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BRIEF COMMUNICATIONS

Homogenization of Hyperbolic Equations: Operator Estimates with Correctors Taken into Account

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To the memory of Izrael Moiseevich Gelfand

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Abstract. An elliptic second-order differential operator $A_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$ on $L_2(\mathbb{R}^d)$ is considered, where $\varepsilon > 0$, $g(\mathbf{x})$ is a positive definite and bounded matrix-valued function periodic with respect to some lattice, and $b(\mathbf{D})$ is a matrix first-order differential operator. Approximations for small ε of the operator-functions $\cos(\tau A_\varepsilon^{1/2})$ and $A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2})$ in various operator norms are obtained. The results can be applied to study the behavior of the solution of the Cauchy problem for the hyperbolic equation $\partial_\tau^2 \mathbf{u}_\varepsilon(\mathbf{x},\tau) = -A_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},\tau)$.

Key words: periodic differential operators, homogenization, hyperbolic equations, operator error estimates.

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1. A class of operators. Let Γ be a lattice in $\mathbb{R}^d$, and let Ω be the elementary cell of this lattice. For functions on $\mathbb{R}^d$, we use the notation $f^\varepsilon(\mathbf{x}) := f(\mathbf{x}/\varepsilon)$, $\varepsilon > 0$. On $L_2(\mathbb{R}^d;\mathbb{C}^n)$ we consider a self-adjoint elliptic second-order differential operator (DO) of the form

$$A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \qquad \varepsilon > 0. \tag{1}$$

Here $g(\mathbf{x})$ is a Γ-periodic Hermitian $(m \times m)$-matrix-valued function such that $g, g^{-1} \in L_\infty$ and $g(\mathbf{x}) > 0$. The operator $b(\mathbf{D})$, $\mathbf{D} = -i\nabla$, is the first-order DO given by $b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$. Here the b_j are constant $m\times n$ matrices such that $m \geqslant n$. It is assumed that the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^d b_j \xi_j$ has maximal rank: $\operatorname{rank} b(\boldsymbol{\xi}) = n$ for $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. The operator (1) is generated by the closed quadratic form $(g^\varepsilon b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u})_{L_2(\mathbb{R}^d)}$, $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$.

2. Operator error estimates. Homogenization problems for the operator (1) were studied in the papers [1]–[3] by Birman and Suslina by using the operator-theoretic (spectral) approach. In [1] it was shown that, as $\varepsilon \to 0$, the resolvent $(A_\varepsilon + I)^{-1}$ converges to the resolvent of the effective operator A^0 in the operator norm on $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and the norm of the difference of the resolvents is of sharp order $O(\varepsilon)$. In [2] a more accurate approximation for the resolvent $(A_\varepsilon + I)^{-1}$ with corrector taken into account was found, the error being of order $O(\varepsilon^2)$. In [3] an approximation for the resolvent of the operator A_ε in the “energy” norm (i.e., the $(L_2 \to H^1)$-norm) with corrector taken into account was obtained, the error being of order $O(\varepsilon)$. Similar results for the semigroup $e^{-\tau A_\varepsilon}$, $\tau > 0$, were obtained in [4]–[7].

Error estimates in operator norms are called *operator error estimates* in homogenization theory. A different approach to such estimates was suggested by Zhikov and Pastukhova ([8], [9]; see also the survey [10]).

The operator-theoretic approach was applied to Schrödinger-type and hyperbolic equations in [11], where the exponential $e^{-i\tau A_\varepsilon}$ and the cosine $\cos(\tau A_\varepsilon^{1/2})$, $\tau \in \mathbb{R}$, were studied. It turned out that it is impossible to approximate these operators in the $(L_2 \to L_2)$-norm. The type of norm must be changed. The result of [11] for the cosine operator is as follows:

$$\| \cos(\tau A_\varepsilon^{1/2}) - \cos(\tau (A^0)^{1/2}) \|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C(1+|\tau|)\varepsilon. \tag{2}$$

