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On some subtheories of strong dependent choice

Juan P. Aguilera[1] · Yudai Suzuki[2] · Keita Yokoyama[3]

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Abstract
In this paper, we give characterizations of the set of Π^1_{e+2}-consequences, Σ^1_{e+1}-consequences and $\mathsf{B}(\Pi^1_{e+1})$-consequences of the axiomatic system of strong dependent choice for Σ^1_i formulas Σ^1_i-SDC_0 for $e < i$. Here, $\mathsf{B}(\Gamma)$ denotes the set generated by $\wedge, \vee, \neg$ starting from Γ.

Keywords Second-order arithmetic · Reverse mathematics · Models of arithmetic

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1 Introduction

In the proof theoretic study of formal systems of arithmetic, we are interested not only in the whole set of the sentences provable from a theory, but also some certain subsets of it. For example, the consistency strength of a theory is characterized by its Π^0_1-consequences, the proof theoretic ordinal is naturally defined by its Π^1_1-consequences, and its Π^1_2-consequences correspond to its proof theoretic dilator.

It is also important in reverse mathematics to study the set of consequences of a theory with restricted complexity. Recently, the Π^1_2-consequences of Π^1_1-CA_0 have been studied thoroughly since Π^1_1-CA_0 is not axiomatizable by any Π^1_2-sentence. The first giant leap for the study of Π^1_2-consequences of Π^1_1-CA_0 was achieved by Towsner [5]. He introduced new Π^1_2-principles whose logical strength is between Π^1_1-CA_0 and

✉ Yudai Suzuki
yudai.suzuki.research@gmail.com

Juan Aguilera
aguilera@logic.at

Keita Yokoyama
keita.yokoyama.c2@tohoku.ac.jp

1 Institute of discrete mathematics and geometry, TU Wien, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria

2 National Institute of Technology, Oyama College, Nakakuki 771, Oyama, Tochigi, Japan

3 Mathematical Institute, Tohoku University, Aramaki Aza-Aoba 6-3, Sendai, Miyagi, Japan

ATR_0. The second and third author extended Towsner's work and gave a hierarchy which covers the set of Π^1_2-consequences of Π^1_1-CA_0 [4].

In this paper, we generalize their work in [4]. For more detail, we will give a characterization of the Π^1_{e+2}-, Σ^1_{e+1}- and $\mathsf{B}(\Pi^1_{e+1})$-consequences of the system strong Σ^1_i dependent choice (Σ^1_i-SDC_0) for $e < i$ where $\mathsf{B}(\Pi^1_{e+1})$ is the set of boolean combinations of Π^1_{e+1} formulas. Since Σ^1_i-SDC_0 is Π^1_4-conservative over Π^1_i-CA_0, we also have a characterization of Π^1_{e+2}-, Σ^1_{e+1}- and $\mathsf{B}(\Pi^1_{e+1})$-consequences of Π^1_i-CA_0 for $e = 0, 1, 2$. We note that the characterization of $\mathsf{B}(\Pi^1_{e+1})$-consequences can be seen as an analogue of the study of the syntactic local reflection principles in first-order arithmetic by Beklemishev [1].

We give a general idea used throughout this paper. It is known that Σ^1_i-SDC_0 is characterized by the β_i-model reflection principle (see Theorem 2.7). Therefore, by extracting a suitable property of β_i-models, we will get a corresponding subtheory of Σ^1_i-SDC_0. For proving a correspondence, we use the Barwise-Schlipf argument. That is, we make a model $\mathcal{M}$ from a given $T \subseteq \Sigma^1_i$-SDC_0 and a T non-provable sentence σ such that $\mathcal{M}$ has a β_i-submodel $\mathcal{M}'$ satisfying Σ^1_i-SDC_0 by the compactness theorem. Then, $\mathcal{M}'$ works as a witness for the non-provability of σ from Σ^1_i-SDC_0.

In Section 2, we give a characterization of the Π^1_{e+2}-consequences of Σ^1_i-SDC_0 for $e < i$. We note that the main result of this section (Theorem 2.14) is essentially mentioned in [2], but our proof is more informative for computability-theoretic aspects (Lemma 2.13). We also give a characterization of the $\mathsf{B}(\Pi^1_{i+1})$- and Σ^1_{i+1}-consequences of Σ^1_i-SDC_0 in Section 3. We then combine this result and the results in Section 2, and give a finer analysis for the Π^1_{e+2}-consequences of Σ^1_i-SDC_0 for $e < i$. By modifying the proofs in Section 3, we give a characterization of the $\mathsf{B}(\Pi^1_{e+1})$- and Σ^1_{e+1}-consequences of Σ^1_i-SDC_0 for $e < i$ in Section 4.

2 Π^1_{e+2} formulas provable from Σ^1_i-SDC_0

In this section, we give a characterization of the set $\{\sigma \in \Pi^1_{e+2} : \Sigma^1_i\text{-}\mathsf{SDC}_0 \vdash \sigma\}$ for $e < i$. As noted in the introduction, we use a characterization of Σ^1_i-SDC based on coded β_i-models for this purpose. We begin with introducing Σ^1_i-SDC_0. We refer the reader to Simpson's textbook [3] for the basic definitions. We note that in [3], the system Σ^1_i-SDC_0 is denoted as $\mathsf{Strong}\ \Sigma^1_i$-$\mathsf{DC}_0$.

Definition 2.1 *(sequence of sets,* RCA_0*)* Let $(\bullet, \bullet) : \mathbb{N}^2 \to \mathbb{N}$ be a fixed primitive recursive pairing function. For $X \subseteq \mathbb{N}$ and $x \in \mathbb{N}$, we define the x-th segment $X_x = \{y : (x, y) \in X\}$. Then, X is regarded as the sequence $\langle X_x \rangle_{x \in \mathbb{N}}$. In this sense, we write $A \in X$ to mean $\exists x (A = X_x)$. For a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of sets, we define $X_{<k}$ as the sequence $\langle X_0, \ldots, X_{k-1} \rangle$.

Definition 2.2 Let φ be a formula. We define strong dependent choice for φ as follows.

$$\varphi\text{-}\mathsf{SDC} \equiv \forall X \exists \langle Y_n \rangle_n (Y_0 = X \wedge \forall n > 0 (\exists Z \varphi(Y_{<n}, Z) \to \varphi(Y_{<n}, Y_n))).$$

We define the axiomatic system Σ^1_i-SDC_0 as $\mathsf{ACA}_0 + \{\varphi\text{-}\mathsf{SDC} : \varphi \in \Sigma^1_i\}$.

We next see the definition of β_i-models, β_i-submodels and coded β_i-models. Intuitively, a β_i-model is a Σ^1_i-elementary submodel of $(\omega, \mathcal{P}(\omega))$.

