Archive for Mathematical Logic
https://doi.org/10.1007/s00153-026-01023-5

Mathematical Logic

On some subtheories of strong dependent choice

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Received: 26 November 2024 / Accepted: 9 June 2026

Abstract
In this paper, we give characterizations of the set of Π^1_{e+2}-consequences, Σ^1_{e+1}-consequences and $\mathsf{B}(\Pi^1_{e+1})$-consequences of the axiomatic system of strong dependent choice for Σ^1_i formulas Σ^1_i-SDC_0 for $e < i$. Here, $\mathsf{B}(\Gamma)$ denotes the set generated by $\wedge, \vee, \neg$ starting from Γ.

Keywords Second-order arithmetic · Reverse mathematics · Models of arithmetic

Mathematics Subject Classification Primary 03F35; Secondary 03B30 · 03C62

1 Introduction

In the proof theoretic study of formal systems of arithmetic, we are interested not only in the whole set of the sentences provable from a theory, but also some certain subsets of it. For example, the consistency strength of a theory is characterized by its Π^0_1-consequences, the proof theoretic ordinal is naturally defined by its Π^1_1-consequences, and its Π^1_2-consequences correspond to its proof theoretic dilator.

It is also important in reverse mathematics to study the set of consequences of a theory with restricted complexity. Recently, the Π^1_2-consequences of Π^1_1-CA_0 have been studied thoroughly since Π^1_1-CA_0 is not axiomatizable by any Π^1_2-sentence. The first giant leap for the study of Π^1_2-consequences of Π^1_1-CA_0 was achieved by Towsner [5]. He introduced new Π^1_2-principles whose logical strength is between Π^1_1-CA_0 and

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ATR_0. The second and third author extended Towsner's work and gave a hierarchy which covers the set of Π^1_2-consequences of Π^1_1-CA_0 [4].

In this paper, we generalize their work in [4]. For more detail, we will give a characterization of the Π^1_{e+2}-, Σ^1_{e+1}- and $\mathsf{B}(\Pi^1_{e+1})$-consequences of the system strong Σ^1_i dependent choice (Σ^1_i-SDC_0) for $e < i$ where $\mathsf{B}(\Pi^1_{e+1})$ is the set of boolean combinations of Π^1_{e+1} formulas. Since Σ^1_i-SDC_0 is Π^1_4-conservative over Π^1_i-CA_0, we also have a characterization of Π^1_{e+2}-, Σ^1_{e+1}- and $\mathsf{B}(\Pi^1_{e+1})$-consequences of Π^1_i-CA_0 for $e = 0, 1, 2$. We note that the characterization of $\mathsf{B}(\Pi^1_{e+1})$-consequences can be seen as an analogue of the study of the syntactic local reflection principles in first-order arithmetic by Beklemishev [1].

We give a general idea used throughout this paper. It is known that Σ^1_i-SDC_0 is characterized by the β_i-model reflection principle (see Theorem 2.7). Therefore, by extracting a suitable property of β_i-models, we will get a corresponding subtheory of Σ^1_i-SDC_0. For proving a correspondence, we use the Barwise-Schlipf argument. That is, we make a model $\mathcal{M}$ from a given $T \subseteq \Sigma^1_i$-SDC_0 and a T non-provable sentence σ such that $\mathcal{M}$ has a β_i-submodel $\mathcal{M}'$ satisfying Σ^1_i-SDC_0 by the compactness theorem. Then, $\mathcal{M}'$ works as a witness for the non-provability of σ from Σ^1_i-SDC_0.

In Section 2, we give a characterization of the Π^1_{e+2}-consequences of Σ^1_i-SDC_0 for $e < i$. We note that the main result of this section (Theorem 2.14) is essentially mentioned in [2], but our proof is more informative for computability-theoretic aspects (Lemma 2.13). We also give a characterization of the $\mathsf{B}(\Pi^1_{i+1})$- and Σ^1_{i+1}-consequences of Σ^1_i-SDC_0 in Section 3. We then combine this result and the results in Section 2, and give a finer analysis for the Π^1_{e+2}-consequences of Σ^1_i-SDC_0 for $e < i$. By modifying the proofs in Section 3, we give a characterization of the $\mathsf{B}(\Pi^1_{e+1})$- and Σ^1_{e+1}-consequences of Σ^1_i-SDC_0 for $e < i$ in Section 4.

2 Π^1_{e+2} formulas provable from Σ^1_i-SDC_0

In this section, we give a characterization of the set $\{\sigma \in \Pi^1_{e+2} : \Sigma^1_i\text{-}\mathsf{SDC}_0 \vdash \sigma\}$ for $e < i$. As noted in the introduction, we use a characterization of Σ^1_i-SDC based on coded β_i-models for this purpose. We begin with introducing Σ^1_i-SDC_0. We refer the reader to Simpson's textbook [3] for the basic definitions. We note that in [3], the system Σ^1_i-SDC_0 is denoted as Strong Σ^1_i-DC_0.

Definition 2.1 *(sequence of sets,* RCA_0*)* Let $(\bullet, \bullet) : \mathbb{N}^2 \to \mathbb{N}$ be a fixed primitive recursive pairing function. For $X \subseteq \mathbb{N}$ and $x \in \mathbb{N}$, we define the x-th segment $X_x = \{y : (x, y) \in X\}$. Then, X is regarded as the sequence $\langle X_x \rangle_{x \in \mathbb{N}}$. In this sense, we write $A \in X$ to mean $\exists x(A = X_x)$. For a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of sets, we define $X_{<k}$ as the sequence $\langle X_0, \ldots, X_{k-1} \rangle$.

Definition 2.2 Let φ be a formula. We define strong dependent choice for φ as follows.

$$\varphi\text{-}\mathsf{SDC} \equiv \forall X \exists \langle Y_n \rangle_n (Y_0 = X \wedge \forall n > 0(\exists Z \varphi(Y_{<n}, Z) \to \varphi(Y_{<n}, Y_n))).$$

We define the axiomatic system Σ^1_i-SDC_0 as $\mathsf{ACA}_0 + \{\varphi\text{-}\mathsf{SDC} : \varphi \in \Sigma^1_i\}$.

We next see the definition of β_i-models, β_i-submodels and coded β_i-models. Intuitively, a β_i-model is a Σ^1_i-elementary submodel of $(\omega, \mathcal{P}(\omega))$.

Definition 2.3 An $\mathcal{L}_2$ structure $\mathcal{M}$ of the form $\mathcal{M} = (\omega, S)$ is called an ω-model. Let $\mathcal{M} = (\omega, S)$ be an ω-model and $i \in \omega$. We say $\mathcal{M}$ is a β_i-model if $\mathcal{M} \models \sigma \Leftrightarrow (\omega, \mathcal{P}(\omega)) \models \sigma$ for any Σ^1_i formula σ with parameters from $\mathcal{M}$.