An approximation for the operator $A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})$ was found by Meshkova [12]:

$$\| A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2}) - (A^0)^{-1/2} \sin(\tau (A^0)^{1/2}) \|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C(1+|\tau|)\varepsilon. \tag{3}$$

In [13] and [14] it was shown that, in the general case, estimates (2) and (3) are sharp with respect to both the type of the operator norm and the dependence on the parameter τ. However, under some additional assumptions (when Condition 1 or Condition 2 specified below holds) the following improvement was obtained:

$$\begin{aligned} \| \cos(\tau A_\varepsilon^{1/2}) - \cos(\tau (A^0)^{1/2}) \|_{H^{3/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} &\leqslant C(1+|\tau|)^{1/2}\varepsilon, \\ \| A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2}) - (A^0)^{-1/2} \sin(\tau (A^0)^{1/2}) \|_{H^{1/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} &\leqslant C(1+|\tau|)^{1/2}\varepsilon. \end{aligned}$$

Similar results for Schrödinger-type equations were obtained in [15] and [16].

In [17]–[19] the question about approximations of the operator exponential with correctors taken into account in the $(H^s \to L_2)$-norm with an error $O(\varepsilon^2)$ and in the $(H^s \to H^1)$-norm with an error $O(\varepsilon)$ (for a fixed τ) was studied. It turned out that it is impossible to obtain such approximations for the operator $e^{-i\tau A_\varepsilon}$; they were found for the operator $e^{-i\tau A_\varepsilon}(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon)$ (Λ and Π_ε are defined in Sections 3 and 4 below).

For the operator $A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})$, it is possible to obtain approximations with correctors due to the presence of the "smoothing" factor $A_\varepsilon^{-1/2}$: an approximation in the energy norm was found in [12] (in [14] the sharpness of the result was confirmed and an improvement under additional assumptions was obtained) and an approximation in the $(H^s \to L_2)$-norm with an error $O(\varepsilon^2)$ was found in [20].

In this paper we present new results on approximations with correctors for the operators $\cos(\tau A_\varepsilon^{1/2})(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon)$ and $\sin(\tau A_\varepsilon^{1/2})$.

3. Effective characteristics. We define a constant $m \times m$ matrix g^0 called the *effective matrix*. Suppose that an $(n \times m)$-matrix-valued function $\Lambda(\mathbf{x})$ is a Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \qquad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0.$$

We put $\widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m)$ and $g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}$. It turns out that the matrix g^0 is positive. The *effective operator* is given by $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ on the domain $H^2(\mathbb{R}^d; \mathbb{C}^n)$.

By using the unitary Gelfand transform the operator $A = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$ can be decomposed into the direct integral $\int_{\widetilde{\Omega}} \oplus A(\mathbf{k})\, d\mathbf{k}$ of operators $A(\mathbf{k})$ acting on $L_2(\Omega; \mathbb{C}^n)$. Here $\widetilde{\Omega}$ is the central Brillouin zone of the dual lattice $\widetilde{\Gamma}$. The parameter $\mathbf{k}$ is called the *quasi-momentum*. The operator $A(\mathbf{k})$ is given by the expression $A(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k})$ with periodic boundary conditions.

The spectrum of the operator $A(\mathbf{k})$ is discrete. We apply perturbation theory. Obviously, $\mathfrak{N} := \operatorname{Ker} A(0) = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \ \mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}$. This means that the number $\lambda = 0$ is an isolated eigenvalue of multiplicity n of the "unperturbed" operator $A(0)$. Therefore, for $|\mathbf{k}| \leqslant t_0$, the "perturbed" operator $A(\mathbf{k})$ has exactly n eigenvalues on the interval $[0, \delta]$ (counted with multiplicities), while the interval $(\delta, 3\delta)$ is free of the spectrum. (We control the numbers δ and t_0 explicitly.) Let $\mathbf{k} = t\boldsymbol{\theta}$, where $t := |\mathbf{k}|$ and $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. According to analytic perturbation theory (see [21]), for $t \leqslant t_0$, there exist real-analytic (in t) branches of the eigenvalues $\lambda_l(t; \boldsymbol{\theta})$ and branches of the eigenvectors $\varphi_l(t; \boldsymbol{\theta})$ of the operator $A(\mathbf{k}) =: A(t; \boldsymbol{\theta})$, $l = 1, \dots, n$. The vectors $\varphi_l(t; \boldsymbol{\theta})$, $l = 1, \dots, n$, form an orthonormal basis in the eigenspace of $A(t;\boldsymbol{\theta})$ corresponding to the interval $[0,\delta]$. For small t, we have the convergent power series expansions

