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SHORT COMMUNICATION

A Note on Quaternionic Möbius Transformations

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Abstract
The group $\mathrm{SL}(2,\mathbb{H})$ naturally acts on the extended quaternionic plane $\mathbb{H}\cup\{\infty\}$ which is conformally identified with the 4-sphere S^4, through quaternionic Möbius transformations. In this paper, we revisit the algebraic characterization of $\mathrm{SL}(2,\mathbb{H})$ and identify a particular case that has not been fully addressed in the existing literature. We obtain a complete characterization of the isometries in $\mathrm{SL}(2,\mathbb{H})$ and $\mathrm{PSL}(2,\mathbb{H})$, providing a unified and systematic classification. Our approach employs geometric and analytic methods to address the exceptional case and thereby completes the existing characterization of the group.

Keywords Quaternionic Möbius transformations · Algebraic characterization · Conjugacy invariants · Dynamical types · Canonical form

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Introduction

We begin by considering the group $\mathrm{SL}(2,\mathbb{H})$, consisting of matrices with quaternionic determinant (see Eq. (2)) one over Hamilton's quaternion $\mathbb{H}$, be the subgroup of the group of all invertible 2×2 quaternionic matrices $\mathrm{GL}(2,\mathbb{H})$. The group $\mathrm{SL}(2,\mathbb{H})$ acts on the extended quaternionic space $\widehat{\mathbb{H}}=\mathbb{H}\cup\{\infty\}$ via quaternionic Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto (az+b)(cz+d)^{-1},$$

where the space $\widehat{\mathbb{H}}$ is identified with the four-sphere $\mathbb{S}^4$ as being conformally diffeomorphic to the one-dimensional quaternionic projective space $\mathbb{P}^1_{\mathbb{H}}$. Following the classical formalism of Ahlfors [1] and Waterman [11], conformal automorphisms of the n-dimensional sphere may be viewed as linear fractional transformations of Vahlen matrices over Clifford algebras. In the specific case of $n=4$, this action extends naturally to the orientation-preserving isometries of the five-dimensional real hyperbolic space $\mathbf{H}^5$ whose conformal boundary at infinity is identified with $\widehat{\mathbb{H}}$, and its orientation-preserving isometry group is identified with the projective linear group:

$$\mathrm{PSL}(2,\mathbb{H})=\mathrm{SL}(2,\mathbb{H})/\{\pm \mathrm{I}_2\}.$$

On applying Lefschetz's fixed-point theorem, an isometry of $\mathbf{H}^5$ must always admit a fixed point in the closure of the hyperbolic space, based on which the following dynamical classifications arise. An element g in $\mathrm{PSL}(2,\mathbb{H})$ is called *elliptic* if it has a fixed point on $\mathbf{H}^5$; it is *parabolic*, resp. *hyperbolic*, if g is non-elliptic and has exactly one, resp. two fixed points on $\widehat{\mathbb{H}}$; see [4] for more details. Over the last couple of years, several authors have explored different ways of characterizing these dynamical classes algebraically using various types of conjugacy invariants for matrices in $\mathrm{SL}(2,\mathbb{H})$, cf. [4, 9], also see [2, 3] and [8, 12]. For more recent work one can see [6].

In accordance with the work of Gongopadhyay in [4], for a given element $A\in\mathrm{GL}(2,\mathbb{H})$ whose associates the characteristic polynomial given by:

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$$\chi_{\mathbb{H}}(A) = x^4 - c_3x^3 + c_2x^2 - c_1x + c_0$$