The notions of ω-models and β_i-models can be extended to a relation of two $\mathcal{L}_2$ structures who share the first-order part.

Definition 2.4 Let $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}})$ and $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}'}, S^{\mathcal{M}'})$ be $\mathcal{L}_2$ structures. We say $\mathcal{M}'$ is an ω-submodel of $\mathcal{M}$ if $\mathbb{N}^{\mathcal{M}'} = \mathbb{N}^{\mathcal{M}}$ and $S^{\mathcal{M}'} \subseteq S^{\mathcal{M}}$.

Let $i \in \omega$. We say $\mathcal{M}'$ is a β_i-submodel of $\mathcal{M}$ if $\mathcal{M}'$ is an ω-submodel of $\mathcal{M}$ and $\mathcal{M}' \models \sigma \Leftrightarrow \mathcal{M} \models \sigma$ for any Σ^1_i formula σ with parameters from $\mathcal{M}'$.

If $S = \{S_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ is countable, then S can be coded by a single set $S_C \subseteq \omega$ by putting $S_C = \{(n, s) : s \in S_n\}$ as in Definition 2.1. This idea can be generalized to the nonstandard models. We call a model $\mathcal{M}' = (\mathbb{N}^{M'}, S_C)$ a coded ω-submodel in $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S)$ when $\mathbb{N}^{\mathcal{M}} = \mathbb{N}^{\mathcal{M}'}$ and $S_C \in S'$. If $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same Σ^1_i-sentences, then $\mathcal{M}'$ is called a coded β_i-submodel in $\mathcal{M}$. Throughout this paper, we omit the word *coded* if it is clear from the context.

Definition 2.5 *(*ACA_0*)* Let $\mathcal{M} = \langle M_x \rangle_x$ be a coded ω-model. We say $\mathcal{M}$ is a β_i-model if

$$\sigma \leftrightarrow \mathcal{M} \models \sigma$$

for any Π^1_i formula σ having parameters from $\mathcal{M}$.

Remark 2.6 We give a few comments on the above definition.

(1) In this paper, we may think that $\mathcal{M} \models \sigma$ is the abbreviation of 'the relativization $\sigma^{\mathcal{M}}$ is true' because our base theory is sufficiently strong. For the difference of $\mathcal{M} \models \sigma$ in the sense that an evaluation for σ exists and $\sigma^{\mathcal{M}}$, see Simpson's textbook and Remark 2.15 in [4].
(2) A β_0-model is just the same as an ω-model.
(3) Let $i > 0$. Then, the condition $\mathcal{M}$ is a β_i-model is equivalent to

$$(\mathcal{M} \models \sigma) \rightarrow \sigma \text{ for any } \Pi^1_i\text{-formula.}$$

Therefore, using a universal Π^1_i-formula, we can find a Π^1_i-formula stating that '$\mathcal{M}$ is a β_i-model'. Throughout this paper, we fix a Π^1_i-formula $\beta_i(\mathcal{M})$ stating '$\mathcal{M}$ is a β_i-model'.

We recall that Σ^1_i-SDC_0 is characterized by β_i-models.

Theorem 2.7 *[3, VII.7.4. and VII.7.6.] For i* > 0*, the following assertions are equivalent over* ACA_0.

(1) Σ^1_i-SDC_0,
(2) *the* β_i*-model reflection:* $\forall X \exists \mathcal{M}(X \in \mathcal{M} \wedge \beta_i(\mathcal{M}))$,
(3) $\forall X(\sigma(X) \to \exists \mathcal{M}(X \in \mathcal{M} \wedge \beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma(X)))$ *for* $\sigma \in \Sigma^1_{i+2}$.

Following these equivalences, it is shown that Σ^1_i-SDC_0 is axiomatizable by a Π^1_{i+2}-sentence but is not axiomatizable by any Σ^1_{i+2}-sentence. In fact, it follows from (3) that Σ^1_i-SDC_0 proves $\sigma \to \mathsf{Con}(\mathsf{ACA}_0 + \sigma)$ for any Σ^1_{i+2}-sentence σ where $\mathsf{Con}(\mathsf{ACA}_0 + \sigma)$ is the Π^0_1-formula asserting the consistency of $\mathsf{ACA}_0 + \sigma$. Hence, if Σ^1_i-SDC_0 were axiomatized by a Σ^1_{i+2}-sentence σ, then σ would prove $\mathsf{Con}(\mathsf{ACA}_0 + \sigma)$. However, this contradicts Gödel's second incompleteness theorem.

Since Σ^1_i-SDC_0 is finitely axiomatizable and not axiomatizable by any Σ^1_{i+2}-sentence, the set $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ of Π^1_{e+2}-consequences of Σ^1_i-SDC_0 is strictly weaker than Σ^1_i-SDC_0 for $e < i$.

In the remaining of this section, we give a characterization of the set $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ based on a weaker variant of (2) in Theorem 2.7 for $i > 1$ and $e < i$. We note that the case $i = 1$ has already been studied in [4].

Definition 2.8 Let $n > 0$. We define a formula $\beta^i(\mathcal{M}_0, \dots, \mathcal{M}_n)$ by

$$\bigwedge_{m<n}(\mathcal{M}_m \in \mathcal{M}_{m+1}) \wedge \bigwedge_{m<n} \mathcal{M}_n \models \beta_i(\mathcal{M}_m).$$

We also define a formula $\beta^i_e(\mathcal{M}_0, \dots, \mathcal{M}_n)$ by $\beta^i(\mathcal{M}_0, \dots, \mathcal{M}_n) \wedge \beta_e(\mathcal{M}_n)$.

We remark that if $\beta^i(\mathcal{M}_0, \dots, \mathcal{M}_n)$, then $\mathcal{M}_{m'} \in \mathcal{M}_m$ and $\mathcal{M}_m \models \beta_i(\mathcal{M}_{m'})$ for any $m' < m < n+1$. If $\beta_e(\mathcal{M}_n)$ in addition, then $\beta_e(\mathcal{M}_m)$ for any $e \leq i$ and $m < n$. In particular, $\beta^i_i(\mathcal{M}_0, \dots, \mathcal{M}_n)$ means that $\mathcal{M}_0 \in \dots \in \mathcal{M}_n$ and $\beta_i(\mathcal{M}_m)$ for all m.

Definition 2.9 Let σ, τ be sentences. For $i, e, n \in \omega$, we define $\beta^i_e\mathsf{RFN}(\sigma; n; \tau)$ as follows.

$$\beta^i_e\mathsf{RFN}(\sigma; n; \tau) \equiv \forall X \exists \mathcal{M}_0, \dots, \mathcal{M}_n \text{ with the following properties.}$$

- $X \in \mathcal{M}_0$,
- $\beta^i_e(\mathcal{M}_0, \dots, \mathcal{M}_n)$,
- $\mathcal{M}_m \models \mathsf{ACA}_0$ for all $m < n+1$,
- $\mathcal{M}_0 \models \sigma$ and $\mathcal{M}_n \models \tau$.