$$\lambda_l(t;\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \mu_l(\boldsymbol{\theta})t^3 + \nu_l(\boldsymbol{\theta})t^4 + \dots, \tag{4}$$

$$\varphi_l(t;\boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta}) + t\psi_l(\boldsymbol{\theta}) + \dots \tag{5}$$

for $l=1,\dots,n$. It is easily seen that $\gamma_l(\boldsymbol{\theta}) \geqslant c_* > 0$. The "embryos" $\omega_l(\boldsymbol{\theta})$, $l=1,\dots,n$, form an orthonormal basis in the subspace $\mathfrak{N}$. The matrix $S(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta})$ is called the *spectral germ* of the operator family $A(t;\boldsymbol{\theta})$ at $t=0$. As shown in [1], the numbers $\gamma_l(\boldsymbol{\theta})$ and the elements $\omega_l(\boldsymbol{\theta})$ are eigenvalues and eigenvectors of the germ: $S(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta})$, $l=1,\dots,n$.

We need the operator $N(\boldsymbol{\theta}) : \mathfrak{N} \to \mathfrak{N}$ given by

$$N(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* L(\boldsymbol{\theta}) b(\boldsymbol{\theta}), \text{ where } L(\boldsymbol{\theta}) := |\Omega|^{-1} \int_\Omega (\Lambda(\mathbf{x})^* b(\boldsymbol{\theta})^* \widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^* b(\boldsymbol{\theta}) \Lambda(\mathbf{x}))\, d\mathbf{x}.$$

The operator $N(\boldsymbol{\theta})$ can also be described in terms of the coefficients of the power series expansions (4) and (5): $N(\boldsymbol{\theta}) = N_0(\boldsymbol{\theta}) + N_*(\boldsymbol{\theta})$, where the first term on the right-hand side is diagonal in the basis $\omega_1(\boldsymbol{\theta}),\dots,\omega_n(\boldsymbol{\theta})$ and is given by

$$N_0(\boldsymbol{\theta}) = \sum_{l=1}^{n} \mu_l(\boldsymbol{\theta}) (\,\cdot\,, \omega_l(\boldsymbol{\theta}))_{L_2(\Omega)} \omega_l(\boldsymbol{\theta})$$

and the second term

$$N_*(\boldsymbol{\theta}) = \sum_{l=1}^{n} \gamma_l(\boldsymbol{\theta}) ((\,\cdot\,, P\psi_l(\boldsymbol{\theta}))_{L_2(\Omega)} \omega_l(\boldsymbol{\theta}) + (\,\cdot\,, \omega_l(\boldsymbol{\theta}))_{L_2(\Omega)} P\psi_l(\boldsymbol{\theta}))$$

has zero diagonal in this basis. Here P is the orthogonal projection onto $\mathfrak{N}$.

We put $S(\mathbf{k}) := t^2 S(\boldsymbol{\theta}) = b(\mathbf{k})^* g^0 b(\mathbf{k})$, $L(\mathbf{k}) := tL(\boldsymbol{\theta})$, and $N(\mathbf{k}) := t^3 N(\boldsymbol{\theta}) = b(\mathbf{k})^* L(\mathbf{k}) b(\mathbf{k})$ for $\mathbf{k} = t\boldsymbol{\theta} \in \mathbb{R}^d$.

4. Main results. We introduce a smoothing operator Π_ε on $L_2(\mathbb{R}^d;\mathbb{C}^n)$ defined by

$$(\Pi_\varepsilon f)(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle} \widehat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi},$$

where $\widehat{f}(\boldsymbol{\xi})$ is the Fourier image of f.

