leqslant \frac{a_1m_2}{a_1 - 1} < 2m_2.$$

Since $m_2 \mid x_0$, then $x_0 = 0$ or $x_0 = m_2$.

- If $x_0 = m_2$, then the equation (3.3.8) is equivalent to

  $$m_3(a_1 - 1)m_2 + m_3m_2x_1 + m_2a_1\tilde{v}_2x_2 + m_2a_1\tilde{v}_3x_3 + m_2a_1\tilde{v}_4x_4 = m_3m_2a_1$$

  Simplifying, we obtain

  $$a_1(\tilde{v}_2x_2 + \tilde{v}_3x_3 + \tilde{v}_4x_4) = m_3(1 - x_1).$$

  Since $a_1(\tilde{v}_2x_2 + \tilde{v}_3x_3 + \tilde{v}_4x_4) \geqslant 0$, we have that $x_1 = 1$ or $x_1 = 0$. If $x_1 = 1$, then $\tilde{v}_2x_2 + \tilde{v}_3x_3 + \tilde{v}_4x_4 = 0$, which implies that $x_2 = x_3 = x_4 = 0$. Thus, we obtain a solution of (3.3.8), which is given by $(m_2, 1, 0, 0, 0)$. On the other hand, if $x_1 = 0$, then $a_1(\tilde{v}_2x_2 + \tilde{v}_3x_3 + \tilde{v}_4x_4) = m_3$, but $\gcd(a_1, m_3) = 1$ and $a_1 \neq 1$. Thus, in this case there is no solution.
- If $x_0 = 0$, then the equation (3.3.8) is given by

  $$m_3m_2x_1 + m_2a_1\tilde{v}_2x_2 + m_2a_1\tilde{v}_3x_3 + m_2a_1\tilde{v}_4x_4 = m_3m_2a_1.$$

  Simplifying, we obtain

  $$m_3x_1 = a_1(m_3 - \tilde{v}_2x_2 - \tilde{v}_3x_3 - \tilde{v}_4x_4).$$

  Since $\gcd(a_1, m_3) = 1$, then we have $m_3 \mid (m_3 - \tilde{v}_2x_2 - \tilde{v}_3x_3 - \tilde{v}_4x_4)$. That is, the expression $\tilde{v}_2x_2 + \tilde{v}_3x_3 + \tilde{v}_4x_4$ can assume two values: m_3 or 0. First, we suppose that $\tilde{v}_2x_2 + \tilde{v}_3x_3 + \tilde{v}_4x_4 = m_3$. This forces $x_1 = 0$. Moreover, since $\tilde{v}_2$, $\tilde{v}_3$ and $\tilde{v}_4$ are given in (3.3.4) and these describe the weight vector $\tilde{\mathbf{w}}$ in (3.3.5), then the equation $\tilde{v}_2x_2 + \tilde{v}_3x_3 + \tilde{v}_4x_4 = m_3$ has exactly three solutions: $(a_2, 1, 0)$, $(0, a_3, 1)$ and $(1, 0, a_4)$. Thus, we obtain three solutions for the equation (3.3.8): $(0, 0, a_2, 1, 0)$, $(0, 0, 0, a_3, 1)$ and $(0, 0, 1, 0, a_4)$. Finally, if we suppose that $\tilde{v}_2x_2 + \tilde{v}_3x_3 + \tilde{v}_4x_4 = 0$, then $x_1 = a_1$. In this case the only solution is $(0, a_1, 0, 0, 0)$.

Summarizing, when $a_1 > 2$, we obtain five solutions for Equation (3.3.8):

$$(m_2, 1, 0, 0, 0), (0, 0, a_2, 1, 0), (0, 0, 0, a_3, 1), (0, 0, 1, 0, a_4) \text{ and } (0, a_1, 0, 0, 0).$$

□

Proposition 3.3.1 *Let f be and invertible polynomial as in (3.3.1) with weights satisfying conditions (3.2.1). If its Berglund-Hübsch transpose f^T with associated weight vector $\tilde{\mathbf{w}}$ satisfies the conditions given in Lemma 3.3.1, then the complex dimension of the local moduli of the orbifold $Z_{f^T} = \{f^T = 0\}/\mathbb{C}^*(\tilde{\mathbf{w}})$ is zero. Moreover, each $H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i))$ is generated by z_i for $i = 0, \ldots, 4$.*