Definition 2.3 An $\mathcal{L}_2$ structure $\mathcal{M}$ of the form $\mathcal{M} = (\omega, S)$ is called an ω-model. Let $\mathcal{M} = (\omega, S)$ be an ω-model and $i \in \omega$. We say $\mathcal{M}$ is a β_i-model if $\mathcal{M} \models \sigma \Leftrightarrow (\omega, \mathcal{P}(\omega)) \models \sigma$ for any Σ^1_i formula σ with parameters from $\mathcal{M}$.

The notions of ω-models and β_i-models can be extended to a relation of two $\mathcal{L}_2$ structures who share the first-order part.

Definition 2.4 Let $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}})$ and $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}'}, S^{\mathcal{M}'})$ be $\mathcal{L}_2$ structures. We say $\mathcal{M}'$ is an ω-submodel of $\mathcal{M}$ if $\mathbb{N}^{\mathcal{M}'} = \mathbb{N}^{\mathcal{M}}$ and $S^{\mathcal{M}'} \subseteq S^{\mathcal{M}}$.

Let $i \in \omega$. We say $\mathcal{M}'$ is a β_i-submodel of $\mathcal{M}$ if $\mathcal{M}'$ is an ω-submodel of $\mathcal{M}$ and $\mathcal{M}' \models \sigma \Leftrightarrow \mathcal{M} \models \sigma$ for any Σ^1_i formula σ with parameters from $\mathcal{M}'$.

If $S = \{S_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ is countable, then S can be coded by a single set $S_C \subseteq \omega$ by putting $S_C = \{(n, s) : s \in S_n\}$ as in Definition 2.1. This idea can be generalized to the nonstandard models. We call a model $\mathcal{M}' = (\mathbb{N}^{M'}, S_C)$ a coded ω-submodel in $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S)$ when $\mathbb{N}^{\mathcal{M}} = \mathbb{N}^{\mathcal{M}'}$ and $S_C \in S'$. If $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same Σ^1_i-sentences, then $\mathcal{M}'$ is called a coded β_i-submodel in $\mathcal{M}$. Throughout this paper, we omit the word *coded* if it is clear from the context.

Definition 2.5 *(*ACA_0*)* Let $\mathcal{M} = \langle M_x \rangle_x$ be a coded ω-model. We say $\mathcal{M}$ is a β_i-model if

$$\sigma \leftrightarrow \mathcal{M} \models \sigma$$

for any Π^1_i formula σ having parameters from $\mathcal{M}$.

Remark 2.6 We give a few comments on the above definition.

(1) In this paper, we may think that $\mathcal{M} \models \sigma$ is the abbreviation of 'the relativization $\sigma^{\mathcal{M}}$ is true' because our base theory is sufficiently strong. For the difference of $\mathcal{M} \models \sigma$ in the sense that an evaluation for σ exists and $\sigma^{\mathcal{M}}$, see Simpson's textbook and Remark 2.15 in [4].
(2) A β_0-model is just the same as an ω-model.
(3) Let $i > 0$. Then, the condition $\mathcal{M}$ is a β_i-model is equivalent to

$$(\mathcal{M} \models \sigma) \to \sigma \text{ for any } \Pi^1_i\text{-formula.}$$

Therefore, using a universal Π^1_i-formula, we can find a Π^1_i-formula stating that '$\mathcal{M}$ is a β_i-model'. Throughout this paper, we fix a Π^1_i-formula $\beta_i(\mathcal{M})$ stating '$\mathcal{M}$ is a β_i-model'.

We recall that Σ^1_i-SDC_0 is characterized by β_i-models.

Theorem 2.7 *[3, VII.7.4. and VII.7.6.] For $i > 0$, the following assertions are equivalent over* ACA_0.

(1) Σ^1_i-SDC_0,
(2) the β_i*-model reflection:* $\forall X \exists \mathcal{M}(X \in \mathcal{M} \wedge \beta_i(\mathcal{M}))$,
(3) $\forall X(\sigma(X) \to \exists \mathcal{M}(X \in \mathcal{M} \wedge \beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma(X)))$ *for* $\sigma \in \Sigma^1_{i+2}$.

Following these equivalences, it is shown that Σ^1_i-SDC_0 is axiomatizable by a Π^1_{i+2}-sentence but is not axiomatizable by any Σ^1_{i+2}-sentence. In fact, it follows from (3) that Σ^1_i-SDC_0 proves $\sigma \to \mathsf{Con}(\mathsf{ACA}_0 + \sigma)$ for any Σ^1_{i+2}-sentence σ where $\mathsf{Con}(\mathsf{ACA}_0 + \sigma)$ is the Π^0_1-formula asserting the consistency of $\mathsf{ACA}_0 + \sigma$. Hence, if Σ^1_i-SDC_0 were axiomatized by a Σ^1_{i+2}-sentence σ, then σ would prove $\mathsf{Con}(\mathsf{ACA}_0 + \sigma)$. However, this contradicts Gödel's second incompleteness theorem.

Since Σ^1_i-SDC_0 is finitely axiomatizable and not axiomatizable by any Σ^1_{i+2}-sentence, the set $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ of Π^1_{e+2}-consequences of Σ^1_i-SDC_0 is strictly weaker than Σ^1_i-SDC_0 for $e < i$.

In the remaining of this section, we give a characterization of the set $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ based on a weaker variant of (2) in Theorem 2.7 for $i > 1$ and $e < i$. We note that the case $i = 1$ has already been studied in [4].

Definition 2.8 Let $n > 0$. We define a formula $\beta^i(\mathcal{M}_0, \ldots, \mathcal{M}_n)$ by

$$\bigwedge_{m<n} (\mathcal{M}_m \in \mathcal{M}_{m+1}) \wedge \bigwedge_{m<n} \mathcal{M}_n \models \beta_i(\mathcal{M}_m).$$

We also define a formula $\beta^i_e(\mathcal{M}_0, \ldots, \mathcal{M}_n)$ by $\beta^i(\mathcal{M}_0, \ldots, \mathcal{M}_n) \wedge \beta_e(\mathcal{M}_n)$.

We remark that if $\beta^i(\mathcal{M}_0, \ldots, \mathcal{M}_n)$, then $\mathcal{M}_{m'} \in \mathcal{M}_m$ and $\mathcal{M}_m \models \beta_i(\mathcal{M}_{m'})$ for any $m' < m < n + 1$. If $\beta_e(\mathcal{M}_n)$ in addition, then $\beta_e(\mathcal{M}_m)$ for any $e \leq i$ and $m < n$. In particular, $\beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_n)$ means that $\mathcal{M}_0 \in \cdots \in \mathcal{M}_n$ and $\beta_i(\mathcal{M}_m)$ for all m.