(cf. (3)), the conjugacy invariants emerging here are furnished below:

$$d_1 = \frac{c_1^2}{4c_0\sqrt{c_0}}, d_2 = \frac{c_2}{c_0}, \quad \text{and} \quad d_3 = \frac{c_3^2}{4c_0}, \tag{1}$$

where $c_0, c_1, c_2, c_3 \in \mathbb{R}$ and c_0 is the quaternionic determinant of the underlying matrix A. The matrix $\begin{pmatrix} \rho e^{i\theta} & 0 \\ 0 & \rho^{-1}e^{i(\pi-\theta)} \end{pmatrix}$, $\rho \neq 1$, $\theta \neq \pi/2$, will be called a 2-rotatory hyperbolic element as per the nomenclature in [4]. In view of the theorem [4, Theorem 1.1(i)], these matrices should satisfy $d_1 \neq d_3$. However, these matrices satisfy $c_0 = 1$ and $c_1 = -c_3$, ensures that $d_1 = d_3$; see case 2 of Theorem 3.2. In fact, these matrices act on $\mathbf{H}^5$ as a 2-rotatory hyperbolic isometry equipped with a rotation angle π and satisfying $c_1 = -c_3$. Furthermore, the 2-rotatory hyperbolic isometries not equipped with a rotation angle π are precisely the non-reversible elements in $\mathrm{PSL}(2, \mathbb{H})$ and must satisfy $d_1 \neq d_2$.

The main focus of this work lies in identifying and resolving this exceptional class (2-rotatory hyperbolic) of quaternionic Möbius transformations that was not captured explicitly in the algebraic characterization of Gongopadhyay [4]. In particular, we show that the family of 2-rotatory hyperbolic elements satisfying

$$c_1 = -c_3,$$

equivalently,

$$\beta_A = -\delta_A,$$

falls outside the scope of the previously stated invariant criterion based on d_1, d_2, d_3. By incorporating this case, we obtain a complete algebraic characterization of the dynamical types of elements in $\mathrm{PSL}(2, \mathbb{H})$.

Our characterization reveals that the condition $c_1 = -c_3$ encodes reversibility information for these transformations, thereby providing an algebraic description of certain reversible classes that were not explicitly detected by the earlier invariant framework. This establishes a direct connection between conjugacy invariants and reversibility phenomena in quaternionic Möbius geometry. This characterization has motivated the recent classification of reversible classes in Gongopadhyay et. al. [5].

Another contribution of this paper is the development of a unified framework based on the conjugacy invariants β_A, γ_A, and δ_A (Lemma 2.3). Within this framework, all dynamical types of quaternionic Möbius transformations are characterized by simple algebraic relations among these invariants. Consequently, our work provides a systematic and unified classification of isometries in both $\mathrm{SL}(2, \mathbb{H})$ and $\mathrm{PSL}(2, \mathbb{H})$. This refinement moves beyond a mere algebraic expansion; it bridges the gap between matrix invariants and hyperbolic dynamics. The resulting classification allows researchers to immediately identify the reversibility and precise dynamical type of any quaternionic Möbius transformation directly from its matrix entries, thereby facilitating more nuanced studies in conjugacy theory and quaternionic geometry.

Table 1 (as well as Sect. 3) provides a complete algebraic classification of elements of $\mathrm{PSL}(2, \mathbb{H})$ in terms of the conjugacy invariants β_A, γ_A, and δ_A. It shows that every quaternionic Möbius transformation can be identified as one of eight dynamical types by examining simple algebraic relations among these invariants. In particular, conditions such as $\beta_A^2 \neq \delta_A^2$, $\beta_A = -\delta_A$, and $\beta_A = \delta_A$, together with inequalities involving γ_A, distinguish the various hyperbolic and elliptic subclasses, while the non-diagonalizable case corresponds to parabolic elements.

Preliminaries

Let us consider the four-dimensional division algebra $\mathbb{H} := \{a_0 + a_1i + a_2j + a_3k : \ i^2 = j^2 = k^2 = ijk = -1\}$. Conjugate of a is $\bar{a} := a_0 - a_1i - a_2j - a_3k$, real part $\Re(a) := a_0$, and imaginary part $\Im(a) := a_1i + a_2j + a_3k$.