Let $G(\mathbf{D})$ and $\widetilde{G}(\mathbf{D})$ be, respectively, the second- and zero-order pseudodifferential operators with symbols

$$G(\boldsymbol{\xi}) = \frac{1}{\pi} \int_0^\infty (S(\boldsymbol{\xi}) + \zeta I)^{-1} N(\boldsymbol{\xi}) (S(\boldsymbol{\xi}) + \zeta I)^{-1} \zeta^{1/2}\, d\zeta,$$

$$\widetilde{G}(\boldsymbol{\xi}) = -\frac{1}{\pi} \int_0^\infty (S(\boldsymbol{\xi}) + \zeta I)^{-1} N(\boldsymbol{\xi}) (S(\boldsymbol{\xi}) + \zeta I)^{-1} \zeta^{-1/2}\, d\zeta.$$

We put

$$J^0_{\cos,\varepsilon}(\tau) := \cos(\tau A_\varepsilon^{1/2})(I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon) - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon)\cos(\tau (A^0)^{1/2}),$$
$$J^0_{\sin,\varepsilon}(\tau) := A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2}) - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon)(A^0)^{-1/2}\sin(\tau (A^0)^{1/2}),$$
$$J_{\cos,\varepsilon}(\tau) := J^0_{\cos,\varepsilon}(\tau) + \varepsilon\int_0^\tau \cos((\tau-\widetilde{\tau})(A^0)^{1/2})G(\mathbf{D})\sin(\widetilde{\tau}(A^0)^{1/2})\,d\tau + \varepsilon\int_0^\tau \sin((\tau-\widetilde{\tau})(A^0)^{1/2})G(\mathbf{D})\cos(\widetilde{\tau}(A^0)^{1/2})\,d\tau,$$
$$J_{\sin,\varepsilon}(\tau) := J^0_{\sin,\varepsilon}(\tau) - \varepsilon\widetilde{G}(\mathbf{D})\sin(\tau(A^0)^{1/2}) - \varepsilon\int_0^\tau (A^0)^{-1/2}\cos((\tau-\widetilde{\tau})(A^0)^{1/2})G(\mathbf{D})\cos(\widetilde{\tau}(A^0)^{1/2})\,d\tau + \varepsilon\int_0^\tau (A^0)^{-1/2}\sin((\tau-\widetilde{\tau})(A^0)^{1/2})G(\mathbf{D})\sin(\widetilde{\tau}(A^0)^{1/2})\,d\tau.$$

Theorem 1. *For $\tau \in \mathbb{R}$ and $\varepsilon > 0$,*

$$\|J_{\cos,\varepsilon}(\tau)\|_{H^4(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(1+|\tau|)^2\varepsilon^2, \tag{6}$$
$$\|J_{\sin,\varepsilon}(\tau)\|_{H^3(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(1+|\tau|)^2\varepsilon^2, \tag{7}$$
$$\|J^0_{\cos,\varepsilon}(\tau)\|_{H^3(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C(1+|\tau|)\varepsilon, \tag{8}$$
$$\|J^0_{\sin,\varepsilon}(\tau)\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C(1+|\tau|)\varepsilon. \tag{9}$$

Estimate (9) was obtained in [12], an analogue of estimate (7) was found in [20] (the difference is in the form of the corrector, which is not uniquely determined). Estimates (6) and (8) are new.

Under some additional assumptions Theorem 1 can be improved.