Proof This result follows from Lemma 3.3.1 and the fact that $\dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i)) \geqslant 1$ for each i and $\sum_{i=0}^{4} \dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i)) \leqslant 5$. □

Lemma 3.3.2 *Let f^T be an invertible polynomial described in (3.3.3) with associated weight vector $\tilde{\mathbf{w}}$ as in (3.3.5). If we add the additional condition $a_1 = 2$, then Equation (3.3.7) has six solutions. Moreover, the set of generators of $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(\tilde{d}))$ is given by the set*

$$\{z_0^{m_2} z_1, z_0^{2m_2}, z_1^2, z_3 z_2^{a_2}, z_4 z_3^{a_3}, z_2 z_4^{a_4}\}.$$

Proof When $a_1 = 2$, Equation (3.3.7) gives

$$m_3 x_0 + m_3 m_2 x_1 + 2m_2 \tilde{v}_2 x_2 + 2m_2 \tilde{v}_3 x_3 + 2m_2 \tilde{v}_4 x_4 = 2m_3 m_2, \tag{3.3.9}$$

which can be written as

$$m_3(x_0 + m_2 x_1) = 2m_2(m_3 - \tilde{v}_2 x_2 - \tilde{v}_3 x_3 - \tilde{v}_4 x_4).$$

As $\gcd(m_2, m_3) = 1$, then $m_2 \mid (x_0 + m_2 x_1)$. This implies that $m_2 \mid x_0$. Moreover, from (3.3.9) we have $m_3 x_0 \leqslant 2m_3 m_2$. Since $m_3 \neq 0$, then $x_0 \leqslant 2m_2$. Thus, x_0 can take three values: 0, m_2 or $2m_2$.

Now, we consider two situations: m_3 is odd or m_3 is even.

a) When m_3 is odd. As $\gcd(a_1, m_3) = 1$, then the method to find the solutions is similar to what we have done for $x_0 = 0$ or $x_0 = m_2$ in the Lemma 3.3.1. For the additional case $x_0 = 2m_2$, we have

$$2m_3 m_2 + m_3 m_2 x_1 + 2m_2 \tilde{v}_2 x_2 + 2m_2 \tilde{v}_3 x_3 + 2m_2 \tilde{v}_4 x_4 = 2m_3 m_2$$

which has solution $x_1 = x_2 = x_3 = x_4 = 0$. So in this case, we add the solution: $(2m_2, 0, 0, 0, 0)$. Thus, we have exactly six solutions.

b) When m_3 is even. Here, we can write $m_3 = 2m_3'$. Replacing this in Equation (3.3.9) and simplifying, we obtain

$$m_3' x_0 + m_3' m_2 x_1 + m_2 \tilde{v}_2 x_2 + m_2 \tilde{v}_3 x_3 + m_2 \tilde{v}_4 x_4 = 2m_3' m_2. \tag{3.3.10}$$

This equation can be written as

$$m_3'(x_0 + m_2 x_1) = m_2(2m_3' - \tilde{v}_2 x_2 - \tilde{v}_3 x_3 - \tilde{v}_4 x_4).$$

By a similar argument used above, we obtain $m_2 \mid x_0$ and $x_0 \leqslant 2m_2$. This implies that $x_0 = 0$, $x_0 = m_2$ or $x_0 = 2m_2$.

- If $x_0 = 0$, then Equation (3.3.10) can be written as

$$m_3' m_2 x_1 + m_2 \tilde{v}_2 x_2 + m_2 \tilde{v}_3 x_3 + m_2 \tilde{v}_4 x_4 = 2m_3' m_2. \tag{3.3.11}$$

Simplifying the expression above, we arrive at the equality

$$\tilde{v}_2 x_2 + \tilde{v}_3 x_3 + \tilde{v}_4 x_4 = m_3'(2 - x_1).$$

We notice that x_1 can assume three values: $x_1 = 0$, $x_1 = 1$ or $x_1 = 2$.

* When $x_1 = 0$, the equation above is

$$\tilde{v}_2 x_2 + \tilde{v}_3 x_3 + \tilde{v}_4 x_4 = 2m_3' = m_3.$$

This equation has exactly three solutions: $(a_2, 1, 0)$, $(0, a_3, 1)$ and $(1, 0, a_4)$. Thus, the solutions of Equation (3.3.9) are $(0, 0, a_2, 1, 0)$, $(0, 0, 0, a_3, 1)$ and $(0, 0, 1, 0, a_4)$.