We sometimes omit σ or τ when they are trivial. For example, we write $\beta^i_e\mathsf{RFN}(n; \tau)$ instead of $\beta^i_e\mathsf{RFN}(0 = 0; n; \tau)$.

Remark 2.10 When we work in a theory including ACA_0 and $e > 0$, then we may omit the condition $\mathcal{M}_m \models \mathsf{ACA}_0$ from the above definition because ACA_0 proves that any β_e-model is a model of ACA_0.

Remark 2.11 We note that $\beta^i_e\mathsf{RFN}(n)$ is first introduced in [2, Section 2] as $\psi_e(i, n)$. In that paper, it is shown that the syntactic reflection of Σ^1_i-SDC_0 for Π^1_{e+2}-formulas is equivalent to $\forall n.\psi_e(i, n)$ over ACA_0. In this section, we consider a standard analogue of this result. We also note that the concept of $\beta^i_e\mathsf{RFN}(n; \tau)$ is introduced in [4].

By iterating (2) in Theorem 2.7 finitely many times, Σ^1_i-SDC_0 proves

$$\forall X \exists \mathcal{M}_0, \ldots, \mathcal{M}_n (X \in \mathcal{M}_0 \wedge \beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_n))$$

for any $n \in \omega$. Notice that $\beta^i_e\mathsf{RFN}(n)$ is a Π^1_{e+2} approximation of the above sentence.

In [4], it is proved that the set $\{\beta^1_0\mathsf{RFN}(n) : n \in \omega\}$ characterizes the Π^1_2-consequences of Π^1_1-CA_0. The key point of this proof is that when a Π^1_3-sentence of the form $\forall X \exists Y \theta(X, Y)$ is provable from Π^1_1-CA_0, then there exists $n \in \omega$ such that a solution Y is computable from any $\mathcal{M}_0, \ldots, \mathcal{M}_n$ with $\beta^1_1(\mathcal{M}_0, \ldots, \mathcal{M}_n)$ and $X \in \mathcal{M}_0$. We extend this result to more general cases.

Lemma 2.12 *Let $i \in \omega$ such that $i \geq 1$ and $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}})$ be a model of* ACA_0. *Let $S' \subseteq S^{\mathcal{M}}$ be such that $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}}, S')$ is a model of* RCA_0 *with the following property: for any $X \in S'$, there exists $Y \in S'$ such that $\mathcal{M} \models \beta_i(Y) \wedge (X \in Y)$. Then,*

- *$\mathcal{M} \models \beta_i(Y)$ implies $\mathcal{M}' \models \beta_i(Y)$ for any $Y \in S'$,*
- *$\mathcal{M}'$ is a model of Σ^1_i-SDC_0, and*
- *$\mathcal{M}'$ is a β_i-submodel of $\mathcal{M}$.*

Proof We prove the lemma by induction. The case $i = 1$ is proved in [4].

Let $i \in \omega$. Assume that for any $X \in S'$, there exists $Y \in S'$ such that $\mathcal{M} \models \beta_{i+1}(Y) \wedge (X \in Y)$. By the induction hypothesis, the following properties hold.

(1) $\mathcal{M} \models \beta_i(Y)$ implies $\mathcal{M}' \models \beta_i(Y)$ for any $Y \in S'$, and
(2) $\mathcal{M}'$ is a β_i submodel of $\mathcal{M}$.

Let $Y \in S'$ such that $\mathcal{M} \models \beta_{i+1}(Y)$. We show that $\mathcal{M}' \models \beta_{i+1}(Y)$. Let $\varphi(X, Z)$ be a Π^1_i-formula and $X \in Y$ such that $\mathcal{M}' \models \exists Z \varphi(X, Z)$. Since $\mathcal{M}'$ is a β_i submodel of $\mathcal{M}$, $\mathcal{M} \models \exists Z \varphi(X, Z)$. Then, $\mathcal{M} \models [Y \models \exists Z \varphi(X, Z)]$ because $\mathcal{M} \models \beta_{i+1}(Y)$. That is, there exists $Z \in Y$ such that $\mathcal{M} \models [Y \models \varphi(X, Z)]$. Then, $\mathcal{M}' \models [Y \models \varphi(X, Z)]$ because $Y \models \varphi(X, Z)$ is an arithmetical condition for X, Y and Z. Thus, $\mathcal{M}' \models [Y \models \exists Z \varphi(X, Z)]$.

We next show that $\mathcal{M}'$ is a β_{i+1} submodel of $\mathcal{M}$. Let $\varphi(X, Z)$ be a Π^1_i-formula and $X \in \mathcal{M}'$ such that $\mathcal{M} \models \exists Z \varphi(X, Z)$. Let $Y \in \mathcal{M}'$ such that $\mathcal{M} \models \beta_{i+1}(Y) \wedge X \in Y$. Then, $\mathcal{M} \models [Y \models \exists Z \varphi(X, Z)]$ and hence $\mathcal{M}' \models [Y \models \exists Z \varphi(X, Z)]$. Therefore $\mathcal{M}' \models \exists Z \varphi(X, Z)$ because $\mathcal{M}' \models \beta_i(Y)$ by the induction hypothesis.

Since $\mathcal{M}' \models \forall X \exists Y (X \in Y \wedge \beta_{i+1}(Y))$, $\mathcal{M}'$ is a model of Σ^1_{i+1}-SDC_0. □

Lemma 2.13 *Let φ be a Π^1_i-formula such that $\forall X \exists Y \varphi$ is provable from Σ^1_i-SDC_0. Then, there exists an $n \in \omega$ such that* ACA_0 *proves that*

$$\forall X, \mathcal{M}_0, \ldots, \mathcal{M}_n ((X \in \mathcal{M}_0 \wedge \beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_n)) \to \exists Y \in \mathcal{M}_n \varphi(X, Y)).$$

Proof Let φ be a Π^1_i-formula such that Σ^1_i-$\mathsf{SDC}_0 \vdash \forall X \exists Y \varphi$. Assume that ACA_0 does not prove

$$\forall X, \mathcal{M}_0, \ldots, \mathcal{M}_n ((X \in \mathcal{M}_0 \wedge \beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_n)) \to \exists Y \in \mathcal{M}_n \varphi(X, Y))$$

for any $n \in \omega$. That is, for any $n \in \omega$, ACA_0 is consistent with

$$\exists X, \mathcal{M}_0, \ldots, \mathcal{M}_n (X \in \mathcal{M}_0 \wedge \beta_i^i(\mathcal{M}_0, \ldots, \mathcal{M}_n) \wedge \forall Y \in \mathcal{M}_n \neg\varphi(X, Y)).$$

Let $\mathcal{L}'$ be the language given by adding new set constant symbols C and $D_0, D_1, \ldots$ into $\mathcal{L}_2$. Define an $\mathcal{L}'$-theory T consisting of the following sentences:

$$\begin{aligned}
&\mathsf{ACA}_0,\\
&C \in D_0,\\
&D_n \in D_{n+1} \text{ for } n \in \omega,\\
&\beta_i(D_n) \text{ for } n \in \omega,\\
&\forall Y \in D_n \neg\varphi(C, Y) \text{ for } n \in \omega.
\end{aligned}$$

Then, each finite subset of T is consistent. Thus, from the compactness theorem, there exists a model $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}}, C, \{D_n\}_{n\in\omega})$ of T. Define $S' \subseteq S^{\mathcal{M}}$ by $S' = \bigcup_{n\in\omega}\{X \in S^{\mathcal{M}} : \mathcal{M} \models X \in D_n\}$.