We begin by examining the algebra $\mathrm{M}(2, \mathbb{H})$ of 2×2 matrices over the quaternions $\mathbb{H}$. A non-zero vector $v \in \mathbb{H}^2$ is a right eigenvector of $A \in \mathrm{M}(2, \mathbb{H})$ with a corresponding right eigenvalue $\lambda \in \mathbb{H}$ if it satisfies the equation $Av = v\lambda$. It is important to note that the eigenvalues of A appear in similarity classes. Within each class, there exists a unique representative in the form of a complex number with a non-negative imaginary part. Unless stated otherwise, we shall refer to these unique complex representatives as the eigenvalues of A. For a more comprehensive treatment of quaternionic linear algebra, we refer the reader to Rodman [10].

Let p be the embedding defined from $\mathrm{M}(2, \mathbb{H})$ into $\mathrm{M}(4, \mathbb{C})$, given by

$$p(A) := \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix},$$

where $A_1, A_2 \in \mathrm{M}(2, \mathbb{C})$ are such that $A = A_1 + A_2j$, and each $\overline{A_i}$ is the complex conjugate of A_i. The respective definitions of the determinant $\det_{\mathbb{H}}(A)$ and the characteristic polynomial $\chi_{\mathbb{H}}(A)$ of $A \in \mathrm{M}(2, \mathbb{H})$[10, p. 113] adopted here are

Table 1 Algebraic characterization of elements in $\mathrm{PSL}(2,\mathbb{H})$

Representative	Invariants	Type
$\begin{pmatrix} \lambda e^{i\theta} & 0 \\ 0 & \lambda^{-1}e^{i\phi} \end{pmatrix}$	$\beta_A^2 \neq \delta_A^2$	Loxodromic ($\lambda \neq 1, \theta \neq \phi, \pi - \phi$)
$\begin{pmatrix} \lambda e^{i\theta} & 0 \\ 0 & \lambda^{-1}e^{i(\pi-\theta)} \end{pmatrix}$	$\beta_A = -\delta_A \neq 0, \gamma_A \geq \beta_A^2 - 2$	2-rotatory hyperbolic ($\lambda \neq 1, \theta \neq \pi/2$)
$\begin{pmatrix} \lambda i & 0 \\ 0 & \lambda^{-1}i \end{pmatrix}$	$\beta_A = \delta_A = 0, \gamma_A > \beta_A^2 + 2$	Special 2-rotatory loxodromic ($\lambda \neq 1$)
$\begin{pmatrix} \lambda e^{i\theta} & 0 \\ 0 & \lambda^{-1}e^{i\theta} \end{pmatrix}$	$\beta_A = \delta_A \neq 0, \gamma_A \geq \beta_A^2 + 2$	Hyperbolic ($\lambda \neq 1, \theta \neq \pi/2$)
$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$	$\beta_A = \delta_A \neq 0, \gamma_A < \beta_A^2 + 2$	2-rotatory Elliptic ($\theta \neq \phi, \pi - \phi$)
$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$	$\beta_A = \delta_A \neq 0, \gamma_A = \beta_A^2 + 2, \lvert\beta_A\rvert \leq 2$	Equal-angle elliptic ($\theta \neq \pi/2$)
$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i(\pi-\theta)} \end{pmatrix}$	$\beta_A = \delta_A = 0, -2 \leq \gamma_A \leq 2$	π-rotatory Elliptic
$\begin{pmatrix} e^{i\theta} & 1 \\ 0 & e^{i\theta} \end{pmatrix}$	$\beta_A = \delta_A, \gamma_A = \beta_A^2 + 2, \lvert\beta_A\rvert \leq 2$	Parabolic

$$\det_{\mathbb{H}}(A) := \det(p(A)) \quad \text{and} \quad \chi_{\mathbb{H}}(A) := \chi_{p(A)}. \tag{2}$$

This enables us to define the groups $\mathrm{GL}(2,\mathbb{H})$ and $\mathrm{SL}(2,\mathbb{H})$ as follows:

$$\mathrm{GL}(2,\mathbb{H}) := \{A \in \mathrm{M}(2,\mathbb{H}) \mid \det_{\mathbb{H}}(A) \neq 0\},$$
$$\mathrm{SL}(2,\mathbb{H}) := \{A \in \mathrm{GL}(2,\mathbb{H}) \mid \det_{\mathbb{H}}(A) = 1\}.$$

Conjugacy Classes in $\mathrm{SL}(2,\mathbb{H})$

With the help of the conjugacy classification given below in Lemma 2.1 for $\mathrm{SL}(2,\mathbb{H})$, the isometries of $\mathrm{PSL}(2,\mathbb{H})$ may be classified into three well-defined mutually exclusive classes which are furnished in the Lemma 2.2 consequently.