* When $x_1 = 1$, Equation (3.3.11) can be simplified:

$$\tilde{v}_2 x_2 + \tilde{v}_3 x_3 + \tilde{v}_4 x_4 = m'_3 \tag{3.3.12}$$

If there exists a solution $(\hat{x}_2, \hat{x}_3, \hat{x}_4)$ of this Diophantine equation, then $(2\hat{x}_2, 2\hat{x}_3, 2\hat{x}_4)$, with $\hat{x}_i \in \mathbb{Z}_0^+$, is solution of the equation

$$\tilde{v}_2 x_2 + \tilde{v}_3 x_3 + \tilde{v}_4 x_4 = 2m'_3 = m_3.$$

Nevertheless, the solutions of this last equation are $(a_2, 1, 0)$, $(0, a_3, 1)$ and $(1, 0, a_4)$, which implies that $2\hat{x}_i = 1$ for any i. That is, $\hat{x}_i$ is not integer, which is not possible.

* When $x_1 = 2$, the equation is

$$\tilde{v}_2 x_2 + \tilde{v}_3 x_3 + \tilde{v}_4 x_4 = 0,$$

which has trivial solution. Therefore, the solution of Equation (3.3.9) is $(0, 2, 0, 0, 0)$.

– If $x_0 = m_2$, then Equation (3.3.10) can be reduced to

$$m'_3 + m'_3 x_1 + \tilde{v}_2 x_2 + \tilde{v}_3 x_3 + \tilde{v}_4 x_4 = 2m'_3 \tag{3.3.13}$$

The above equation can be written as

$$\tilde{v}_2 x_2 + \tilde{v}_3 x_3 + \tilde{v}_4 x_4 = m'_3(1 - x_1).$$

Then x_1 can take two values: $x_1 = 0$ or $x_1 = 1$.

* If $x_1 = 0$, then we obtain Equation (3.3.12), which has no solution.
* If $x_1 = 1$, we obtain the equation

$$\tilde{v}_2 x_2 + \tilde{v}_3 x_3 + \tilde{v}_4 x_4 = 0,$$

which has trivial solution. In this case the solution obtained is $(m_2, 1, 0, 0, 0)$.

– If $x_0 = 2m_2$. Here, Equation (3.3.10) can be reduced to

$$m'_3 m_2 x_1 + m_2 \tilde{v}_2 x_2 + m_2 \tilde{v}_3 x_3 + m_2 \tilde{v}_4 x_4 = 0$$

which has trivial solution. In this case, the solution of Equation (3.3.9) is given by $(2m_2, 0, 0, 0, 0)$.

In view of the foregoing discussion, we find six solutions for Equation (3.3.9):

$$(0, 0, a_2, 1, 0), (0, 0, 0, a_3, 1), (0, 0, 1, 0, a_4), (0, 2, 0, 0, 0), (m_2, 1, 0, 0, 0), \text{ and } (2m_2, 0, 0, 0, 0).$$

□

Proposition 3.3.2 *Let f be and invertible polynomial as in (3.3.1) with weights satisfying conditions (3.2.1). If its Berglund-Hübsch transpose f^T with associated weight vector $\tilde{\mathbf{w}}$ satisfies the conditions given in Lemma 3.3.2. Then the complex dimension of the moduli of the orbifold $Z_{f^T} = \{f^T = 0\}/\mathbb{C}^*(\tilde{\mathbf{w}})$ is zero. Moreover, $H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_1))$ is generated by $z_0^{m_2}$ and z_1 while $H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i))$ is generated by z_i, for $i \neq 1$.*

Proof First, we will prove that $\dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_1)) \geqslant 2$. In fact, it is equivalent to show that the quantity of solutions of the following Diophantine equation is not less that two:

$$\tilde{w}_0 x_0 + \tilde{w}_1 x_1 + \tilde{w}_2 x_2 + \tilde{w}_3 x_3 + \tilde{w}_4 x_4 = \tilde{w}_1.$$

Using the expression for $\tilde{\mathbf{w}}$ given in (3.3.5) and replacing $a_1 = 2$ in the equation above, we arrive to the equation

$$m_3x_0 + m_3m_2x_1 + 2m_2\tilde{v}_2x_2 + 2m_2\tilde{v}_2x_3 + 2m_2\tilde{v}_4x_4 = m_3m_2.$$

Here, we can exhibit at least two solutions: $(m_2, 0, 0, 0, 0)$ and $(0, 1, 0, 0, 0)$. Therefore, we obtain $\dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_1)) \geqslant 2$.