Definition 2.9 Let σ, τ be sentences. For $i, e, n \in \omega$, we define $\beta^i_e\mathsf{RFN}(\sigma; n; \tau)$ as follows.

$$\beta^i_e\mathsf{RFN}(\sigma; n; \tau) \equiv \forall X \exists \mathcal{M}_0, \ldots, \mathcal{M}_n \text{ with the following properties.}$$

- $X \in \mathcal{M}_0$,
- $\beta^i_e(\mathcal{M}_0, \ldots, \mathcal{M}_n)$,
- $\mathcal{M}_m \models \mathsf{ACA}_0$ for all $m < n + 1$,
- $\mathcal{M}_0 \models \sigma$ and $\mathcal{M}_n \models \tau$.

We sometimes omit σ or τ when they are trivial. For example, we write $\beta^i_e\mathsf{RFN}(n; \tau)$ instead of $\beta^i_e\mathsf{RFN}(0 = 0; n; \tau)$.

Remark 2.10 When we work in a theory including ACA_0 and $e > 0$, then we may omit the condition $\mathcal{M}_m \models \mathsf{ACA}_0$ from the above definition because ACA_0 proves that any β_e-model is a model of ACA_0.

Remark 2.11 We note that $\beta^i_e\mathsf{RFN}(n)$ is first introduced in [2, Section 2] as $\psi_e(i, n)$. In that paper, it is shown that the syntactic reflection of Σ^1_i-SDC_0 for Π^1_{e+2}-formulas is equivalent to $\forall n.\psi_e(i, n)$ over ACA_0. In this section, we consider a standard analogue of this result. We also note that the concept of $\beta^i_e\mathsf{RFN}(n; \tau)$ is introduced in [4].

By iterating (2) in Theorem 2.7 finitely many times, Σ^1_i-SDC_0 proves

$$\forall X \exists \mathcal{M}_0, \ldots, \mathcal{M}_n (X \in \mathcal{M}_0 \wedge \beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_n))$$

for any $n \in \omega$. Notice that $\beta^i_e\mathsf{RFN}(n)$ is a Π^1_{e+2} approximation of the above sentence.

In [4], it is proved that the set $\{\beta^1_0\mathsf{RFN}(n) : n \in \omega\}$ characterizes the Π^1_2-consequences of Π^1_1-CA_0. The key point of this proof is that when a Π^1_3-sentence of the form $\forall X \exists Y \theta(X, Y)$ is provable from Π^1_1-CA_0, then there exists $n \in \omega$ such that a solution Y is computable from any $\mathcal{M}_0, \ldots, \mathcal{M}_n$ with $\beta^1_1(\mathcal{M}_0, \ldots, \mathcal{M}_n)$ and $X \in \mathcal{M}_0$. We extend this result to more general cases.

Lemma 2.12 *Let $i \in \omega$ such that $i \geq 1$ and $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}})$ be a model of ACA_0. Let $S' \subseteq S^{\mathcal{M}}$ be such that $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}}, S')$ is a model of RCA_0 with the following property: for any $X \in S'$, there exists $Y \in S'$ such that $\mathcal{M} \models \beta_i(Y) \wedge (X \in Y)$. Then,*

- *$\mathcal{M} \models \beta_i(Y)$ implies $\mathcal{M}' \models \beta_i(Y)$ for any $Y \in S'$,*
- *$\mathcal{M}'$ is a model of Σ^1_i-SDC_0, and*
- *$\mathcal{M}'$ is a β_i-submodel of $\mathcal{M}$.*

Proof We prove the lemma by induction. The case $i = 1$ is proved in [4].

Let $i \in \omega$. Assume that for any $X \in S'$, there exists $Y \in S'$ such that $\mathcal{M} \models \beta_{i+1}(Y) \wedge (X \in Y)$. By the induction hypothesis, the following properties hold.

(1) $\mathcal{M} \models \beta_i(Y)$ implies $\mathcal{M}' \models \beta_i(Y)$ for any $Y \in S'$, and
(2) $\mathcal{M}'$ is a β_i submodel of $\mathcal{M}$.

Let $Y \in S'$ such that $\mathcal{M} \models \beta_{i+1}(Y)$. We show that $\mathcal{M}' \models \beta_{i+1}(Y)$. Let $\varphi(X, Z)$ be a Π^1_i-formula and $X \in Y$ such that $\mathcal{M}' \models \exists Z \varphi(X, Z)$. Since $\mathcal{M}'$ is a β_i submodel of $\mathcal{M}$, $\mathcal{M} \models \exists Z \varphi(X, Z)$. Then, $\mathcal{M} \models [Y \models \exists Z \varphi(X, Z)]$ because $\mathcal{M} \models \beta_{i+1}(Y)$. That is, there exists $Z \in Y$ such that $\mathcal{M} \models [Y \models \varphi(X, Z)]$. Then, $\mathcal{M}' \models [Y \models \varphi(X, Z)]$ because $Y \models \varphi(X, Z)$ is an arithmetical condition for X, Y and Z. Thus, $\mathcal{M}' \models [Y \models \exists Z \varphi(X, Z)]$.

We next show that $\mathcal{M}'$ is a β_{i+1} submodel of $\mathcal{M}$. Let $\varphi(X, Z)$ be a Π^1_i-formula and $X \in \mathcal{M}'$ such that $\mathcal{M} \models \exists Z \varphi(X, Z)$. Let $Y \in \mathcal{M}'$ such that $\mathcal{M} \models \beta_{i+1}(Y) \wedge X \in Y$. Then, $\mathcal{M} \models [Y \models \exists Z \varphi(X, Z)]$ and hence $\mathcal{M}' \models [Y \models \exists Z \varphi(X, Z)]$. Therefore $\mathcal{M}' \models \exists Z \varphi(X, Z)$ because $\mathcal{M}' \models \beta_i(Y)$ by the induction hypothesis.