Lemma 2.1 *(cf. [10, Theorem 5.5.3]) Every element of* $\mathrm{SL}(2,\mathbb{H})$ *is conjugate to one of the following matrices:*

(i) $\begin{pmatrix} \mu e^{i\theta} & 0 \\ 0 & \mu^{-1}e^{i\phi} \end{pmatrix}$, *where* $\mu \in \mathbb{R}, \mu > 0$, *and* $\theta, \phi \in [0,\pi]$.

(ii) $\begin{pmatrix} e^{i\theta} & 1 \\ 0 & e^{i\theta} \end{pmatrix}$, *where* $\theta \in [0,\pi]$.

Lemma 2.2 *Let* $\tilde{A} \in \mathrm{PSL}(2,\mathbb{H})$ *associate a lift* $A \in \mathrm{SL}(2,\mathbb{H})$. *Then*

(i) *$\tilde{A}$ is hyperbolic if A has at least one eigenvalue with a non-unit modulus.*
(ii) *$\tilde{A}$ is elliptic if A is diagonalizable and each eigenvalue of A has a unit modulus.*
(iii) *$\tilde{A}$ is parabolic if A is non-diagonalizable and each eigenvalue of A has a unit modulus.*

Conjugacy Invariants of a Quaternionic Matrix

Let us recall some well-known conjugacy invariants of quaternionic matrices in $\mathrm{M}(2,\mathbb{H})$ which are basically an immediate consequence of [3, 7].

Lemma 2.3 *(cf. [7, Section 4]) Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an element of* $\mathrm{M}(2,\mathbb{H})$. *Then the following functions on* $\mathrm{M}(2,\mathbb{H})$ *are conjugation invariants.*

(i) $\alpha_A := |a|^2|d|^2 + |b|^2|c|^2 - 2\Re(a\bar{c}d\bar{b})$.
(ii) $\beta_A := \Re((ad - bc)\bar{a} + (da - cb)\bar{d})$.
(iii) $\gamma_A := |a + d|^2 + 2\Re(ad - bc)$.
(iv) $\delta_A := \Re(a + d)$.

Observe that $\beta_A = -\phi_{-A}$ and $\delta_A = -\delta_{-A}$, for all $A \in \mathrm{M}(2,\mathbb{H})$. Since A and $-A$ are two possible lifts of element $\tilde{A} \in \mathrm{PSL}(2,\mathbb{H})$, using the conjugacy invariants β_A and δ_A for $\tilde{A}$ will leave ambiguity, though the conjugacy invariants $\alpha_A, \beta_A^2, \gamma_A$, and δ_A^2 are surely well-defined for $\tilde{A} \in \mathrm{PSL}(2,\mathbb{H})$ as because they do not depend on the choice of lift of $\tilde{A}$ in $\mathrm{PSL}(2,\mathbb{H})$ and this has been incorporated in the statements of Theorem 3.2 as one can see.

Characteristic Polynomial of a Quaternionic Matrix

Suppose that the characteristic polynomial of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2, \mathbb{H})$ is given by

$$\chi_{\mathbb{H}}(A) = x^4 - c_3 x^3 + c_2 x^2 - c_1 x + c_0, \tag{3}$$

where $c_0 = \det_{\mathbb{H}}(A)$. In view of [7, Section 4], we have

$$c_3 = 2\delta_A, \; c_2 = \gamma_A, \; c_1 = 2\beta_A, \; c_0 = \alpha_A. \tag{4}$$