It is easy to see that $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}}, S')$ is a model of RCA_0. Moreover, for any $X \in S'$, there exists $Y \in S'$ such that $\mathcal{M} \models X \in Y \wedge \beta_i(Y)$. Thus, by Lemma 2.12, $\mathcal{M}'$ is a model of Σ_i^1-SDC_0 and a β_i submodel of $\mathcal{M}$. Thus $\mathcal{M}' \models \forall Y \neg\varphi(C, Y)$ because $\mathcal{M} \models \neg\varphi(C, Y)$ for any $Y \in \mathcal{M}'$. However, this contradicts the assumption Σ_i^1-$\mathsf{SDC}_0 \vdash \forall X \exists Y \varphi(X, Y)$. □

Theorem 2.14 *Let $e, i \in \omega$ such that $i > e$. Then, the theories $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$ and $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma_i^1\text{-}\mathsf{SDC}_0)$ prove the same sentences.*

Proof Since ACA_0 is Π^1_2 axiomatizable, ACA_0 is in $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma_i^1\text{-}\mathsf{SDC}_0)$. It is easy to see that $\beta_e^i\mathsf{RFN}(n)$ is in $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma_i^1\text{-}\mathsf{SDC}_0)$ for each $n \in \omega$. Therefore, any sentence provable from $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$ is also provable from $\mathrm{Th}_{\Pi^1_{e+2}}(\Sigma_i^1\text{-}\mathsf{SDC}_0)$.

For the converse, let σ be a Π^1_{e+2} sentence provable from Σ_i^1-SDC_0. Write $\sigma \equiv \forall X \exists Y \varphi(X, Y)$ by a Π^1_e-formula φ. Then, there exists $k \in \omega$ such that ACA_0 proves

$$\forall X, \mathcal{M}_0, \ldots, \mathcal{M}_k((X \in \mathcal{M}_0 \wedge \beta_i^i(\mathcal{M}_0, \ldots, \mathcal{M}_k)) \to \exists Y \in \mathcal{M}_k \varphi(X, Y)).$$

We show that $\forall X \exists Y \varphi(X, Y)$ is provable from $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$. We reason in $\mathsf{ACA}_0 + \{\beta_e^i\mathsf{RFN}(n) : n \in \omega\}$. Take an arbitrary X. By $\beta_e^i\mathsf{RFN}(k+1)$, take $\mathcal{M}_0, \ldots, \mathcal{M}_{k+1}$ such that

$$X \in \mathcal{M}_0 \wedge \beta_e^i(\mathcal{M}_0, \ldots, \mathcal{M}_{k+1}) \wedge \bigwedge_{m<k+2} \mathcal{M}_m \models \mathsf{ACA}_0.$$

Then, $\mathcal{M}_{k+1}$ satisfies $\mathsf{ACA}_0 \wedge X \in \mathcal{M}_0 \wedge \beta_i^i(\mathcal{M}_0, \ldots, \mathcal{M}_k)$. Thus, $\mathcal{M}_{k+1} \models \exists Y \in \mathcal{M}_k \varphi(X, Y)$ and hence $\mathcal{M}_{k+1} \models \exists Y \varphi(X, Y)$. Since $\mathcal{M}_{k+1}$ is a β_e model and φ is a Π^1_e-formula, $\exists Y \varphi(X, Y)$ holds. □

Corollary 2.15 $\mathsf{ACA}_0 + \{\beta^i_e\mathsf{RFN}(n) : n \in \omega\}$ *is the same as* $\mathrm{Th}_{\Pi^1_{e+2}}(\Pi^1_i\text{-}\mathsf{CA}_0)$ *for* $e = 0, 1, 2$ *and* $i > e$.

Proof The case $i \geq 3$ is immediate from the fact that $\Sigma^1_i\text{-}\mathsf{SDC}_0$ is Π^1_4-conservative over $\Pi^1_i\text{-}\mathsf{CA}_0$ [3, VII.6.20]. The cases $i = 1$ and $i = 2$ are from the fact that $\Sigma^1_i\text{-}\mathsf{SDC}_0$ is equivalent to $\Pi^1_i\text{-}\mathsf{CA}_0$ [3, VII.6.9]. □

3 Boolean combinations of Π^1_{i+1}-formulas provable from Σ^1_i-SDC

In this section, we give a characterization of the set $\{\sigma \in \mathsf{B}(\Pi^1_{i+1}) : \Sigma^1_i\text{-}\mathsf{SDC}_0 \vdash \sigma\}$ for $i > 0$. Here, $\mathsf{B}(\Gamma)$ is the set of boolean combinations of formulas in Γ. In other words, $\mathsf{B}(\Gamma)$ denotes the set generated by $\wedge, \vee, \neg$ starting from Γ.

Definition 3.1 For a sentence σ, we define $\beta_i\mathsf{Rfn}(\sigma)$ by

$$\sigma \to \exists \mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma).$$

For a class Γ of formulas, we define an axiomatic system $\beta_i\mathsf{Rfn}(\Gamma)_0$ by $\mathsf{ACA}_0 + \{\beta_i\mathsf{Rfn}(\sigma) : \sigma \in \Gamma, \sigma \text{ is a sentence}\}$.

Remark 3.2 Notice that $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is a weaker variant of (3) in Theorem 2.7. The difference between those is whether σ allows (set) parameters. Since each instance of $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is Σ^1_{i+2}, $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is strictly weaker than $\Sigma^1_i\text{-}\mathsf{SDC}_0$.

We show that the theories $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ and $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ are equivalent. We start with showing a basic property of $\mathsf{B}(\Pi^1_i)$.

Lemma 3.3 *Let* $\rho \in \mathsf{B}(\Pi^1_i)$. *Then, there exist* $\pi_0, \ldots, \pi_{n-1} \in \Pi^1_i$ *and* $\sigma_0, \ldots, \sigma_{n-1} \in \Sigma^1_i$ *such that* ρ *is equivalent to* $\bigwedge_{n'<n}(\pi_{n'} \vee \sigma_{n'})$ *over* RCA_0.

Proof We reason in RCA_0. Then, Π^1_i and Σ^1_i are closed under $\vee$.