Since $\mathcal{M}' \models \forall X \exists Y (X \in Y \wedge \beta_{i+1}(Y))$, $\mathcal{M}'$ is a model of Σ^1_{i+1}-SDC_0. □

Lemma 2.13 *Let φ be a Π^1_i-formula such that $\forall X \exists Y \varphi$ is provable from Σ^1_i-SDC_0. Then, there exists an $n \in \omega$ such that ACA_0 proves that*

$$\forall X, \mathcal{M}_0, \ldots, \mathcal{M}_n ((X \in \mathcal{M}_0 \wedge \beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_n)) \to \exists Y \in \mathcal{M}_n \varphi(X, Y)).$$

Proof Let φ be a Π^1_i-formula such that Σ^1_i-$\mathsf{SDC}_0 \vdash \forall X \exists Y \varphi$. Assume that ACA_0 does not prove

$$\forall X, \mathcal{M}_0, \ldots, \mathcal{M}_n ((X \in \mathcal{M}_0 \wedge \beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_n)) \to \exists Y \in \mathcal{M}_n \varphi(X, Y))$$

for any $n \in \omega$. That is, for any $n \in \omega$, ACA_0 is consistent with

$$\exists X, \mathcal{M}_0, \ldots, \mathcal{M}_n (X \in \mathcal{M}_0 \wedge \beta_i^i(\mathcal{M}_0, \ldots, \mathcal{M}_n) \wedge \forall Y \in \mathcal{M}_n \neg\varphi(X, Y)).$$

Let $\mathcal{L}'$ be the language given by adding new set constant symbols C and $D_0, D_1, \ldots$ into $\mathcal{L}_2$. Define an $\mathcal{L}'$-theory T consisting of the following sentences:

$$\begin{aligned}
&\mathsf{ACA}_0,\\
&C \in D_0,\\
&D_n \in D_{n+1} \text{ for } n \in \omega,\\
&\beta_i(D_n) \text{ for } n \in \omega,\\
&\forall Y \in D_n \neg\varphi(C, Y) \text{ for } n \in \omega.
\end{aligned}$$

Then, each finite subset of T is consistent. Thus, from the compactness theorem, there exists a model $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}}, C, \{D_n\}_{n\in\omega})$ of T. Define $S' \subseteq S^{\mathcal{M}}$ by $S' = \bigcup_{n\in\omega}\{X \in S^{\mathcal{M}} : \mathcal{M} \models X \in D_n\}$.

It is easy to see that $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}}, S')$ is a model of RCA_0. Moreover, for any $X \in S'$, there exists $Y \in S'$ such that $\mathcal{M} \models X \in Y \wedge \beta_i(Y)$. Thus, by Lemma 2.12, $\mathcal{M}'$ is a model of Σ^1_i-SDC_0 and a β_i submodel of $\mathcal{M}$. Thus $\mathcal{M}' \models \forall Y \neg\varphi(C, Y)$ because $\mathcal{M} \models \neg\varphi(C, Y)$ for any $Y \in \mathcal{M}'$. However, this contradicts the assumption Σ^1_i-$\mathsf{SDC}_0 \vdash \forall X \exists Y \varphi(X, Y)$. □

Theorem 2.14 *Let $e, i \in \omega$ such that $i > e$. Then, the theories $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$ and $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ prove the same sentences.*

Proof Since ACA_0 is Π^1_2 axiomatizable, ACA_0 is in $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$. It is easy to see that $\beta_e^i\mathsf{RFN}(n)$ is in $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ for each $n \in \omega$. Therefore, any sentence provable from $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$ is also provable from $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$.

For the converse, let σ be a Π^1_{e+2} sentence provable from Σ^1_i-SDC_0. Write $\sigma \equiv \forall X \exists Y \varphi(X, Y)$ by a Π^1_e-formula φ. Then, there exists $k \in \omega$ such that ACA_0 proves

$$\forall X, \mathcal{M}_0, \ldots, \mathcal{M}_k ((X \in \mathcal{M}_0 \wedge \beta_i^i(\mathcal{M}_0, \ldots, \mathcal{M}_k)) \to \exists Y \in \mathcal{M}_k \varphi(X, Y)).$$

We show that $\forall X \exists Y \varphi(X, Y)$ is provable from $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$. We reason in $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$. Take an arbitrary X. By $\beta_e^i\mathsf{RFN}(k+1)$, take $\mathcal{M}_0, \ldots, \mathcal{M}_{k+1}$ such that

$$X \in \mathcal{M}_0 \wedge \beta_e^i(\mathcal{M}_0, \ldots, \mathcal{M}_{k+1}) \wedge \bigwedge_{m<k+2} \mathcal{M}_m \models \mathsf{ACA}_0.$$

Then, $\mathcal{M}_{k+1}$ satisfies $\mathsf{ACA}_0 \wedge X \in \mathcal{M}_0 \wedge \beta_i^i(\mathcal{M}_0, \ldots, \mathcal{M}_k)$. Thus, $\mathcal{M}_{k+1} \models \exists Y \in \mathcal{M}_k \varphi(X, Y)$ and hence $\mathcal{M}_{k+1} \models \exists Y \varphi(X, Y)$. Since $\mathcal{M}_{k+1}$ is a β_e model and φ is a Π^1_e-formula, $\exists Y \varphi(X, Y)$ holds. □

Corollary 2.15 $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$ *is the same as* $\mathrm{Th}_{\Pi^1_{e+2}}(\Pi^1_i\text{-}\mathsf{CA}_0)$ *for* $e = 0, 1, 2$ *and* $i > e$.

Proof The case $i \geq 3$ is immediate from the fact that $\Sigma^1_i\text{-}\mathsf{SDC}_0$ is Π^1_4-conservative over $\Pi^1_i\text{-}\mathsf{CA}_0$ [3, VII.6.20]. The cases $i = 1$ and $i = 2$ are from the fact that $\Sigma^1_i\text{-}\mathsf{SDC}_0$ is equivalent to $\Pi^1_i\text{-}\mathsf{CA}_0$ [3, VII.6.9]. □

3 Boolean combinations of Π^1_{i+1}-formulas provable from Σ^1_i-SDC

In this section, we give a characterization of the set $\{\sigma \in \mathsf{B}(\Pi^1_{i+1}) : \Sigma^1_i\text{-}\mathsf{SDC}_0 \vdash \sigma\}$ for $i > 0$. Here, $\mathsf{B}(\Gamma)$ is the set of boolean combinations of formulas in Γ. In other words, $\mathsf{B}(\Gamma)$ denotes the set generated by $\wedge, \vee, \neg$ starting from Γ.

Definition 3.1 For a sentence σ, we define $\beta_i\mathsf{Rfn}(\sigma)$ by

$$\sigma \to \exists \mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma).$$

For a class Γ of formulas, we define an axiomatic system $\beta_i\mathsf{Rfn}(\Gamma)_0$ by $\mathsf{ACA}_0 + \{\beta_i\mathsf{Rfn}(\sigma) : \sigma \in \Gamma, \sigma \text{ is a sentence}\}$.

Remark 3.2 Notice that $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is a weaker variant of (3) in Theorem 2.7. The difference between those is whether σ allows (set) parameters. Since each instance of $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is Σ^1_{i+2}, $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is strictly weaker than $\Sigma^1_i\text{-}\mathsf{SDC}_0$.

We show that the theories $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ and $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ are equivalent. We start with showing a basic property of $\mathsf{B}(\Pi^1_i)$.

Lemma 3.3 *Let* $\rho \in \mathsf{B}(\Pi^1_i)$. *Then, there exist* $\pi_0, \ldots, \pi_{n-1} \in \Pi^1_i$ *and* $\sigma_0, \ldots, \sigma_{n-1} \in \Sigma^1_i$ *such that* ρ *is equivalent to* $\bigwedge_{n'<n}(\pi_{n'} \vee \sigma_{n'})$ *over* RCA_0.

