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Eigenvalue problem versus Casimir functions for Lie algebras

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Abstract

We present a new perspective on the invariants of Lie algebras (Casimir functions). Our approach is based on the connection of a linear mapping $F \in End(V)$, which has a given eigenvector v, to a Lie algebra. We obtain a solvable Lie algebra by considering a single pair (F, v). However, by considering a set of such pairs (F_i, v_i), $i = 1, 2, \ldots, s$, we can obtain any finite-dimensional Lie algebra. We also describe the Casimir function equations in terms of pairs, since the eigenvalue problem of (F, v) yields a Lie bracket. We outline the criterion for the quantity of Casimirs and their formulas for any Lie algebra, which depends on the decomposability of the tensor built from the pairs (F_i, v_i). In addition, we present the meaning of lifting Lie algebras in this context and explain how to construct Casimir functions for the lifted Lie algebra based on Casimir functions for the initial Lie algebra. One of the main results of the paper is to present the method to identify all Casimirs for a lifted Lie algebra starting from the initial one.

Keywords Lie algebra · Lie bracket · Poisson bracket · Casimir function · Decomposable tensor · Nambu bracket · $ax + b$-group · Complete and vertical lifts

1 Introduction

In the theory of Lie algebras, various invariants are a useful tool, which allows to characterise such algebras. Casimir operators, or polynomial invariants, are one of the most important invariants for the study of Lie algebras. The study of quadratic invariants for given Lie algebras was started by the prominent scientists H.B.G. Casimir,

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B.L. van der Waerden in the 1930 s, [1, 4]. In the 1950, G. Racah published a paper in which he gave a construction of polynomial invariants, called Casimir invariants, for semisimple Lie algebras [20]. Also C. Chevalley investigated invariant polynomials in the enveloping algebra of semisimple Lie algebras [6]. In the recent past, a number of researchers have been involved in the study of Casimir invariants for non-simple Lie algebras, [16, 17, 19].

More generally, we are looking for functions, not necessarily polynomials, which commute with the general element of the given Lie algebra. Such functions will be called generalised Casimir invariants. There are two general methods for the determination of these functions, see [21]. Recall that one of them is the infinitesimal method, which involves a Poisson structure on the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. In a fixed basis $e_1, e_2, \ldots, e_N$ the Lie algebra is determined by the structure constants c_{ij}^k. Vector fields

$$X_i = \sum_{j,k=1}^{N} c_{ij}^k x_k \frac{\partial}{\partial x_j}, \quad i = 1, 2, \ldots, N$$

can be associated with them, where x_k are coordinates in $\mathfrak{g}^*$. The generalised Casimir invariants c are solutions of a system of partial differential equations

$$X_i\, c(x) = 0, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_N)$. This system can be written in matrix form

$$\pi(x)\nabla c(x) = 0, \tag{2}$$

where $\pi(x) = (\pi_{ij}(x)) = \left(\sum_{k=1}^N c_{ij}^k x_k\right)$ is an antisymmetric matrix of dimension $N \times N$ and ∇ is a gradient operator $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N}\right)^T$. It is easy to see that $\pi(x)$ is the matrix of the Poisson tensor $\pi \in \Gamma\left(\bigwedge^2 T\mathfrak{g}^*\right)$ of the Lie–Poisson bracket $\{\cdot, \cdot\}$ on $\mathfrak{g}^*$. The rank r of this matrix naturally determines the number of Casimirs, since there are $N - r$ of them.

The paper is organized as follows: in Sect. 2 we recall some information about a lift of multivector field, introduce complete and vertical lifts of such vector field, present properties of Schouten–Nijenhuis bracket and define Casimir function in terms of Poisson geometry. In Sect. 3 we present the method for computing Casimir functions for Lie algebras. The approach presented here is based on relation between a linear mapping $F \in End(V)$ with a fixed eigenvector v and a Lie algebra. A detailed discussion of the case of the eigenvalue problem given by pairs (F_i, v_i) are provided and relevant examples are given. The final result of these considerations is Theorem 3, which gives the criterion for the number of Casimir functions and their formulas in any Lie algebra, relating it to the decomposability of the tensor $\star \sum_{i=1}^N (F_i x \wedge v_i)$. Also we introduce the formulas for Casimir functions for a lifted Lie algebra. Considering an N-dimensional Lie algebra given by one pair (F, v), Theorem 2 says that it contains at most $N-2$ invariant functions. The other results of the paper are given in Theorems 4 and 5. They consist in giving formulas for all Casimirs on $2N$-dimensional lifted Lie algebra.