Let ρ be a $\mathsf{B}(\Pi^1_i)$ sentence. Taking a conjunction normal form,

$$\rho \leftrightarrow \bigwedge_{n'<n} \bigvee_{m'<m} (\pi_{n',m'} \vee \sigma_{n',m'})$$

where each $\pi_{n',m'}$ is Π^1_i and each $\sigma_{n',m'}$ is Σ^1_i. Put $\pi_{n'} \equiv \bigvee_{m'<m} \pi_{n',m'}$ and $\sigma_{n'} \equiv \bigvee_{m'<m} \sigma_{n',m'}$. Then, $\pi_{n'} \in \Pi^1_i$, $\sigma_{n'} \in \Sigma^1_i$ and

$$\rho \leftrightarrow \bigwedge_{n'<n} (\pi_{n'} \vee \sigma_{n'}).$$

This completes the proof. □

Definition 3.4 We define $\beta^i_e\mathsf{RFN}^-(\varphi; n)$ as the Σ^1_{e+1}-sentence stating that there exist $Y_0, \dots Y_n$ such that $Y_0 \models \varphi$, and $\beta^i_e(Y_0, \dots, Y_n)$. We omit φ if it is trivial.

Lemma 3.5 *Let $i \in \omega$ such that $i > 0$. Then, for any $n \in \omega$, $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ proves $\varphi \to \beta^i_i\mathsf{RFN}^-(\varphi; n)$ for any sentence $\varphi \in \Sigma^1_{i+1}$.*

Proof We prove the lemma by induction. In the case $n = 0$, $\varphi \to \beta^i_i\mathsf{RFN}^-(\varphi; n)$ is the same as $\varphi \to \exists Y(\beta_i(Y) \wedge Y \models \varphi)$. Therefore, it is an instance of $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ and hence provable from $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$.

For the induction step, take an arbitrary n and assume that $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ proves $\varphi \to \beta^i_i\mathsf{RFN}^-(\varphi; n)$ for any sentence $\varphi \in \Sigma^1_{i+1}$. Take an arbitrary sentence $\varphi \in \Sigma^1_{i+1}$. We reason in $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ and show that $\varphi \to \beta^i_i\mathsf{RFN}^-(\varphi; n+1)$ holds.

Assume that φ holds. Then, we have $\beta^i_i\mathsf{RFN}^-(\varphi; n)$ because of the induction hypothesis. Since $\beta^i_i\mathsf{RFN}^-(\varphi; n)$ is a Σ^1_{i+1}-sentence,

$$\beta^i_i\mathsf{RFN}^-(\varphi; n) \to \exists Y(\beta_i(Y) \wedge Y \models \beta^i_i\mathsf{RFN}^-(\varphi; n))$$

also holds by the induction hypothesis. Thus we have $\exists Y(\beta_i(Y) \wedge Y \models \beta^i_i\mathsf{RFN}^-(\varphi; n))$. Take a β_i-model Y such that $Y \models \beta^i_i\mathsf{RFN}^-(\varphi; n)$. Then, there exist $Y_0, \dots, Y_n$ such that

$$Y \models \left[(Y_0 \models \varphi) \wedge \bigwedge_{m<n+1} \beta_i(Y_m) \wedge \bigwedge_{m<n} Y_m \in Y_{m+1}\right].$$

Since the condition in the square bracket is Π^1_i,

$$(Y_0 \models \varphi) \wedge \bigwedge_{m<n+1} \beta_i(Y_m) \wedge \bigwedge_{m<n} Y_m \in Y_{m+1}$$

actually holds. Take $Y_{n+1} = Y$, then $Y_0, \dots, Y_n, Y_{n+1}$ witness $\beta^i_i\mathsf{RFN}^-(\varphi; n+1)$. □

Lemma 3.6 *Let $i \in \omega$, $i > 0$. Let π be a Π^1_{i+1}-sentence and σ a Σ^1_{i+1}-sentence such that Σ^1_i-$\mathsf{SDC}_0 \vdash \pi \vee \sigma$. Then, $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0 \vdash \pi \vee \sigma$.*

Proof For the sake of contradiction, assume that Σ^1_i-SDC_0 proves $\pi \vee \sigma$ but $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ does not prove it. Then, the theory $T = \beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0 \cup \{\neg\pi, \neg\sigma\}$ is consistent.

Let $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}})$ be a model of T. We take new set constant symbols C_n for each $n \in \omega$. Define a theory T' by

$$T' = \mathsf{ACA}_0 + \neg\sigma + \{\beta^i_i(C_0, \dots, C_n) : n \in \omega\} + C_0 \models \neg\pi.$$

Then, $\mathcal{M}$ satisfies every finite subtheory of T' because $\mathcal{M}$ is a model of ACA_0, $\neg\sigma$ and $\beta^i_i\mathsf{RFN}^-(\neg\pi; n)$ for all n. Here, the condition $\mathcal{M} \models \beta^i_i\mathsf{RFN}^-(\neg\pi; n)$ is obtained from the previous lemma by putting $\varphi \equiv \neg\pi$. Therefore, T' is a consistent theory.

Let $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}'}, S, \{Y_n : n \in \omega\})$ be a model of T'. Define $S' = \bigcup_{n\in\omega}\{X \in S : \mathcal{M}' \models X \in Y_n\}$. Put $\mathcal{M}'' = (\mathbb{N}^{\mathcal{M}'}, S')$. Then, by Lemma 2.12,

- $\mathcal{M}''$ is a model of Σ^1_i-SDC_0,
- $\mathcal{M}''$ is a β_i-submodel of $\mathcal{M}'$ and
- for any $n \in \omega$, $\mathcal{M}'' \models \beta_i(Y_n)$.

We note that $\mathcal{M}''$ satisfies $\neg\sigma$ because $\mathcal{M}'$ satisfies $\neg\sigma$ and $\mathcal{M}''$ is a β_i-submodel of $\mathcal{M}'$. We claim that $\mathcal{M}''$ also satisfies $\neg\pi$. Indeed, $\mathcal{M}'' \models [Y_0 \models \neg\pi]$ because $\mathcal{M}' \models [Y_0 \models \neg\pi]$ and the condition $Y_0 \models \neg\pi$ is arithmetical. Since $\mathcal{M}'' \models \beta_i(Y_0)$, $\mathcal{M}'' \models \neg\pi$.

Consequently, $\mathcal{M}''$ is a model of Σ^1_i-$\mathsf{SDC}_0 \cup \{\neg\pi, \neg\sigma\}$. However, this contradicts the assumption Σ^1_i-SDC_0 proves $\pi \vee \sigma$. □

Theorem 3.7 *The theories $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ and $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ prove the same sentences.*

Proof It is enough to show that each axiom in $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is provable from $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ and vice verse.