Proof We reason in RCA_0. Then, Π^1_i and Σ^1_i are closed under $\vee$.

Let ρ be a $\mathsf{B}(\Pi^1_i)$ sentence. Taking a conjunction normal form,

$$\rho \leftrightarrow \bigwedge_{n'<n} \bigvee_{m'<m} (\pi_{n',m'} \vee \sigma_{n',m'})$$

where each $\pi_{n',m'}$ is Π^1_i and each $\sigma_{n',m'}$ is Σ^1_i. Put $\pi_{n'} \equiv \bigvee_{m'<m} \pi_{n',m'}$ and $\sigma_{n'} \equiv \bigvee_{m'<m} \sigma_{n',m'}$. Then, $\pi_{n'} \in \Pi^1_i$, $\sigma_{n'} \in \Sigma^1_i$ and

$$\rho \leftrightarrow \bigwedge_{n'<n} (\pi_{n'} \vee \sigma_{n'}).$$

This completes the proof. □

Definition 3.4 We define $\beta_e^i\mathsf{RFN}^-(\varphi; n)$ as the Σ^1_{e+1}-sentence stating that there exist $Y_0, \dots Y_n$ such that $Y_0 \models \varphi$, and $\beta_e^i(Y_0, \dots, Y_n)$. We omit φ if it is trivial.

Lemma 3.5 *Let $i \in \omega$ such that $i > 0$. Then, for any $n \in \omega$, $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ proves $\varphi \to \beta_i^i\mathsf{RFN}^-(\varphi; n)$ for any sentence $\varphi \in \Sigma^1_{i+1}$.*

Proof We prove the lemma by induction. In the case $n = 0$, $\varphi \to \beta_i^i\mathsf{RFN}^-(\varphi; n)$ is the same as $\varphi \to \exists Y(\beta_i(Y) \wedge Y \models \varphi)$. Therefore, it is an instance of $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ and hence provable from $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$.

For the induction step, take an arbitrary n and assume that $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ proves $\varphi \to \beta_i^i\mathsf{RFN}^-(\varphi; n)$ for any sentence $\varphi \in \Sigma^1_{i+1}$. Take an arbitrary sentence $\varphi \in \Sigma^1_{i+1}$. We reason in $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ and show that $\varphi \to \beta_i^i\mathsf{RFN}^-(\varphi; n+1)$ holds.

Assume that φ holds. Then, we have $\beta_i^i\mathsf{RFN}^-(\varphi; n)$ because of the induction hypothesis. Since $\beta_i^i\mathsf{RFN}^-(\varphi; n)$ is a Σ^1_{i+1}-sentence,

$$\beta_i^i\mathsf{RFN}^-(\varphi; n) \to \exists Y(\beta_i(Y) \wedge Y \models \beta_i^i\mathsf{RFN}^-(\varphi; n))$$

also holds by the induction hypothesis. Thus we have $\exists Y(\beta_i(Y) \wedge Y \models \beta_i^i\mathsf{RFN}^-(\varphi; n))$. Take a β_i-model Y such that $Y \models \beta_i^i\mathsf{RFN}^-(\varphi; n)$. Then, there exist $Y_0, \dots, Y_n$ such that

$$Y \models \left[(Y_0 \models \varphi) \wedge \bigwedge_{m<n+1} \beta_i(Y_m) \wedge \bigwedge_{m<n} Y_m \in Y_{m+1} \right].$$

Since the condition in the square bracket is Π^1_i,

$$(Y_0 \models \varphi) \wedge \bigwedge_{m<n+1} \beta_i(Y_m) \wedge \bigwedge_{m<n} Y_m \in Y_{m+1}$$

actually holds. Take $Y_{n+1} = Y$, then $Y_0, \dots, Y_n, Y_{n+1}$ witness $\beta_i^i\mathsf{RFN}^-(\varphi; n+1)$. □

Lemma 3.6 *Let $i \in \omega, i > 0$. Let π be a Π^1_{i+1}-sentence and σ a Σ^1_{i+1}-sentence such that Σ^1_i-$\mathsf{SDC}_0 \vdash \pi \vee \sigma$. Then, $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0 \vdash \pi \vee \sigma$.*

Proof For the sake of contradiction, assume that Σ^1_i-SDC_0 proves $\pi \vee \sigma$ but $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ does not prove it. Then, the theory $T = \beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0 \cup \{\neg\pi, \neg\sigma\}$ is consistent.

Let $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}})$ be a model of T. We take new set constant symbols C_n for each $n \in \omega$. Define a theory T' by

$$T' = \mathsf{ACA}_0 + \neg\sigma + \{\beta_i^i(C_0, \dots, C_n) : n \in \omega\} + C_0 \models \neg\pi.$$

Then, $\mathcal{M}$ satisfies every finite subtheory of T' because $\mathcal{M}$ is a model of ACA_0, $\neg\sigma$ and $\beta_i^i\mathsf{RFN}^-(\neg\pi; n)$ for all n. Here, the condition $\mathcal{M} \models \beta_i^i\mathsf{RFN}^-(\neg\pi; n)$ is obtained from the previous lemma by putting $\varphi \equiv \neg\pi$. Therefore, T' is a consistent theory.

Let $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}'}, S, \{Y_n : n \in \omega\})$ be a model of T'. Define $S' = \bigcup_{n\in\omega}\{X \in S : \mathcal{M}' \models X \in Y_n\}$. Put $\mathcal{M}'' = (\mathbb{N}^{\mathcal{M}'}, S')$. Then, by Lemma 2.12,

- $\mathcal{M}''$ is a model of Σ^1_i-SDC_0,
- $\mathcal{M}''$ is a β_i-submodel of $\mathcal{M}'$ and
- for any $n \in \omega$, $\mathcal{M}'' \models \beta_i(Y_n)$.

We note that $\mathcal{M}''$ satisfies $\neg\sigma$ because $\mathcal{M}'$ satisfies $\neg\sigma$ and $\mathcal{M}''$ is a β_i-submodel of $\mathcal{M}'$. We claim that $\mathcal{M}''$ also satisfies $\neg\pi$. Indeed, $\mathcal{M}'' \models [Y_0 \models \neg\pi]$ because $\mathcal{M}' \models [Y_0 \models \neg\pi]$ and the condition $Y_0 \models \neg\pi$ is arithmetical. Since $\mathcal{M}'' \models \beta_i(Y_0)$, $\mathcal{M}'' \models \neg\pi$.