2 Preliminaries

The main purpose of this section is to recall some facts from the field of differential geometry and, in particular, Poisson geometry. We briefly review some definitions of complete and vertical lifts of multivector fields, Poisson structures, Lie algebras, and their relationship.

Let M be a finite dimensional smooth manifold with a local coordinate system $x = (x_1, x_2, \dots, x_N)$. Let k-vector fields on M be denoted by $\mathfrak{X}^k(M) = \Gamma(\bigwedge^k TM)$. Any multivector field on M in the local coordinate system has a form

$$X = \sum_{i_1, i_2, \dots, i_k = 1}^{N} v_{i_1, i_2, \dots, i_k}(x) \frac{\partial}{\partial x_{i_1}} \wedge \frac{\partial}{\partial x_{i_2}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_k}}. \tag{3}$$

Let $(x, y) = (x_1, x_2, \dots, x_N, y_1, y_2, \dots, y_N)$ denotes the induced local coordinate system on a tangent bundle TM of M. A lift of a k-vector field of $\mathfrak{X}^k(M)$ is a k-vector field belonging to $\mathfrak{X}^k(TM)$. The complete and vertical lifts of (3) on TM are given by

$$\begin{aligned}
X^C = &\sum_{i_1, i_2, \dots, i_k = 1}^{N} \left(\sum_{l=1}^{k} v_{i_1, i_2, \dots, i_k}(x) \frac{\partial}{\partial y_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y_{i_{l-1}}} \wedge \frac{\partial}{\partial x_{i_l}} \wedge \frac{\partial}{\partial y_{i_{l+1}}} \wedge \dots \wedge \frac{\partial}{\partial y_{i_k}} \right. \\
&\left. + \sum_{s=1}^{N} \frac{\partial v_{i_1, i_2, \dots, i_k}}{\partial x_s}(x) y_s \frac{\partial}{\partial y_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y_{i_k}} \right), \\
X^V = &\sum_{i_1, i_2, \dots, i_k = 1}^{N} v_{i_1, i_2, \dots, i_k}(x) \frac{\partial}{\partial y_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y_{i_k}}.
\end{aligned}$$

The space of multivectors $\mathfrak{X}(M) = \bigoplus_{k=0}^{\infty} \mathfrak{X}^k(M)$, where $\mathfrak{X}^0(M) = C^\infty(M)$ is endowed with a graded Lie algebra structure. This structure is given by the Schouten–Nijenhuis bracket, which is a bilinear map $[\cdot, \cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M)$ defined by the properties

$$[X, Y] = -(-1)^{(k-1)(l-1)}[Y, X], \tag{4}$$

$$\begin{aligned}
&(-1)^{(k-1)(m-1)}[X, [Y, Z]] + (-1)^{(l-1)(k-1)}[Y, [Z, X]] \\
&\quad + (-1)^{(l-1)(m-1)}[Z, [X, Y]] = 0,
\end{aligned} \tag{5}$$

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(k-1)l} Y \wedge [X, Z], \tag{6}$$

$$[W, Y] = \pounds_W Y, \tag{7}$$

$$[f, g] = 0, \tag{8}$$

where $X \in \mathfrak{X}^k(M), Y \in \mathfrak{X}^l(M), Z \in \mathfrak{X}^m(M),\ W \in \mathfrak{X}^1(M),\ f, g \in \mathfrak{X}^0(M) = C^\infty(M)$ and $\pounds_W$ is a Lie derivative by W. In particular $[W, f] = W(f)$.