We first show that each axiom in $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ is provable from the theory $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$. It is enough to show that $\beta_i\mathsf{Rfn}(\sigma)$ is in $\mathsf{B}(\Pi^1_{i+1})$ for $\sigma \in \Sigma^1_{i+1}$. Moreover, it is sufficient to show that $\exists\mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma) \in \mathsf{B}(\Pi^1_{i+1})$ because $\beta_i\mathsf{RFN}(\sigma)$ is of the form $\sigma \to \exists\mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma)$ and $\sigma \in \mathsf{B}(\Pi^1_{i+1})$. As in Remark 2.6 (3), $\beta_i(\mathcal{M})$ is Π^1_i and hence $\exists\mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \sigma)$ is Σ^1_{i+1}.

We then show that each $\mathsf{B}(\Pi^1_{i+1})$-formula provable from Σ^1_i-SDC_0 is already provable from $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$.

Let $\rho \in \mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$. We show that ρ is provable from $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$. There exist $\pi_0, \ldots, \pi_{n-1} \in \Pi^1_{i+1}, \sigma_0, \ldots, \sigma_{n-1} \in \Sigma^1_{i+1}$ such that ρ is equivalent to $\bigwedge_{m<n} \pi_m \vee \sigma_m$ by Lemma 3.3. Since ρ is provable from Σ^1_i-SDC_0, $\pi_m \vee \sigma_m$ is provable from Σ^1_i-SDC_0 for any $m < n$. Therefore, $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ proves $\pi_m \vee \sigma_m$ for any $m < n$ by Lemma 3.6. Hence $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ proves $\bigwedge_{m<n} \pi_m \vee \sigma_m$. □

Corollary 3.8 *The theories $\beta_i\mathsf{Rfn}(\Sigma^1_{i+1})_0$ and $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$ prove the same sentences.*

Proof It is enough to show that each instance of $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))$ is a $\mathsf{B}(\Pi^1_{i+1})$-sentence provable from Σ^1_i-SDC_0. This is immediate from Theorem 2.7. □

Corollary 3.9 *For any $i > 0$, $\mathrm{Th}_{\mathsf{B}(\Pi^1_{i+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$ is not finitely axiomatizable.*

Proof We show that $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$ is not finitely axiomatizable. For the sake of contradiction, suppose that $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$ is finitely axiomatizable. Then, there are finitely many instances $\rho_0, \ldots, \rho_{n-1}$ of $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$ such that $\mathsf{ACA}_0 + \bigwedge_{i<n} \rho_i$ axiomatizes $\beta_i\mathsf{Rfn}(\mathsf{B}(\Pi^1_{i+1}))_0$. Let ρ be the conjunction of $\rho_0, \ldots, \rho_n$. Since ρ is in $\mathsf{B}(\Pi^1_{i+1})$, we have $\mathsf{ACA}_0 + \rho \vdash \exists\mathcal{M}(\beta_i(\mathcal{M}) \wedge \mathcal{M} \models \rho)$. Therefore, $\mathsf{ACA}_0 + \rho$ proves the consistency of itself, a contradiction. □

Theorem 3.10 Σ^1_i-SDC_0 *is* Σ^1_{i+1}*-conservative over* $\mathsf{ACA}_0 + \{\beta^i_i\mathsf{RFN}^-(n) : n \in \omega\}$*. In particular,* Π^1_1-CA_0 *is* Σ^1_2*-conservative over the theory* $\mathsf{ACA}_0 + \{\exists Y(Y = \mathrm{HJ}^n(\varnothing)) : n \in \omega\}$.

Proof For the former statement, the same argument for Lemma 3.6 works. From this result and the equivalence of the existence of a coded β-model and a hyperjump [3, VII.2.9], we have the latter holds. □

In [4], it is proved that $\{\exists Y(Y = \mathrm{HJ}^n(\varnothing)) : n \in \omega\}$ is equivalent to $\{(\Sigma^{0,\varnothing}_1)_n\text{-}\mathsf{Det} : n \in \omega\}$ over ACA_0. Here, $(\Sigma^{0,\varnothing}_1)_n$-$\mathsf{Det}$ is the determinacy of Gale-Stewart games with payoff sets of complexity $(\Sigma^{0,\varnothing}_1)_n$. Thus, we have the following characterization of Σ^1_2-consequences of Π^1_1-CA_0.

Corollary 3.11 Π^1_1-CA_0 *is* Σ^1_2*-conservative over* $\mathsf{ACA}_0 + \{(\Sigma^{0,\varnothing}_1)_n\text{-}\mathsf{Det} : n \in \omega\}$.

We then see the relationship between $\beta^i_e\mathsf{RFN}^-$ and RFN^i_e.

Definition 3.12 Let Γ be a class of formulas. For theories T and T', we say T is a Γ-extension of T' (written $T' \subseteq_\Gamma T$) if any sentence in Γ which is provable from T' is also provable from T. We write $T' \subsetneq_\Gamma T$ if there is a sentence in Γ which is provable from T but not provable from T' in addition. We also write $T =_\Gamma T'$ when T and T' are Γ-equivalent, that is, they prove the same Γ-sentence.

As $\mathsf{ACA}_0 + \{\beta^1_0\mathsf{RFN}(n) : n \in \omega\} =_{\Pi^1_2} \Pi^1_1$-$\mathsf{CA}_0$ and $\mathsf{ACA}_0 + \{\exists Y(Y = \mathrm{HJ}^n(\varnothing)) : n \in \omega\} =_{\Sigma^1_2} \Pi^1_1$-$\mathsf{CA}_0$, these three theories are Π^1_1-equivalent. Thus, we have two approximations of the Π^1_1-consequences of Π^1_1-CA_0; one is the hierarchy of $\beta^1_0\mathsf{RFN}(0), \beta^1_0\mathsf{RFN}(1), \beta^1_0\mathsf{RFN}(2), \dots,$ and the other is the hierarchy of the iterated hyperjumps of the empty set. In what follows, we compare these two approximations.

Lemma 3.13 *Let T and T' be theories such that T is recursively axiomatizable and T' includes* ACA_0*. If T' proves the β_i-model reflection of T, then* $T \subsetneq_{\Pi^1_{i+2}} T'$ *and T' proves the consistency of T.*

Proof Let $\sigma \equiv \forall X \exists Y \theta(X, Y)$ be a Π^1_{i+2}-sentence provable from T. We work in T'. Take an arbitrary X and a β_i-model $\mathcal{M}$ such that $X \in \mathcal{M}$ and $\mathcal{M} \models T$. Then, $\mathcal{M} \models \exists Y \theta(X, Y)$. Since θ is Π^1_i and $\mathcal{M}$ is a β_i-model, $\exists Y \theta(X, Y)$ holds. □

By a similar proof, we also have the following.