Consequently, $\mathcal{M}''$ is a model of Σ^1_i-$\mathsf{SDC}_0 \cup \{\neg\pi, \neg\sigma\}$. However, this contradicts the assumption Σ^1_i-SDC_0 proves $\pi \vee \sigma$. □

Theorem 3.7 *The theories* $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ *and* $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ *prove the same sentences.*

Proof It is enough to show that each axiom in $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is provable from $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ and vice verse.

We first show that each axiom in $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is provable from the theory $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$. It is enough to show that $\beta_i\mathsf{Rfn}(\sigma)$ is in $\mathsf{B}(\Pi^1_{i+1})$ for $\sigma \in \Sigma^1_{i+1}$. Moreover, it is sufficient to show that $\exists\mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma) \in \mathsf{B}(\Pi^1_{i+1})$ because $\beta_i\mathsf{RFN}(\sigma)$ is of the form $\sigma \to \exists\mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma)$ and $\sigma \in \mathsf{B}(\Pi^1_{i+1})$. As in Remark 2.6 (3), $\beta_i(\mathcal{M})$ is Π^1_i and hence $\exists\mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma)$ is Σ^1_{i+1}.

We then show that each $\mathsf{B}(\Pi^1_{i+1})$-formula provable from Σ^1_i-SDC_0 is already provable from $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$.

Let $\rho \in \mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$. We show that ρ is provable from $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$. There exist $\pi_0, \ldots, \pi_{n-1} \in \Pi^1_{i+1}, \sigma_0, \ldots, \sigma_{n-1} \in \Sigma^1_{i+1}$ such that ρ is equivalent to $\bigwedge_{m<n}\pi_m \vee \sigma_m$ by Lemma 3.3. Since ρ is provable from Σ^1_i-SDC_0, $\pi_m \vee \sigma_m$ is provable from Σ^1_i-SDC_0 for any $m < n$. Therefore, $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ proves $\pi_m \vee \sigma_m$ for any $m < n$ by Lemma 3.6. Hence $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ proves $\bigwedge_{m<n}\pi_m \vee \sigma_m$. □

Corollary 3.8 *The theories* $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ *and* $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$ *prove the same sentences.*

Proof It is enough to show that each instance of $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))$ is a $\mathsf{B}(\Pi^1_{i+1})$-sentence provable from Σ^1_i-SDC_0. This is immediate from Theorem 2.7. □

Corollary 3.9 *For any* $i > 0$, $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ *is not finitely axiomatizable.*

Proof We show that $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$ is not finitely axiomatizable. For the sake of contradiction, suppose that $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$ is finitely axiomatizable. Then, there are finitely many instances $\rho_0, \ldots, \rho_{n-1}$ of $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$ such that $\mathsf{ACA}_0 + \bigwedge_{i<n}\rho_i$ axiomatizes $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$. Let ρ be the conjunction of $\rho_0, \ldots, \rho_n$. Since ρ is in $\mathsf{B}(\Pi^1_{i+1})$, we have $\mathsf{ACA}_0 + \rho \vdash \exists\mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \rho)$. Therefore, $\mathsf{ACA}_0 + \rho$ proves the consistency of itself, a contradiction. □

Theorem 3.10 *Σ^1_i-SDC_0 is Σ^1_{i+1}-conservative over $\mathsf{ACA}_0 + \{\beta^i_i\mathsf{RFN}^-(n) : n \in \omega\}$. In particular, Π^1_1-CA_0 is Σ^1_2-conservative over the theory $\mathsf{ACA}_0 + \{\exists Y(Y = \mathrm{HJ}^n(\varnothing)) : n \in \omega\}$.*

Proof For the former statement, the same argument for Lemma 3.6 works. From this result and the equivalence of the existence of a coded β-model and a hyperjump [3, VII.2.9], we have the latter holds. □

In [4], it is proved that $\{\exists Y(Y = \mathrm{HJ}^n(\varnothing)) : n \in \omega\}$ is equivalent to $\{(\Sigma^{0,\varnothing}_1)_n\text{-}\mathsf{Det} : n \in \omega\}$ over ACA_0. Here, $(\Sigma^{0,\varnothing}_1)_n$-$\mathsf{Det}$ is the determinacy of Gale-Stewart games with payoff sets of complexity $(\Sigma^{0,\varnothing}_1)_n$. Thus, we have the following characterization of Σ^1_2-consequences of Π^1_1-CA_0.

Corollary 3.11 *Π^1_1-CA_0 is Σ^1_2-conservative over $\mathsf{ACA}_0 + \{(\Sigma^{0,\varnothing}_1)_n\text{-}\mathsf{Det} : n \in \omega\}$.*

We then see the relationship between $\beta^i_e\mathsf{RFN}^-$ and RFN^i_e.

Definition 3.12 Let Γ be a class of formulas. For theories T and T', we say T is a Γ-extension of T' (written $T' \subseteq_\Gamma T$) if any sentence in Γ which is provable from T' is also provable from T. We write $T' \subsetneq_\Gamma T$ if there is a sentence in Γ which is provable from T but not provable from T' in addition. We also write $T =_\Gamma T'$ when T and T' are Γ-equivalent, that is, they prove the same Γ-sentence.

As $\mathsf{ACA}_0 + \{\beta^1_0\mathsf{RFN}(n) : n \in \omega\} =_{\Pi^1_2} \Pi^1_1$-$\mathsf{CA}_0$ and $\mathsf{ACA}_0 + \{\exists Y(Y = \mathrm{HJ}^n(\varnothing)) : n \in \omega\} =_{\Sigma^1_2} \Pi^1_1$-$\mathsf{CA}_0$, these three theories are Π^1_1-equivalent. Thus, we have two approximations of the Π^1_1-consequences of Π^1_1-CA_0; one is the hierarchy of $\beta^1_0\mathsf{RFN}(0)$, $\beta^1_0\mathsf{RFN}(1)$, $\beta^1_0\mathsf{RFN}(2)$, ..., and the other is the hierarchy of the iterated hyperjumps of the empty set. In what follows, we compare these two approximations.

Lemma 3.13 *Let T and T' be theories such that T is recursively axiomatizable and T' includes ACA_0. If T' proves the β_i-model reflection of T, then $T \subsetneq_{\Pi^1_{i+2}} T'$ and T' proves the consistency of T.*

Proof Let $\sigma \equiv \forall X \exists Y \theta(X, Y)$ be a Π^1_{i+2}-sentence provable from T. We work in T'. Take an arbitrary X and a β_i-model $\mathcal{M}$ such that $X \in \mathcal{M}$ and $\mathcal{M} \models T$. Then, $\mathcal{M} \models \exists Y \theta(X, Y)$. Since θ is Π^1_i and $\mathcal{M}$ is a β_i-model, $\exists Y \theta(X, Y)$ holds. □

By a similar proof, we also have the following.