If $A \in \mathrm{SL}(2, \mathbb{H})$, then $c_0 = \alpha_A = 1$. Furthermore, for $A \in \mathrm{SL}(2, \mathbb{H})$, we have

$$\begin{aligned} &\chi_{\mathbb{H}}(-A) = x^4 + c_3 x^3 + c_2 x^2 + c_1 x + 1 \\ &\text{and} \quad \chi_{\mathbb{H}}(A^{-1}) = x^4 - c_1 x^3 + c_2 x^2 - c_3 x + 1. \end{aligned} \tag{5}$$

In view of Lemma 2.3 and Equation (4), each coefficient $c_i \in \mathbb{R}$ is a conjugacy invariant of A and can be expressed in terms of the entries a, b, c and d of the matrix A.

Algebraic Characterization for Elements in $\mathrm{PSL}(2, \mathbb{H})$

First, we state the following lemma which demonstrates the relationships in between the conjugacy invariants (i.e. the co-efficients of the characteristic polynomial) and the entries of a conjugacy class representative of a semisimple element in $\mathrm{SL}(2, \mathbb{H})$.

Lemma 3.1 *Let* $A = \begin{pmatrix} \lambda e^{i\theta} & 0 \\ 0 & \lambda^{-1} e^{i\phi} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{H})$, *where* $\lambda \in \mathbb{R}, \lambda > 0$ *and* $\theta, \phi \in [0, \pi]$. *Let* $\chi_{\mathbb{H}}(A) = x^4 - c_3 x^3 + c_2 x^2 - c_1 x + 1$ *be the characteristic polynomial of A as given in Equation (3). Then the following statements are true.*

(1) $c_3 = 2(\lambda \cos(\theta) + \lambda^{-1} \cos(\phi))$.
(2) $c_2 = 4 \cos(\theta) \cos(\phi) + (\lambda^2 + \lambda^{-2})$.
(3) $c_1 = 2(\lambda^{-1} \cos(\theta) + \lambda \cos(\phi))$.
(4) $c_1 = c_3$ *if and only if* $\lambda = 1$ *or* $\theta = \phi$.
(5) $c_1 = -c_3$ *if and only if* $\theta = \pi - \phi$.
(6) $c_1 = 0$ *if and only if either* $\theta = \phi = \pi/2$ *with* $\lambda > 0$ *or,* $\theta = \pi - \phi$ *with* $\lambda = 1$.
(7) $c_2 - \left(\frac{c_1^2}{4} + 2\right) = -(\cos(\theta) - \cos(\phi))^2$ *if and only if either* $\lambda = 1$ *or,* $\theta, \phi \in \{0, \pi\}$.
(8) $c_2 - \left(\frac{c_1^2}{4} + 2\right) = (\lambda - \lambda^{-1})^2 (1 - \cos^2(\theta))$ *if and only if* $\theta = \phi$.
(9) $c_2 - \left(\frac{c_1^2}{4} - 2\right) = (\lambda + \lambda^{-1})^2 (1 - \cos^2(\theta))$ *if and only if* $\theta = \pi - \phi$.

Proof Using the Equation (2), the characteristic polynomial $\chi_{\mathbb{H}}(A)$ of A is given by:

$$\chi_{\mathbb{H}}(A) := \chi_{p(A)} = (x - \lambda e^{i\theta})(x - \lambda e^{-i\theta})(x - \lambda^{-1} e^{i\phi})(x - \lambda^{-1} e^{-i\phi})$$

Expanding the factors, we obtain

$$\begin{aligned} \chi_{\mathbb{H}}(A) &= (x^2 - \lambda(e^{i\theta} + e^{-i\theta})x + \lambda^2)(x^2 - \lambda^{-1}(e^{i\phi} + e^{-i\phi})x + \lambda^{-2}) \\ &= x^4 - (\lambda(e^{i\theta} + e^{-i\theta}) + \lambda^{-1}(e^{i\phi} + e^{-i\phi}))x^3 \\ &+ ((e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi}) + (\lambda^2 + \lambda^{-2}))x^2 \\ &- (\lambda^{-1}(e^{i\theta} + e^{-i\theta}) + \lambda(e^{i\phi} + e^{-i\phi})x + 1. \end{aligned}$$