Lemma 3.14 *Let T and T' be theories such that T is recursively axiomatizable and T' includes* ACA_0*. If T' proves the existence of a β_i-model of T, then* $T \subsetneq_{\Sigma^1_{i+1}} T'$ *and T' proves the consistency of T.*

Theorem 3.15 *Let $i, e \in \omega$ such that $e < i$. Then, for each $n \in \omega$,* $\beta^i_e\mathsf{RFN}(n+1)$ *proves the β_e-model reflection of* $\beta^i_i\mathsf{RFN}^-(n)$ *over* ACA_0*. In particular, we have* $\beta^i_i\mathsf{RFN}^-(n) \subsetneq_{\Pi^1_{e+2}} \beta^i_e\mathsf{RFN}(n+1)$ *over* ACA_0.

Proof Take an arbitrary X. By $\beta^i_e\mathsf{RFN}(n+1)$, take $\mathcal{M}_0, \ldots, \mathcal{M}_{n+1}$ such that $X \in \mathcal{M}_0$ and $\beta^i_e(\mathcal{M}_0, \ldots, \mathcal{M}_{n+1})$. Then, $\mathcal{M}_{n+1}$ is a β_e-model and $\mathcal{M}_{n+1} \models \beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_n)$. In particular, $\mathcal{M}_{n+1} \models \beta^i_i\mathsf{RFN}^-(n)$. The inclusions $\beta^i_i\mathsf{RFN}^-(n) \subsetneq_{\Pi^1_{e+2}} \beta^i_e\mathsf{RFN}(n+1)$ follows from Lemma 3.13. □

Theorem 3.16 *Let $i \in \omega$ such that $i > 0$. Then, for each $n \in \omega$, $\beta^i_i\mathsf{RFN}^-(n+1)$ proves the existence of a β_i-model of $\beta^i_{i-1}\mathsf{RFN}(n+1)$. In particular, we have $\beta^i_{i-1}\mathsf{RFN}(n+1) \subsetneq_{\Sigma^1_{i+1}} \beta^i_i\mathsf{RFN}^-(n+1)$ over Σ^1_{i-1}-SDC_0.*

Proof Let $\mathcal{M}_0, \ldots, \mathcal{M}_{n+1}$ be such that $\beta^i_i(\mathcal{M}_0, \ldots, \mathcal{M}_{n+1})$. Then, $\mathcal{M}_0$ is a β_i-model, and $\mathcal{M}_{n+1}$ is a model of Σ^1_{i-1}-SDC_0. We prove that $\mathcal{M}_0$ is a model of $\beta^i_{i-1}\mathsf{RFN}(n+1)$. Since $\mathcal{M}_{n+1}$ is a model of Σ^1_{i-1}-SDC, there exists an $\mathcal{M}' \in \mathcal{M}_{n+1}$ such that $\mathcal{M}_n \in \mathcal{M}'$ and $\mathcal{M}_{n+1} \models \beta_{i-1}(\mathcal{M}')$. We show that $\mathcal{M}'$ believes that $\mathcal{M}_n$ is a β_i-model. Take a Π^1_i-sentence σ with parameters from $\mathcal{M}_n$ such that $\mathcal{M}_n \models \sigma$. Then, $\mathcal{M}_{n+1} \models \sigma$ because $\mathcal{M}_n$ is a β_i-submodel of $\mathcal{M}_{n+1}$. Since $\mathcal{M}'$ is a β_{i-1}-submodel of $\mathcal{M}_{n+1}$, $\mathcal{M}' \models \sigma$.

Now, for any $X \in \mathcal{M}_0$, $\mathcal{M}_{n+1}$ satisfies

$$(*)\ \exists Y_0, \ldots, Y_{n+1}(X \in Y_0 \wedge \beta^i_{i-1}(Y_0, \ldots, Y_n, Y_{n+1}))$$

via taking $Y_m = \mathcal{M}_m$ for $m < n+1$ and $Y_{n+1} = \mathcal{M}'$. Since $(*)$ is a Σ^1_i condition and $\mathcal{M}_0$ is a β_i-submodel of $\mathcal{M}_{n+1}$, $\mathcal{M}_0$ also satisfies $(*)$. The inclusion $\beta^i_{i-1}\mathsf{RFN}(n+1) \subsetneq_{\Sigma^1_{i+1}} \beta^i_i\mathsf{RFN}^-(n+1)$ follows from Lemma 3.14. □

Corollary 3.17 *Over ACA^+_0,*

$$\exists Y(Y = \mathrm{HJ}^{n+1}(\varnothing)) \subsetneq_{\Pi^1_2} \beta^1_0\mathsf{RFN}(n+1) \subsetneq_{\Sigma^1_2} \exists Y(Y = \mathrm{HJ}^{n+2}(\varnothing)).$$

Proof It is immediate from the above theorems and the equivalence of the existence of a β-model and a hyperjump. □

Remark 3.18 We note that the inclusions in the previous corollary also hold over a theory T such that T includes ACA^+_0 and is a proper subtheory of $\beta^1_0\mathsf{RFN}(n+1)$. For example, the inclusions also hold over ATR_0.

4 Boolean combinations of Π^1_{e+1}-formulas provable from Σ^1_i-SDC_0

In this section, we generalize the result in the previous sections to give a characterization of $\mathrm{Th}_{\mathsf{B}(\Pi^1_{e+1})}(\Sigma^1_i\text{-}\mathsf{SDC})$ for $e < i$. Most arguments of this section are almost the same as those in the previous section. That is, we use statements of the form $\sigma \to \beta^i_e\mathsf{RFN}(\sigma; n)$ for the characterization. The difference between this section and the previous section is that

- $\{\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; 1) : \sigma \in \Sigma^1_{e+1}\}$ does not include $\{\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n) : n \in \omega, \sigma \text{ is a } \Sigma^1_{e+1}\text{-sentence}\}$,

- by our compactness argument, the constructed model may not be a model of Σ^1_i-SDC_0.

To address the first problem, we adopt $\{\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n) : n \in \omega,\ \sigma$ is a Σ^1_{e+1}- sentence$\}$ for the characterization. To address the second problem, we slightly modify the proof in the previous section.

We give a comment concerning the first item.

Proposition 4.1 *Let $i, e \in \omega$ such that $e < i$. Let $n \in \omega$. Then, the theory $\mathsf{ACA}_0 + \{\beta^i_e\mathsf{RFN}^-(\sigma; n+1) : \sigma$ is a Σ^1_{e+1}-sentence$\}$ proves the consistency of $\mathsf{ACA}_0 + \{\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n) : \sigma$ is a Σ^1_{e+1}-sentence$\}$.*

Proof Let $\mathcal{M}_0, \dots, \mathcal{M}_{n+1}$ be coded ω-models such that $\beta^i_e(\mathcal{M}_0, \dots, \mathcal{M}_{n+1})$. We show that $\mathcal{M}_0 \models [\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n)]$ for $\sigma \in \Sigma^1_{e+1}$.