On the other hand, we know that $\dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i)) \geqslant 1$ for $i \neq 1$. This implies that

$$6 \leqslant \sum_{i=0}^{4} \dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i)) \leqslant \dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{d})) = 6.$$

From here, we can conclude that $H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_1))$ has dimension 2 and the set of generators is $\{z_0^{m_2}, z_1\}$ and each $H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i))$ is one dimensional and it is generated by z_i for $i \neq 1$.

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{d})) - \sum_{i=0}^{4} \dim_{\mathbb{C}} H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i)) = 0.$$

□

Next we discuss the moduli of the orbifold associated to the polynomials considered in this subsection.

Proposition 3.3.3 *For $\mathbb{P}(\tilde{\mathbf{w}}) = \mathrm{Proj}(S(\tilde{\mathbf{w}}))$, the complex automorphisms of the graded ring $S(\tilde{\mathbf{w}})$ can be defined on generators for the two types of polynomials given in this subsection thanks to Propositions 3.3.1 and 3.3.2. We can describe the moduli of the corresponding orbifold, as done previously. Indeed, let $\alpha_i, \beta_1 \in \mathbb{C}^*$, then we have*

- *If f is a polynomial as in Lemma 3.3.1, then $G(\tilde{\mathbf{w}})$ is given on generators by*

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

  and the moduli of the orbifold Z_{f^T} is included in the quotient

$$Span\{z_0^{m_2} z_1, z_1^{a_1}, z_3 z_2^{a_2}, z_4 z_3^{a_3}, z_2 z_4^{a_4}\}/G(\tilde{\mathbf{w}})$$

  a zero-dimensional quotient space.
- *If f is a polynomial as in Lemma 3.3.2, then $G(\tilde{\mathbf{w}}))$ is given on generators by*

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 + \beta_1 z_0^{m_2} \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

  and the moduli of the orbifold Z_{f^T} is included in the quotient

$$Span\left\{z_0^{m_2} z_1, z_0^{2m_2}, z_1^2, z_3 z_2^{a_2}, z_4 z_3^{a_3}, z_2 z_4^{a_4}\right\}/G(\tilde{\mathbf{w}})$$

again a zero-dimensional quotient space.

□

In [16], Theorem 4.1 we established the existence of 75 new rational homology 7-spheres admitting Sasaki-Einstein metrics with not well-formed quotients, that is, with some group elements having codimension 1 fixed point sets. All these links, which are listed in a table in the Appendix in [16], are given by polynomials of chain-cycle type of type II with weights satisfying the conditions given in (3.2.1). So we have:

Proposition 3.3.4 *Let f be and invertible polynomial as in (3.3.1) with weights satisfying conditions (3.2.1). If its Berglund-Hübsch transpose f^T with associated weight vector $\tilde{\mathbf{w}}$ satisfies the conditions given in either Lemma 3.3.1 or Lemma 3.3.2 and additionally f belongs to one of the 236 rational homology 7-spheres admitting Sasaki-Einstein metrics found in [5] and [16], then the dimension of the local moduli space of Sasaki-Einstein metrics of the rational homology 7-spheres at L_{f^T} equals zero. Thus rational homology 7-spheres given as links coming from polynomials f^T as above do not admit inequivalent families of Sasaki-Einstein structures.*

Proof This result follows from Propositions 3.3.1 and 3.3.2 □

Remark 3.3.1 From the list of 75 new examples of Sasaki-Einstein rational homology 7-sphere given in the Appendix of [16], there are 7 weight vectors $\tilde{\mathbf{w}}$ that do not satisfy the additional conditions of either Lemma 3.3.1 or Lemma 3.3.2, actually all of the elements of this table satisfy $a_1 > 2$ but $\gcd(m_3, a_1) \neq 1$. For these, we have the following table where $b_{\mu_{\mathbb{R}}}$ denotes the upper bound for the real dimension of the local moduli of Sasaki-Einstein metrics of rational homology 7-spheres at L_{f^T}.