Using the identity $e^{i\psi} + e^{-i\psi} = 2\cos(\psi)$, where $\psi \in \mathbb{R}$, we express the coefficients of $\chi_{\mathbb{H}}(A)$ as

$$\begin{aligned} c_3 &= 2(\lambda \cos(\theta) + \lambda^{-1} \cos(\phi)), \; c_2 = 4\cos(\theta)\cos(\phi) \\ &+ (\lambda^2 + \lambda^{-2}), \; c_1 = 2(\lambda^{-1} \cos(\theta) + \lambda \cos(\phi)). \end{aligned} \tag{6}$$

From these, we derive the following equations:

$$\begin{aligned} c_3 - c_1 &= 2(\lambda - \lambda^{-1})(\cos(\theta) - \cos(\phi)) \quad \text{and} \\ c_3 + c_1 &= 2(\lambda + \lambda^{-1})(\cos(\theta) + \cos(\phi)). \end{aligned} \tag{7}$$

Note that statements $(1) - (6)$ follows from Equations (6) and (7).

Now, using Equation (6), we derive the following relationships between c_1 and c_2:

$$\begin{aligned} c_2 - \left(\frac{c_1^2}{4} + 2\right) &= (\lambda - \lambda^{-1})^2 - (\lambda^{-1} \cos(\theta) - \lambda \cos(\phi))^2, \quad \text{and} \\ c_2 - \left(\frac{c_1^2}{4} - 2\right) &= (\lambda + \lambda^{-1})^2 - (\lambda^{-1} \cos(\theta) - \lambda \cos(\phi))^2. \end{aligned}$$

To prove statements $(7) - (9)$, we analyze the following equivalences:

(A) $c_2 - \left(\frac{c_1^2}{4} + 2\right) = -(\cos(\theta) - \cos(\phi))^2$ if and only if

$$\begin{aligned} &(\lambda - \lambda^{-1})^2 - (\lambda^{-1} \cos(\theta) - \lambda \cos(\phi))^2 = -(\cos(\theta) - \cos(\phi))^2 \\ &\Leftrightarrow \lambda^2 + \lambda^{-2} - 2 - \lambda^{-2} \cos^2(\theta) - \lambda^2 \cos^2(\phi) = -\cos^2(\theta) - \cos^2(\phi) \\ &\Leftrightarrow (\lambda^2 - 1)(1 - \cos^2(\phi)) = -(\lambda^{-2} - 1)(1 - \cos^2(\theta)) \\ &\Leftrightarrow 2(\lambda^2 - 1)\sin^2(\phi) = 2(\lambda^2 - 1)(\lambda^{-2} \sin^2(\theta)). \end{aligned}$$

Also, if $\lambda \neq 1$, then

$$\lambda^2 \sin^2(\phi) = \sin^2(\theta) \Leftrightarrow \sin^2(\phi) = \sin^2(\theta) = 0 \Leftrightarrow \theta, \phi \in \{0, \pi\}$$

.

(B) $c_2 - \left(\frac{c_1^2}{4} + 2\right) = (\lambda - \lambda^{-1})^2 (1 - \cos^2(\theta))$ if and only if

$$\begin{aligned}
&(\lambda-\lambda^{-1})^2-(\lambda^{-1}\cos(\theta)-\lambda\cos(\phi))^2\\
&\quad=(\lambda-\lambda^{-1})^2(1-\cos^2(\theta))\\
&\Leftrightarrow(\lambda^{-1}\cos(\theta)-\lambda\cos(\phi))^2=(\lambda-\lambda^{-1})^2\cos^2(\theta)\\
&\Leftrightarrow\lambda^{-2}\cos^2(\theta)+\lambda^2\cos^2(\phi)-2\cos(\theta)\cos(\phi)\\
&\quad=\lambda^{-2}\cos^2(\theta)+\lambda^2\cos^2(\theta)-2\cos^2(\theta)\\
&\Leftrightarrow\lambda^2(\cos(\theta)+\cos(\phi))(\cos(\theta)-\cos(\phi))\\
&\quad=2\cos(\theta)(\cos(\theta)-\cos(\phi))\\
&\Leftrightarrow\cos(\theta)=\cos(\phi)\Leftrightarrow\theta=\phi.
\end{aligned}$$