Assume $\mathcal{M}_0 \models \sigma$. Then, $\mathcal{M}_{n+1} \models [\exists \mathcal{N}_0, \dots, \mathcal{N}_n(\beta^i_i(\mathcal{N}_0, \dots, \mathcal{N}_n) \wedge \mathcal{N}_0 \models \sigma)]$ by taking $\mathcal{N}_0 = \mathcal{M}_0, \dots, \mathcal{N}_n = \mathcal{M}_n$. Then we also have $\mathcal{M}_{n+1} \models [\exists \mathcal{N}_0, \dots, \mathcal{N}_n(\beta^i_e(\mathcal{N}_0, \dots, \mathcal{N}_n) \wedge \mathcal{N}_0 \models \sigma)]$ because $e < i$. Since $\mathcal{M}_0$ is a β_i-submodel of $\mathcal{M}_{n+1}$ and $[\exists \mathcal{N}_0, \dots, \mathcal{N}_n(\beta^i_e(\mathcal{N}_0, \dots, \mathcal{N}_n) \wedge \mathcal{N}_0 \models \sigma)]$ is Σ^1_{e+1}, which is included Σ^1_i, $\mathcal{M}_0 \models [\exists \mathcal{N}_0, \dots, \mathcal{N}_n(\beta^i_e(\mathcal{N}_0, \dots, \mathcal{N}_n) \wedge \mathcal{N}_0 \models \sigma)]$. □

Theorem 4.2 *Let $i, e \in \omega$, $e < i$. The following theories are equivalent.*

(1) $\mathsf{ACA}_0 + \{\sigma \to \beta^i_e\mathsf{RFN}^-(\sigma; n) : n \in \omega, \sigma$ *is a* Σ^1_{e+1}*-sentence*$\}$.
(2) $\mathrm{Th}_{\mathsf{B}(\Pi^1_{e+1})}(\Sigma^1_i\text{-}\mathsf{SDC}_0)$.

Proof It is trivial that any instance of (1) is in (2). Therefore, (1) is a subtheory of (2). We show the converse inclusion.

Let T be the theory of (1) As in the previous section, it is enough to show that for any $\sigma \in \Sigma^1_{e+1}$ and $\pi \in \Pi^1_{e+1}$, if Σ^1_i-SDC_0 proves $\sigma \vee \pi$ then so does T.

Let $\sigma \in \Sigma^1_{e+1}$ and $\pi \in \Pi^1_{e+1}$ such that Σ^1_i-SDC_0 proves $\sigma \vee \pi$. Assume that T does not prove $\sigma \vee \pi$ for the sake of contradiction. Let $\{C_n\}_{n \in \omega}$ be new set constants. Then, the theory

$$T' = T + \neg\sigma + \{\beta^i_e(C_0, \dots, C_n) : n \in \omega\} + C_0 \models \neg\pi$$

is consistent. Let $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, S^{\mathcal{M}}, \{A_n\}_{n \in \omega})$ be a model of T'. Put $S' = \bigcup_{n \in \omega}\{X \in S^{\mathcal{M}} : \mathcal{M} \models X \in A_n\}$ and $\mathcal{M}' = (\mathbb{N}^{\mathcal{M}}, S')$. We note that

- $\mathcal{M}' \models \beta_e(A_n)$ for all $n \in \omega$,
- $\mathcal{M}'$ is a β_e-submodel of $\mathcal{M}$

by Lemma 2.12. We show that $\mathcal{M}'$ satisfies both $\sigma \vee \pi$ and $\neg\sigma \wedge \neg\pi$, a contradiction.

Claim 4.2.1 $\mathcal{M}' \models \sigma \vee \pi$.

Proof of the claim Since $\sigma \vee \pi$ is a Π^1_{e+2}-sentence, there exists a Π^1_e-formula ρ such that $\sigma \vee \pi \leftrightarrow \forall X \exists Y \rho$. Since $e < i$, we can apply Lemma 2.13. Thus, we have $n \in \omega$ such that

$$\mathsf{ACA}_0 \vdash \forall X, \mathcal{M}_0, \dots, \mathcal{M}_n((X \in \mathcal{M}_0 \wedge \beta^i_i(\mathcal{M}_0, \dots, \mathcal{M}_n)) \to \exists Y \in \mathcal{M}_n \rho(X, Y)).$$

To prove $\mathcal{M}' \models \forall X \exists Y \rho$, take an arbitrary $X \in \mathcal{M}'$. Then, for sufficiently large k, we have $X \in A_k$. Within $\mathcal{M}'$, the following properties holds.

- $\beta_e(A_{k+n+1})$ and $A_{k+n+1} \models \mathsf{ACA}_0$,
- A_{k+n+1} satisfies $X \in A_k \wedge \beta_i^i(A_k, \dots, A_{k+n})$.

Therefore, $\mathcal{M}' \models \left[A_{k+n+1} \models \exists Y \rho(X, Y)\right]$. Since $\mathcal{M}' \models \beta_e(A_{k+n+1})$ and ρ is a Π^1_e-formula, $\mathcal{M}' \models \exists Y \rho(X, Y)$. Consequently, $\mathcal{M}' \models \forall X \exists Y \rho(X, Y)$ and hence $\mathcal{M}' \models \sigma \vee \tau$. □

Claim 4.2.2 $\mathcal{M}' \models \neg\sigma \wedge \neg\pi$.

Proof of the claim Since $\neg\sigma$ is a Π^1_{e+1}-sentence true in $\mathcal{M}$ and $\mathcal{M}'$ is a β_e-submodel of $\mathcal{M}$, $\mathcal{M}'$ also satisfies $\neg\sigma$. Since $\neg\pi$ is a Σ^1_{e+1}-sentence true in A_0 and A_0 is a β_i-submodel of $\mathcal{M}'$, $\mathcal{M}'$ also satisfies $\neg\pi$. □

The above claims complete the proof. □

By a similar argument, we have the following characterization of Σ^1_{e+1}-consequences of Σ^1_i-SDC_0.

Theorem 4.3 *Let $i, e \in \omega$, $e < i$. Then, Σ^1_i-SDC_0 is Σ^1_{e+1} conservative over* $\mathsf{ACA}_0 + \{\beta^i_e\mathsf{RFN}^-(n) : n \in \omega\}$.

From the Π^1_4-equivalence of Σ^1_i-SDC_0 and Π^1_i-CA_0, we also have the following.

Corollary 4.4 *Let $i, e \in \omega$, $e < i$ and $e < 2$. Then, Π^1_e-CA_0 is Σ^1_{e+1} conservative over* $\mathsf{ACA}_0 + \{\beta^i_e\mathsf{RFN}^-(n) : n \in \omega\}$.

Acknowledgements Aguilera would like to acknowledge support from the Austrian Science Foundation through grant STA-139. Yokoyama's work is partially supported by JSPS KAKENHI grant numbers JP19K03601, JP21KK0045 and JP23K03193.

Author Contributions Y.S. wrote the main manuscript text. All authors reviewed the manuscript. The main results in this manuscript are obtained from discussions among all authors.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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