$\tilde{\mathbf{w}}$	$\tilde{d}$	m_3	$\dim_{\mathbb{C}} H^0$ $(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{d}))$	$\sum_{i=0}^{4} \dim_{\mathbb{C}} H^0$ $(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i))$	$b_{\mu_{\mathbb{R}}}$
(177,295,270,370,70)	1180	118	5	5	0
(52,663,867,1581,153)	3315	65	9	6	6
(148,777,987,1911,63)	3885	185	9	6	6
(86,3655,5185,595,1445)	10965	129	6	5	2
(86,3655,4165,2635,425)	10965	129	6	5	2
(438,4161,6175,133,1577)	12483	657	6	5	2
(438,4161,4693,3097,95)	12483	657	6	5	2

In contrast to the previous cases obtained by the Berglund-Hübsch rule, with the exception of the first member of the above table, all the generating monomials of the local moduli are such that the weight vectors admit blocks of the form $z_\alpha^{n_\alpha} z_\beta^{n_\beta}$ or $z_\alpha^{n_\alpha} z_\beta^{n_\beta} z_\lambda$. For instance, the weight vector $\tilde{\mathbf{w}} = (52, 663, 867, 1581, 153)$ determines as the set of generators of $H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{d}))$ the following

$$\{z_1 z_0^{51}, z_1^5, z_1^3 z_2 z_4^3, z_1^2 z_4^{13}, z_1 z_2^2 z_4^6, z_1 z_3 z_4^7, z_2^2 z_3, z_2 z_4^{16}, z_3^2 z_4\}.$$

The set $H^0(\mathbb{P}(\tilde{\mathbf{w}}), \mathcal{O}(\tilde{w}_i))$ is spanned by the automorphisms

$$\varphi_{\tilde{\mathrm{w}}}\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 \\ \alpha_2 z_2 \\ \alpha_3 z_3 + \beta_3 z_1 z_4^6 \\ \alpha_4 z_4 \end{pmatrix}.$$

Hence the complex dimension of the moduli of the corresponding orbifold has three as an upper bound.

4 Application: links of non-isolated singularities and klt singularities

The explicit description of the monomials generating the vector space $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ yields arguments to the study of the non-quasismooth polynomials generated by monomials in this set. Then, the weighted hypersurfaces determined by these sort of polynomials can be considered as points in the boundary of a compactification of the moduli of quasismooth polynomials since the subset of all quasismooth elements is dense in the set of monomials generating $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$. Below, we explain via examples, how to produce non-quasismooth hypersurfaces with klt singularities which are the candidates to give rise to non-smooth links whose metric cones can be considered as some sort of degenerating CalabiYau cones [34, 41]. For precise definitions of the singularities of the minimal model appearing in this section and their possible relations see [27, 31]. In particular, at the end of page 42 in [31], Kollár gives conditions needed to ensure that canonical singularities are klt, these conditions are trivially satisfied in our setting.

First, recall the notion of the Newton polyhedra: let us write the monomial $x_0^{a_0} \cdots x_n^{a_n}$ as $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$. Given a polynomial function $f(x) = \sum_{\mathbf{a}\in\mathbb{Z}_+^{n+1}} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$, then the support of f is defined as $\mathrm{supp}(f) := \left\{\mathbf{a} \in \mathbb{Z}_+^{n+1} \mid c_{\mathbf{a}} \neq 0\right\}$. The Newton polyhedron $\Gamma_+(f)$ of f is defined to be the convex hull of the following set $\bigcup_{\mathbf{a}\in\mathrm{supp}(f)} \left(\mathbf{a} + \mathbb{R}_{\geqslant 0}^{n+1}\right)$. For each face γ of $\Gamma_+(f)$, we define the polynomial f_γ as follows:

$$f_\gamma = \sum_{\mathbf{a}\in\gamma} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}.$$

A power series f is said to be Newton non-degenerate, if for every face γ the equation $f_\gamma = 0$ defines a hypersurface smooth in the complement of the hypersurface $x_0 \cdots x_n = 0$.

Now we state the following criterion which we will use below, following [47], to determine whether the singularity is klt. See [22] for a proof of the following lemma.