Here, we used that $(\lambda^2-2)\cos(\theta)=-\lambda^2\cos(\phi)\Leftrightarrow\lambda=1$ and $\cos(\theta)=\cos(\phi)$, where $\lambda>0$ and $\theta,\phi\in[0,\pi]$.

(C) $c_2-\left(\frac{c_1^2}{4}-2\right)=(\lambda+\lambda^{-1})^2(1-\cos^2(\theta))$ if and only if

$$\begin{aligned}
&(\lambda+\lambda^{-1})^2-(\lambda^{-1}\cos(\theta)-\lambda\cos(\phi))^2\\
&\quad=(\lambda+\lambda^{-1})^2(1-\cos^2(\theta))\\
&\Leftrightarrow(\lambda^{-1}\cos(\theta)-\lambda\cos(\phi))^2=(\lambda+\lambda^{-1})^2\cos^2(\theta)\\
&\Leftrightarrow\lambda^{-2}\cos^2(\theta)+\lambda^2\cos^2(\phi)-2\cos(\theta)\cos(\phi)\\
&\quad=\lambda^{-2}\cos^2(\theta)+\lambda^2\cos^2(\theta)+2\cos^2(\theta)\\
&\Leftrightarrow\lambda^2(\cos(\theta)-\cos(\phi))(\cos(\theta)+\cos(\phi))\\
&\quad=-2\cos(\theta)(\cos(\theta)+\cos(\phi)).\\
&\Leftrightarrow\cos(\theta)=-\cos(\phi)\Leftrightarrow\theta=\pi-\phi.
\end{aligned}$$

Here, we used that $(\lambda^2+2)\cos(\theta)=\lambda^2\cos(\phi)\Leftrightarrow\cos(\theta)=\cos(\phi)=0$, where $\lambda>0$ and $\theta,\phi\in[0,\pi]$.

This completes the proof. □

Now we state our main result in the following that provides a modified algebraic characterization of elements in $\mathrm{PSL}(2,\mathbb{H})$ compared to [4, Theorem 1.1].

Theorem 3.2 *Let $\tilde{A}$ be an element of $\mathrm{PSL}(2,\mathbb{H})$, where $A\in\mathrm{SL}(2,\mathbb{H})$ is a lift of it. Suppose that $\lambda\in\mathbb{R}$ with $\lambda>0$ and $\theta,\phi\in[0,\pi]$. Then A is conjugate to one of the matrices given below.*

(1) $\begin{pmatrix}\lambda e^{i\theta} & 0\\ 0 & \lambda^{-1}e^{i\phi}\end{pmatrix}$, $\lambda\neq1$, $\theta\neq\phi,\pi-\phi$ *if and only if* $\beta_A^2\neq\delta_A^2$.

(2) $\begin{pmatrix}\lambda e^{i\theta} & 0\\ 0 & \lambda^{-1}e^{i(\pi-\theta)}\end{pmatrix}$, $\lambda\neq1$, $\theta\neq\pi/2$ *if and only if* $\beta_A=-\delta_A\neq0$, $\gamma_A\geq\beta_A^2-2$. *Moreover,* $\theta\in\{0,\pi\}$ *if and only if* $\gamma_A=\beta_A^2-2$.

(3) $\begin{pmatrix}\lambda i & 0\\ 0 & \lambda^{-1}i\end{pmatrix}$, $\lambda\neq1$ *if and only if* $\beta_A=\delta_A=0$, $\gamma_A>\beta_A^2+2$.