Lemma 3.14 *Let T and T' be theories such that T is recursively axiomatizable and T' includes ACA_0. If T' proves the existence of a β_i-model of T, then $T \subsetneq_{\Sigma^1_{i+1}} T'$ and T' proves the consistency of T.*

Theorem 3.15 *Let $i, e \in \omega$ such that $e < i$. Then, for each $n \in \omega$, $\beta^i_e\mathsf{RFN}(n + 1)$ proves the β_e-model reflection of $\beta^i_i\mathsf{RFN}^-(n)$ over ACA_0. In particular, we have $\beta^i_i\mathsf{RFN}^-(n) \subsetneq_{\Pi^1_{e+2}} \beta^i_e\mathsf{RFN}(n + 1)$ over ACA_0.*

Proof Take an arbitrary X. By $\beta_e^i\mathsf{RFN}(n+1)$, take $\mathcal{M}_0, \ldots, \mathcal{M}_{n+1}$ such that $X \in \mathcal{M}_0$ and $\beta_e^i(\mathcal{M}_0, \ldots, \mathcal{M}_{n+1})$. Then, $\mathcal{M}_{n+1}$ is a β_e-model and $\mathcal{M}_{n+1} \models \beta_i^i(\mathcal{M}_0, \ldots, \mathcal{M}_n)$. In particular, $\mathcal{M}_{n+1} \models \beta_i^i\mathsf{RFN}^-(n)$. The inclusions $\beta_i^i\mathsf{RFN}^-(n) \subsetneq_{\Pi^1_{e+2}} \beta_e^i\mathsf{RFN}(n+1)$ follows from Lemma 3.13. □

Theorem 3.16 *Let $i \in \omega$ such that $i > 0$. Then, for each $n \in \omega$, $\beta_i^i\mathsf{RFN}^-(n+1)$ proves the existence of a β_i-model of $\beta_{i-1}^i\mathsf{RFN}(n+1)$. In particular, we have $\beta_{i-1}^i\mathsf{RFN}(n+1) \subsetneq_{\Sigma^1_{i+1}} \beta_i^i\mathsf{RFN}^-(n+1)$ over Σ^1_{i-1}-SDC_0.*

Proof Let $\mathcal{M}_0, \ldots, \mathcal{M}_{n+1}$ be such that $\beta_i^i(\mathcal{M}_0, \ldots, \mathcal{M}_{n+1})$. Then, $\mathcal{M}_0$ is a β_i-model, and $\mathcal{M}_{n+1}$ is a model of Σ^1_{i-1}-SDC_0. We prove that $\mathcal{M}_0$ is a model of $\beta_{i-1}^i\mathsf{RFN}(n+1)$. Since $\mathcal{M}_{n+1}$ is a model of Σ^1_{i-1}-SDC, there exists an $\mathcal{M}' \in \mathcal{M}_{n+1}$ such that $\mathcal{M}_n \in \mathcal{M}'$ and $\mathcal{M}_{n+1} \models \beta_{i-1}(\mathcal{M}')$. We show that $\mathcal{M}'$ believes that $\mathcal{M}_n$ is a β_i-model. Take a Π^1_i-sentence σ with parameters from $\mathcal{M}_n$ such that $\mathcal{M}_n \models \sigma$. Then, $\mathcal{M}_{n+1} \models \sigma$ because $\mathcal{M}_n$ is a β_i-submodel of $\mathcal{M}_{n+1}$. Since $\mathcal{M}'$ is a β_{i-1}-submodel of $\mathcal{M}_{n+1}$, $\mathcal{M}' \models \sigma$.

Now, for any $X \in \mathcal{M}_0$, $\mathcal{M}_{n+1}$ satisfies

$$(*)\ \exists Y_0, \ldots, Y_{n+1}(X \in Y_0 \wedge \beta_{i-1}^i(Y_0, \ldots, Y_n, Y_{n+1}))$$

via taking $Y_m = \mathcal{M}_m$ for $m < n+1$ and $Y_{n+1} = \mathcal{M}'$. Since $(*)$ is a Σ^1_i condition and $\mathcal{M}_0$ is a β_i-submodel of $\mathcal{M}_{n+1}$, $\mathcal{M}_0$ also satisfies $(*)$. The inclusion $\beta_{i-1}^i\mathsf{RFN}(n+1) \subsetneq_{\Sigma^1_{i+1}} \beta_i^i\mathsf{RFN}^-(n+1)$ follows from Lemma 3.14. □

Corollary 3.17 *Over* ACA_0^+,

$$\exists Y(Y = \mathrm{HJ}^{n+1}(\varnothing)) \subsetneq_{\Pi^1_2} \beta_0^1\mathsf{RFN}(n+1) \subsetneq_{\Sigma^1_2} \exists Y(Y = \mathrm{HJ}^{n+2}(\varnothing)).$$

Proof It is immediate from the above theorems and the equivalence of the existence of a β-model and a hyperjump. □

Remark 3.18 We note that the inclusions in the previous corollary also hold over a theory T such that T includes ACA_0^+ and is a proper subtheory of $\beta_0^1\mathsf{RFN}(n+1)$. For example, the inclusions also hold over ATR_0.

4 Boolean combinations of Π^1_{e+1}-formulas provable from Σ^1_i-SDC_0

In this section, we generalize the result in the previous sections to give a characterization of $\mathrm{Th}_{\mathrm{B}(\Pi^1_{e+1})}(\Sigma^1_i\text{-}\mathsf{SDC})$ for $e < i$. Most arguments of this section are almost the same as those in the previous section. That is, we use statements of the form $\sigma \to \beta_e^i\mathsf{RFN}(\sigma; n)$ for the characterization. The difference between this section and the previous section is that

- $\{\sigma \to \beta_e^i\mathsf{RFN}^-(\sigma; 1) : \sigma \in \Sigma^1_{e+1}\}$ does not include $\{\sigma \to \beta_e^i\mathsf{RFN}^-(\sigma; n) : n \in \omega, \sigma \text{ is a } \Sigma^1_{e+1}\text{-sentence}\}$,

- by our compactness argument, the constructed model may not be a model of Σ^1_i-SDC_0.

To address the first problem, we adopt $\{\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n) : n \in \omega, \sigma \text{ is a } \Sigma^1_{e+1}\text{- sentence}\}$ for the characterization. To address the second problem, we slightly modify the proof in the previous section.

We give a comment concerning the first item.

Proposition 4.1 *Let $i, e \in \omega$ such that $e < i$. Let $n \in \omega$. Then, the theory $\mathsf{ACA}_0 + \{\beta^i_e\mathsf{RFN}^-(\sigma; n+1) : \sigma$ is a Σ^1_{e+1}-sentence$\}$ proves the consistency of $\mathsf{ACA}_0 + \{\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n) : \sigma$ is a Σ^1_{e+1}-sentence$\}$.*

Proof Let $\mathcal{M}_0, \dots, \mathcal{M}_{n+1}$ be coded ω-models such that $\beta^i_e(\mathcal{M}_0, \dots, \mathcal{M}_{n+1})$. We show that $\mathcal{M}_0 \models [\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n)]$ for $\sigma \in \Sigma^1_{e+1}$.