Lemma 4.0.1 *Let $S \subset \mathbb{C}^{n+1}$ be a normal hypersurface defined as the set of zeros of a Newton non-degenerate polynomial f. If the point $(1, 1, \ldots, 1)$ is in the interior of the Newton polyhedron $\Gamma_+(f)$, then S has canonical singularities.*

□

- Now, consider Example 3.2.1 (1): the polynomial of type I given by

$$f = z_0^9 + z_1^9 + z_4 z_2^2 + z_2 z_3^2 + z_3 z_4^{19},$$

has associated weight vector $\mathbf{w} = (77, 77, 333, 180, 27)$ and degree $d = 693$, which determines a well-formed Fano Kähler-Einstein 3-fold with at worst cyclic singularities [24]. The set $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by the set of thirteen monomials

$$\{z_0^9, z_0^8 z_1, \ldots, z_0 z_1^8, z_1^9, z_4 z_2^2, z_2 z_3^2, z_3 z_4^{19}\}.$$

One can select a certain subset of these monomials and consider the non-quasismooth hypersurface

$$X_0 : z_0^9 + z_0^2 z_1^7 + z_4 z_2^2 + z_2 z_3^2 + z_3 z_4^{19} = 0$$

which belongs to the family of weighted hypersurfaces

$$X_t : z_0^9 + (1-t) z_0^2 z_1^7 + t z_1^9 + z_4 z_2^2 + z_2 z_3^2 + z_3 z_4^{19} = 0 \subset \mathbb{P}(77, 77, 333, 180, 27) \times \mathbb{C}.$$

From Lemma 2.1.1, it follows that X_t is quasismooth for all $t \neq 0$. As we show below, the subvariety X_0 is klt. Indeed, since the number of monomials defining X_0 is equal to the number of variables, any linear combination of these monomials with nonzero coefficients defines a hypersurface isomorphic to X_0. Thus we can take X_0 to be a general divisor in the linear system defined by the monomials of the defining equation. The base locus of this linear system is contained in the following set of points

$$B = \big\{[0:0:1:0:0], [0:0:0:1:0], [0:0:0:0:1], [0:1:1:0:0], [0:1:0:1:0], [0:1:0:0:1], [0:1:0:0:0]\big\}$$

By Bertini's Theorem on $\mathbb{C}^5 - \{0\}$, we know that X_0 is quasismooth outside these points. Moreover, from setting the gradient of f equal to zero, we find that X_0 is quasismooth at all elements of B except at the point $[0:1:0:0:0]$. Actually X_0 is klt at $[0:1:0:0:0]$. As mentioned above, a general linear combination of the monomials defining X_0, determines a hypersurface

$$\tilde{X_0} = c_0 z_0^9 + c_1 z_0^2 z_1^7 + c_2 z_4 z_2^2 + c_3 z_2 z_3^2 + c_4 z_3 z_4^{19} = 0$$

isomorphic to X_0 for $c_i{}'s$ general complex numbers different than zero. So we will show that a general hypersurface $\tilde{X_0}$ is klt at $[0:1:0:0:0]$. In the affine chart $z_1 \neq 0$ we take $z_1 = 1$ and locally $\tilde{X_0}$ is the quotient of the hypersurface

$$S : c_0 z_0^9 + c_1 z_0^2 + c_2 z_4 z_2^2 + c_3 z_2 z_3^2 + c_4 z_3 z_4^{19} = 0$$

in $\mathbb{C}^4$ by the group $\mathbb{Z}_{77}$. Since klt is a property preserved by finite quotients ([31], Corollary 2.43), it suffices to show that such the general hypersurface $S \subset \mathbb{C}^4$ has canonical singularities. Clearly, S is normal (it is a hypersurface and its singular set has codimension at least 2). Also, since the coefficients of the monomials in this example are taken to be general, the Newton non-degeneracy is satisfied since the base locus of any collection of the monomials in the equation $c_0 z_0^9 + c_1 z_0^2 + c_2 z_4 z_2^2 + c_3 z_2 z_3^2 + c_4 z_3 z_4^{19} = 0$ is contained in the hypersurface $x_0 \cdots x_n = 0$. So we can apply the criterion given above to determine whether the singularity is klt: the singularity in S is canonical if the point $(1, 1, 1, 1)$ is in the interior of the Newton polyhedron $\Gamma_+(g)$ generated by support of the polynomial g defining $\tilde{X_0}$, that is, generated by the set

$$\mathrm{Supp}(g) = \big\{(9,0,0,0), (2,0,0,0), (0,2,0,1), (0,1,2,0), (0,0,1,19)\big\}.$$

We notice that it is enough to show that there exists some point in the Newton polyhedron $\Gamma_+(g)$ with all its entries less than 1 . For this, we consider the following point

$$P_0 = \frac{43}{539}(9,0,0,0)+\frac{76}{539}(2,0,0,0)+\frac{20}{77}(0,2,0,1)+\frac{37}{77}(0,1,2,0)+\frac{3}{77}(0,0,1,19)$$

which has all its entries equal to 1. Thus, the point $(1, 1, 1, 1)$ is in the interior of $\Gamma_+(f)$. We conclude that X_0 is klt.