Condition 1. *$N(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*

Condition 2. *$N_0(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, i.e., $\mu_l(\boldsymbol{\theta}) \equiv 0$ for $l = 1, \dots, n$. Moreover, the number p of different eigenvalues of the spectral germ $S(\boldsymbol{\theta})$ does not depend on $\boldsymbol{\theta}$.*

Theorem 2. *Suppose that Condition 1 or 2 is satisfied. Then, for any $\tau \in \mathbb{R}$ and $\varepsilon > 0$,*

$$\|J_{\cos,\varepsilon}(\tau)\|_{H^3(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(1+|\tau|)\varepsilon^2, \tag{10}$$
$$\|J_{\sin,\varepsilon}(\tau)\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(1+|\tau|)\varepsilon^2, \tag{11}$$
$$\|J^0_{\cos,\varepsilon}(\tau)\|_{H^{5/2}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C(1+|\tau|)^{1/2}\varepsilon, \tag{12}$$
$$\|J^0_{\sin,\varepsilon}(\tau)\|_{H^{3/2}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C(1+|\tau|)^{1/2}\varepsilon. \tag{13}$$

Estimate (13) was proved by the authors in [14], and estimates (10)–(12) are new.

Note that under Condition 1 we have $J_{\cos,\varepsilon}(\tau) = J^0_{\cos,\varepsilon}(\tau)$ and $J_{\sin,\varepsilon}(\tau) = J^0_{\sin,\varepsilon}(\tau)$. Some sufficient conditions ensuring the fulfillment of Condition 1 or 2 can be found in [2; Sec. 4].

Proposition 3. 1. *Suppose that $A_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$, where $g(\mathbf{x})$ is a symmetric matrix with real entries. Then Condition 1 is satisfied.*

2. *Suppose that the matrices $g(\mathbf{x})$ and $b(\boldsymbol{\theta})$ have real entries and the spectrum of the germ $S(\boldsymbol{\theta})$ is simple for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then Condition 2 is satisfied.*

Next, we confirm the sharpness of the results with respect to both the type of the norm and the dependence of the estimates on τ. The following result demonstrates the sharpness of Theorem 1.

Theorem 4. *Suppose that $N_0(\boldsymbol{\theta}_0) \neq 0$ for some $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$ (i.e., $\mu_l(\boldsymbol{\theta}_0) \neq 0$ for some l and $\boldsymbol{\theta}_0$).*

1. *If $\tau \neq 0$ and $s < 4$, then there does not exist a constant $C(\tau) > 0$ such that, for small ε, the following inequality holds*:

$$\|J_{\cos,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon^2. \tag{14}$$

2. *If $\tau \neq 0$ and $s < 3$, then there does not exist a constant $C(\tau) > 0$ such that, for small ε, the following inequality holds*:

$$\|J_{\sin,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon^2. \tag{15}$$

3. *If $\tau \neq 0$ and $s < 3$, then there does not exist a constant $C(\tau) > 0$ such that, for small ε, the following inequality holds*:

$$\|J^0_{\cos,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon. \tag{16}$$

4. *If $\tau \neq 0$ and $s < 2$, then there does not exist a constant $C(\tau) > 0$ such that, for small ε, the following inequality holds*:

$$\|J^0_{\sin,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon. \tag{17}$$

5. *If $s \geqslant 4$, then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau\to\infty} C(\tau)/\tau^2 = 0$ and inequality* (14) *holds for $\tau \in \mathbb{R}$ and small ε.*

6. *If $s \geqslant 3$, then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau\to\infty} C(\tau)/\tau^2 = 0$ and inequality* (15) *holds for $\tau \in \mathbb{R}$ and small ε.*

7. *If $s \geqslant 3$, then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau\to\infty} C(\tau)/|\tau| = 0$ and inequality* (16) *holds for $\tau \in \mathbb{R}$ and small ε.*

8. *If $s \geqslant 2$, then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau\to\infty} C(\tau)/|\tau| = 0$ and inequality* (17) *holds for $\tau \in \mathbb{R}$ and small ε.*

Statements 4 and 8 were proved in [14], and the other statements are new. There are examples of operators satisfying the assumptions of Theorem 4; see [2; Sec. 10.4], [15; Example 8.7], and [14; Sec. 14.3].

The following statement shows that Theorem 2 is sharp as well.