(4) $\begin{pmatrix}\lambda e^{i\theta} & 0\\ 0 & \lambda^{-1}e^{i\theta}\end{pmatrix}$, $\lambda\neq1$, $\theta\neq\pi/2$ *if and only if* $\beta_A=\delta_A\neq0$, $\gamma_A\geq\beta_A^2+2$. *Moreover,* $\theta\in\{0,\pi\}$ *if and only if* $\gamma_A=\beta_A^2+2$, $|\beta_A|>2$.

(5) $\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{i\phi}\end{pmatrix}$, $\theta\neq\phi,\pi-\phi$ *if and only if* $\beta_A=\delta_A\neq0$, $\gamma_A<\beta_A^2+2$.

(6) $\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{i\theta}\end{pmatrix}$, $\theta\neq\pi/2$ *if and only if* $\beta_A=\delta_A\neq0$, $\gamma_A=\beta_A^2+2$, *and* $|\beta_A|\leq2$. *Moreover,* $\theta=0$ *if and only if* $\beta_A=2$; *and* $\theta=\pi$ *if and only if* $\beta_A=-2$.

(7) $\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{i(\pi-\theta)}\end{pmatrix}$ *if and only if* $\beta_A=\delta_A=0$, $-2\leq\gamma_A\leq2$. *Moreover,* $\theta\in\{0,\pi\}$ *if and only if* $\gamma_A=-2$ *and* $\theta=\pi/2$ *if and only if* $\gamma_A=2$.

(8) $\begin{pmatrix}e^{i\theta} & 1\\ 0 & e^{i\theta}\end{pmatrix}$ *if and only if* $\beta_A=\delta_A$, $\gamma_A=\beta_A^2+2$, $|\beta_A|\leq2$, *and A is non-diagonalizable.*

Proof Let $\chi_{\mathbb{H}}(A)=x^4-c_3x^3+c_2x^2-c_1x+1$ be the characteristic polynomial of A. Using Equation (4), we have

$$c_3=2\delta_A,\quad c_2=\gamma_A,\quad\text{and}\quad c_1=2\beta_A.$$

Recall that A is conjugate to one of the matrices given in Lemma 2.1. By Lemma 3.1, we can express the coefficients c_1,c_2, and c_3 in terms of the eigenvalues of A to establish explicit equivalence relations among them.

For example, Lemma 3.1 states that $c_1=-c_3$ if and only if $\theta=\pi-\phi$. This condition algebraically translates to $2\beta_A=-2\delta_A$, or equivalently, $\beta_A=-\delta_A$.

By substituting the respective conjugacy invariants back into these equivalences, a straightforward algebraic calculations directly yield the necessary and sufficient criteria for the various dynamical classes, thereby completing the proof of the theorem. □

An Illustrative Example

We consider the matrix

$$A=\begin{pmatrix}2e^{\frac{i\pi}{3}} & 0\\ 0 & \frac{1}{2}e^{\frac{i2\pi}{3}}\end{pmatrix}\in\mathrm{SL}(2,\mathbb{H}),$$

which is a 2-rotatory hyperbolic element (see Table 1). By Lemma 3.1, the associated characteristic polynomial

$$\chi_{\mathbb{H}}(A) = x^4 - c_3x^3 + c_2x^2 - c_1x + 1$$

has coefficients

$$c_3 = \frac{3}{2}, \qquad c_1 = -\frac{3}{2}, \qquad c_2 = \frac{13}{4}.$$

Using the earlier invariant formulation, we obtain

$$d_1 = \frac{c_1^2}{4c_0} = \frac{9}{16} \text{ and } d_3 = \frac{c_3^2}{4c_0} = \frac{9}{16} \text{ which gives us } d_1 = d_3.$$

In this situation, the previous invariant criterion does not distinguish the relevant dynamical behavior in a decisive manner. The refined criterion resolves this directly. Since $c_1 = -c_3 \neq 0$, and equivalently $\beta_A = -\delta_A \neq 0$, Theorem 3.2 applies and identifies the element as reversible.

This example demonstrates how the relation $c_1 = -c_3$ captures reversibility information that is not detected by the earlier invariant condition alone and highlights the conceptual advantage of our characterization given in Theorem 3.2.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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