Assume $\mathcal{M}_0 \models \sigma$. Then, $\mathcal{M}_{n+1} \models [\exists \mathcal{N}_0, \dots, \mathcal{N}_n(\beta^i_i(\mathcal{N}_0, \dots, \mathcal{N}_n) \wedge \mathcal{N}_0 \models \sigma)]$ by taking $\mathcal{N}_0 = \mathcal{M}_0, \dots, \mathcal{N}_n = \mathcal{M}_n$. Then we also have $\mathcal{M}_{n+1} \models [\exists \mathcal{N}_0, \dots, \mathcal{N}_n(\beta^i_e(\mathcal{N}_0, \dots, \mathcal{N}_n) \wedge \mathcal{N}_0 \models \sigma)]$ because $e < i$. Since $\mathcal{M}_0$ is a β_i-submodel of $\mathcal{M}_{n+1}$ and $[\exists \mathcal{N}_0, \dots, \mathcal{N}_n(\beta^i_e(\mathcal{N}_0, \dots, \mathcal{N}_n) \wedge \mathcal{N}_0 \models \sigma)]$ is Σ^1_{e+1}, which is included Σ^1_i, $\mathcal{M}_0 \models [\exists \mathcal{N}_0, \dots, \mathcal{N}_n(\beta^i_e(\mathcal{N}_0, \dots, \mathcal{N}_n) \wedge \mathcal{N}_0 \models \sigma)]$. □

Theorem 4.2 *Let $i, e \in \omega, e < i$. The following theories are equivalent.*

(1) $\mathsf{ACA}_0 + \{\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n) : n \in \omega, \sigma$ *is a* Σ^1_{e+1}*-sentence*$\}$.

(2) $\mathrm{Th}_{\mathrm{B}(\Pi^1_{e+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$.

Proof It is trivial that any instance of (1) is in (2). Therefore, (1) is a subtheory of (2). We show the converse inclusion.

Let T be the theory of (1) As in the previous section, it is enough to show that for any $\sigma \in \Sigma^1_{e+1}$ and $\pi \in \Pi^1_{e+1}$, if Σ^1_i-SDC_0 proves $\sigma \vee \pi$ then so does T.

Let $\sigma \in \Sigma^1_{e+1}$ and $\pi \in \Pi^1_{e+1}$ such that Σ^1_i-SDC_0 proves $\sigma \vee \pi$. Assume that T does not prove $\sigma \vee \pi$ for the sake of contradiction. Let $\{C_n\}_{n\in\omega}$ be new set constants. Then, the theory

$$T' = T + \neg\sigma + \{\beta^i_e(C_0, \dots, C_n) : n \in \omega\} + C_0 \models \neg\pi$$

is consistent. Let $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}}, \{A_n\}_{n\in\omega})$ be a model of T'. Put $S' = \bigcup_{n\in\omega}\{X \in S^{\mathcal{M}} : \mathcal{M} \models X \in A_n\}$ and $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}}, S')$. We note that

- $\mathcal{M}' \models \beta_e(A_n)$ for all $n \in \omega$,
- $\mathcal{M}'$ is a β_e-submodel of $\mathcal{M}$

by Lemma 2.12. We show that $\mathcal{M}'$ satisfies both $\sigma \vee \pi$ and $\neg\sigma \wedge \neg\pi$, a contradiction.

Claim 4.2.1 $\mathcal{M}' \models \sigma \vee \pi$.

Proof of the claim Since $\sigma \vee \pi$ is a Π^1_{e+2}-sentence, there exists a Π^1_e-formula ρ such that $\sigma \vee \pi \leftrightarrow \forall X \exists Y \rho$. Since $e < i$, we can apply Lemma 2.13. Thus, we have $n \in \omega$ such that

$$\mathsf{ACA}_0 \vdash \forall X, \mathcal{M}_0, \dots, \mathcal{M}_n((X \in \mathcal{M}_0 \wedge \beta^i_i(\mathcal{M}_0, \dots, \mathcal{M}_n)) \to \exists Y \in \mathcal{M}_n \rho(X, Y)).$$

To prove $\mathcal{M}' \models \forall X \exists Y \rho$, take an arbitrary $X \in \mathcal{M}'$. Then, for sufficiently large k, we have $X \in A_k$. Within $\mathcal{M}'$, the following properties holds.

- $\beta_e(A_{k+n+1})$ and $A_{k+n+1} \models \mathsf{ACA}_0$,
- A_{k+n+1} satisfies $X \in A_k \wedge \beta_i^i(A_k, \ldots, A_{k+n})$.

Therefore, $\mathcal{M}' \models \left[A_{k+n+1} \models \exists Y \rho(X, Y)\right]$. Since $\mathcal{M}' \models \beta_e(A_{k+n+1})$ and ρ is a Π^1_e-formula, $\mathcal{M}' \models \exists Y \rho(X, Y)$. Consequently, $\mathcal{M}' \models \forall X \exists Y \rho(X, Y)$ and hence $\mathcal{M}' \models \sigma \vee \tau$. □

Claim 4.2.2 $\mathcal{M}' \models \neg\sigma \wedge \neg\pi$.

Proof of the claim Since $\neg\sigma$ is a Π^1_{e+1}-sentence true in $\mathcal{M}$ and $\mathcal{M}'$ is a β_e-submodel of $\mathcal{M}$, $\mathcal{M}'$ also satisfies $\neg\sigma$. Since $\neg\pi$ is a Σ^1_{e+1}-sentence true in A_0 and A_0 is a β_i-submodel of $\mathcal{M}'$, $\mathcal{M}'$ also satisfies $\neg\pi$. □

The above claims complete the proof. □

By a similar argument, we have the following characterization of Σ^1_{e+1}-consequences of Σ^1_i-SDC_0.

Theorem 4.3 *Let $i, e \in \omega$, $e < i$. Then, Σ^1_i-SDC_0 is Σ^1_{e+1} conservative over $\mathsf{ACA}_0 + \{\beta_e^i \mathsf{RFN}^-(n) : n \in \omega\}$.*

From the Π^1_4-equivalence of Σ^1_i-SDC_0 and Π^1_i-CA_0, we also have the following.

Corollary 4.4 *Let $i, e \in \omega$, $e < i$ and $e < 2$. Then, Π^1_e-CA_0 is Σ^1_{e+1} conservative over $\mathsf{ACA}_0 + \{\beta_e^i \mathsf{RFN}^-(n) : n \in \omega\}$.*

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