- Similar arguments can be used to study the non-quasismooth hypersurfaces for the two remaining types of polynomials producing non-trivial moduli. For instance, consider Example 3.2.1 (2): the polynomial

$$f = z_0^{125} + z_0z_1^4 + z_4z_2^2 + z_2z_3^7 + z_3z_4^3$$

of type II with weight vector $\boldsymbol{w} = (43, 1333, 1875, 500, 1625)$ and degree $d = 5375$ which determines a well-formed Fano Kähler-Einstein 3-fold with at worst cyclic singularities [24]. The set $H^0(\mathbb{P}(\boldsymbol{w}), \mathcal{O}(d))$ is given by

$$\left\{z_0^{125}, z_0^{94}z_1, z_0^{63}z_1^2, z_0^{32}z_1^3, z_0z_1^4, z_4z_2^2, z_2z_3^7, z_3z_4^3\right\}.$$

From this set of generators, we consider the family

$$X_t : (1-t)z_0^{63}z_1^2+z_0z_1^4+tz_0^{125}+z_4z_2^2+z_2z_3^7+z_3z_4^3 = 0 \subset \mathbb{P}(43, 1333, 1875, 500, 1625)\times\mathbb{C}$$

with central fiber

$$X_0 : z_0^{63}z_1^2 + z_0z_1^4 + z_4z_2^2 + z_2z_3^7 + z_3z_4^3 = 0,$$

and, using Lemma 2.1.1, with X_t quasismooth for all $t \neq 0$. One can verify that X_0 is either smooth or quasismooth in all points except at the point $[1 : 0 : 0 : 0 : 0]$. With identical arguments as the ones used above, one can show that the weighted subvariety is klt at the point $[1 : 0 : 0 : 0 : 0]$.

- Of course, one also can describe non-quasismooth hypersurfaces from orbifolds not belonging to the list of anticanonically embedded Kähler-Einstein Fano 3-folds given by Johnson and Kollár [24], that is, where the index corresponding to the quasismooth weighted variety has index $I > 1$. For instance, consider the polynomial

$$f = z_0^5 + z_1^5 + z_4z_2^3 + z_2z_3^3 + z_3z_4^9$$

which determines a weighted hypersurface X with weight vector $\mathbf{w} = (82, 82, 125, 95, 35)$ and degree $d = 410$. We notice that in this case the index is $I = 9$. Since $Id = 9(410) < \frac{4}{3}(82)(35)$, it follows from Theorem 2.1.1 that the weighted hypersurface $X \subset \mathbb{P}(\mathbf{w})$ admits a Kähler-Einstein orbifold metric and hence the link L_f admits a Sasaki-Einstein metric as well. By Proposition 3.2.2, the set of monomials generating $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is given by

$$\{z_0^5, z_0^4z_1, \dots, z_0z_1^4, z_1^5, z_4z_2^3, z_2z_3^3, z_3z_4^9\}.$$

One considers the following family generated by some elements of this set:

$$X_t : (1-t)z_0^2z_1^3 + z_0^5 + tz_1^5 + z_4z_2^3 + z_2z_3^3 + z_3z_4^9 = 0 \subset \mathbb{P}(82, 82, 125, 95, 35) \times \mathbb{C}$$

with central fiber

$$X_0 : g = z_0^5 + z_0^2z_1^3 + z_4z_2^3 + z_2z_3^3 + z_3z_4^9 = 0.$$

From Lemma 2.1.1, X_t is quasismooth for all $t \neq 0$ and X_0 is non-quasismooth. As before, it is not difficult to show that X_0 is either smooth or quasismooth in all points except at the point $[0 : 1 : 0 : 0 : 0]$. Actually X_0 is klt at $[0 : 1 : 0 : 0 : 0]$. Let us consider a general linear combination of the monomials defining X_0:

$$\tilde{X_0} = c_0 z_0^5 + c_1 z_0^2 z_1^3 + c_2 z_4 z_2^3 + c_3 z_2 z_3^3 + c_4 z_3 z_4^9 = 0$$