Theorem 5. *Suppose that $N_0(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ (i.e., $\mu_l(\boldsymbol{\theta}) \equiv 0$ for $l = 1,\dots,n$) and $\nu_j(\boldsymbol{\theta}_0) \neq 0$ for some j and $\boldsymbol{\theta}_0$.*

1. *If $\tau \neq 0$ and $s < 3$, then there does not exist a constant $C(\tau) > 0$ such that inequality* (14) *holds for small ε.*

2. *If $\tau \neq 0$ and $s < 2$, then there does not exist a constant $C(\tau) > 0$ such that inequality* (15) *holds for small ε.*

3. *If $\tau \neq 0$ and $s < 5/2$, then there does not exist a constant $C(\tau) > 0$ such that inequality* (16) *holds for small ε.*

4. *If $\tau \neq 0$ and $s < 3/2$, then there does not exist a constant $C(\tau) > 0$ such that inequality* (17) *holds for small ε.*

5. *If $s \geqslant 3$, then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau\to\infty} C(\tau)/|\tau| = 0$ and inequality* (14) *holds for $\tau \in \mathbb{R}$ and small ε.*

6. *If $s \geqslant 2$, then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau\to\infty} C(\tau)/|\tau| = 0$ and inequality* (15) *holds for $\tau \in \mathbb{R}$ and small ε.*

7. *If $s \geqslant 5/2$, then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau\to\infty} C(\tau)/|\tau|^{1/2} = 0$ and inequality* (16) *holds for $\tau \in \mathbb{R}$ and small ε.*

8. *If $s \geqslant 3/2$, then there does not exist a positive function $C(\tau)$ such that $\lim_{\tau\to\infty} C(\tau)/|\tau|^{1/2} = 0$ and inequality* (17) *holds for $\tau \in \mathbb{R}$ and small ε.*

Remark 6. According to [16; Lemma 5.8], in the one-dimensional case, for the operator $A_\varepsilon = -\frac{d}{dx} g^\varepsilon(x) \frac{d}{dx}$, the expansion (4) of $\lambda_1(k)$ takes the form $\lambda_1(k) = \gamma k^2 + \nu k^4 + \dots$, where $\nu \neq 0$, provided that the periodic function $g(x)$ is nonconstant. The authors believe that, in the multidimensional case, as a rule, $\nu_j(\boldsymbol{\theta}) \neq 0$.

Remark 7. 1. Using interpolation, we can deduce "intermediate" results from Theorems 1 and 2. For instance, under the assumptions of Theorem 1 we have $\|J_{\cos,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_s(1+|\tau|)^{s/2}\varepsilon^{s/2}$ for $0 \leqslant s \leqslant 4$.

2. Theorems 1 and 2 make it possible to deduce qualified error estimates for large values of time $\tau = O(\varepsilon^{-\alpha})$, where $0 < \alpha < 1$ in the general case and $0 < \alpha < 2$ if Condition 1 or 2 is fulfilled.

5. Application to the Cauchy problem. The results can be applied to study the behavior of the solution $\mathbf{u}_\varepsilon(\mathbf{x},\tau)$, $\mathbf{x} \in \mathbb{R}^d$, $\tau \in \mathbb{R}$, of the Cauchy problem for the hyperbolic equation with initial data in a special class:

$$\partial_\tau^2 \mathbf{u}_\varepsilon(\mathbf{x},\tau) = -A_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},\tau),$$
$$\mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}) + \varepsilon\Lambda^\varepsilon(\mathbf{x})b(\mathbf{D})(\Pi_\varepsilon\boldsymbol{\phi})(\mathbf{x}), \quad \partial_\tau \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}),$$

where $\boldsymbol{\phi}, \boldsymbol{\psi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. The solution can be represented as

$$\mathbf{u}_\varepsilon(\,\cdot\,,\tau) = \cos(\tau A_\varepsilon^{1/2})(I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon)\boldsymbol{\phi} + A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2})\boldsymbol{\psi}. \tag{18}$$

Representation (18) and Theorems 1 and 2 allow us to obtain approximations for the solution $\mathbf{u}_\varepsilon(\,\cdot\,,\tau)$ in the norm on $L_2(\mathbb{R}^d;\mathbb{C}^n)$ or $H^1(\mathbb{R}^d;\mathbb{C}^n)$, provided that the functions $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ belong to suitable Sobolev classes.

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