which is isomorphic to X_0 for $c_i' s$ general complex numbers different than zero. This general hypersurface $\tilde{X_0}$ is klt at $[0 : 1 : 0 : 0 : 0]$. Indeed, in the affine chart $z_1 \neq 0$ we take $z_1 = 1$ and locally $\tilde{X_0}$ is the quotient of the hypersurface

$$S : c_0 z_0^5 + c_1 z_0^2 + c_2 z_4 z_2^3 + c_3 z_2 z_3^3 + c_4 z_3 z_4^9 = 0$$

in $\mathbb{C}^4$ by the group $\mathbb{Z}_{82}$. Since klt is a property preserved by finite quotients, it suffices to show that such the general hypersurface $S \subset \mathbb{C}^4$ has canonical singularities. Clearly, S is normal and since the coefficients of the monomials in this example are taken to be general, the Newton non-degeneracy is satisfied since the base locus of any collection of the monomials in the equation $c_0 z_0^5 + c_1 z_0^2 + c_2 z_4 z_2^2 + c_3 z_2 z_3^2 + c_4 z_3 z_4^{19} = 0$ is contained in the hypersurface $x_0 \cdots x_n = 0$. So we can apply Lemma 4.0.1 to determine whether the singularity is klt: the singularity in S is canonical if the point $(1, 1, 1, 1)$ is in the interior of the Newton polyhedron $\Gamma_+(h)$ generated by support of the polynomial h defining $\tilde{X_0}$, that is, generated by the set

$$\mathrm{Supp}(h) = \big\{(5, 0, 0, 0), (2, 0, 0, 0), (0, 3, 0, 1), (0, 1, 3, 0), (0, 0, 1, 9)\big\}.$$

As before, it is enough to show that there exists some point in the Newton polyhedron $\Gamma_+(h)$ with all its entries less than 1 . For this, we consider the following point

$$P_0 = \frac{44}{205}(3, 0, 0, 0) + \frac{73}{410}(2, 0, 0, 0) + \frac{19}{82}(0, 3, 0, 1) + \frac{25}{82}(0, 1, 3, 0) + \frac{7}{82}(0, 0, 1, 9)$$

which has all its entries equal to 1. Thus, the point $(1, 1, 1, 1)$ is in the interior of $\Gamma_+(h)$, so X_0 is klt.

- In the context of Sasakian geometry, the links determined by the klt Fano varieties described above are not smooth anymore. For instance the hypersurface

$$f = z_0^9 + z_0^2 z_1^7 + z_4 z_2^2 + z_2 z_3^2 + z_3 z_4^{19} = 0$$

on $\mathbb{C}^5 - \{0\}$ has a one dimensional singular set $\Sigma = \{(0, z_1, 0, 0, 0)\}$ which intersects the unit sphere S^9 in the circle S^1 so the link L_f is non-smooth. Since the contact 1-form on the link $L_f = V_f \cap S^9$ is given by the contact 1-form on the weighted sphere $\eta_{\mathbf{w}} = \frac{\eta}{\sum_{i=0}^4 w_i |z_i|^2}$ restricted to L_f, one notices that this contact 1-form degenerates on the singular set $\Sigma \cap S^9$. If one considers the open manifold resulting from excluding this singular set we still obtain a Reeb vector field ξ on the regular part L_f^{reg} and a Riemannian metric on it. This natural way to produce singular links with Sasaki metrics on the regular part can be interpreted as the horizon (or base) of a metric cone $C(L_f) = L_f \times \mathbb{R}^+$ with a function $r : C(L_f) \to \mathbb{R}^+$ that determines the radial coordinate and hence a Liouville vector field $\Psi = r\partial_r$ on the regular part of $C(L_f)$. In accordance to the Yau-Tian-Donaldson conjecture for singular varieties ([35], Theorem 1.6) in order to extend this argument to the realm of Sasaki-Einstein metrics on these singular links and on the corresponding weak Ricci-flat metrics on the associated metric cone one needs to show the exceptionality of the klt singularity ([43], Theorem 1.4), that is, determine whether the pair (X, D) is klt for every effective $\mathbb{Q}$-divisor D that is $\mathbb{Q}$-linearly equivalent to $-K_X$, or equivalently, show that the α-invariant of X is greater than 1 [2]. Based on the explicit generators found in this article and the method of the weighted tangent cone developed by Totaro in [47], we do this with certain generality in [15].

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Declarations

Competing interests The authors declare no competing interests.

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