Let us recall that a Poisson manifold (M, π) is a pair of a manifold M and a bivector field $\pi \in \Gamma(\bigwedge^2 TM)$ with the following property

$$[\pi, \pi] = 0. \tag{9}$$

The bivector π is called a Poisson tensor and it defines the Poisson bracket $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on M by the formula

$$\{f, g\} = \pi(df, dg).$$

The Poisson bracket is a bilinear, antisymmetric map, which satisfies the Leibniz property and the Jacobi identity.

The Poisson tensor π in a local coordinate system on M has a form

$$\pi(x) = \sum_{1 \le i < j}^{N} \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{10}$$

where $\pi_{ij}(x) = -\pi_{ji}(x) = \{x_i, x_j\}$. By means of the complete and vertical lifts of the multivectors, the tensor π can be lifted to the tensors on TM

$$\pi^C(x, y) = \sum_{1 \le i < j}^{N} \left(\pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + \pi_{ij}(x) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_j} + \sum_{s=1}^{N} \frac{\partial \pi_{ij}}{\partial x_s}(x) y_s \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \right),$$
$$\pi^V(x, y) = \sum_{1 \le i < j}^{N} \left(\pi_{ij}(x) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \right).$$

Matrices of these tensors are the following

$$\pi^C(x, y) = \left(\begin{array}{c|c} 0 & \pi(x) \\ \hline \pi(x) & \sum_{s=1}^{N} \frac{\partial \pi}{\partial x_s}(x) y_s \end{array} \right), \qquad \pi^V(x, y) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \pi(x) \end{array} \right).$$

The bivector π^C is also called a tangent Poisson tensor, it appears in a natural way in the theory of Lie algebroids, see [18]. Note that for a decomposable Poisson tensor, i.e., $\pi = X \wedge Y$, where $X, Y \in \mathfrak{X}^1(M)$, the formula holds

$$\pi^C = (X \wedge Y)^C = X^C \wedge Y^V + X^V \wedge Y^C.$$

It is shown in [9], that if only $[X, Y] = \alpha Y, \alpha \in \mathbb{R}$, then also $X^C \wedge Y^V$ gives a Poisson structure on TM (see for example [13]).

A Lie–Poisson structure on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a natural linear Poisson structure defined by

$$\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle$$

for $f, g \in C^\infty(\mathfrak{g}^*)$ and $x \in \mathfrak{g}^*$, where $df(x), dg(x) \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$. There is a one-to-one correspondence between N-dimensional linear Poisson structures and N-dimensional Lie algebras. Let $\{e_1, e_2, \ldots, e_N\}$ be a basis of $\mathfrak{g}$ and $x = (x_1, x_2, \ldots, x_N)$ be a linear system of coordinates on the dual space $\mathfrak{g}^*$. The commutator relations of the Lie algebra $[e_i, e_j] = \sum_{k=1}^N c_{ij}^k e_k$, where c_{ij}^k are the structure constants, give the Lie–Poisson bracket $\{x_i, x_j\} = \sum_{k=1}^N c_{ij}^k x_k$.

A smooth function c is called a Casimir function, if

$$\{f, c\} = 0 \tag{11}$$

for any function $f \in C^\infty(M)$. For $M = \mathfrak{g}^*$ the condition (11) is equivalent to (1) or (2).

3 Eigenvalue problem and invariants

Let V be a finite dimensional linear space over $\mathbb{R}$, $\dim V = N$. Consider a pair (F, v), where $F \in End(V)$ and v is an eigenvector of a map F corresponding to eigenvalue $\lambda = 0$ $(Fv = 0)$. By V^* we denote a dual space to V. A pair (F, v) gives a Lie bracket on a dual space V^*, as it was shown in [10]

$$[\psi, \phi]_{(F,v)} := \phi(v)F^*(\psi) - \psi(v)F^*(\phi), \tag{12}$$

where $\psi, \phi \in V^*$ (see also [12]).

Remark 1 The eigenvalue λ of the endomorphism F can be non-zero. However, this does not affect the Lie bracket (12).

It is easy to verify that Lie algebra $(V^*, [\cdot, \cdot]_{(F,v)})$ is solvable and if only operator F is nilpotent, then Lie algebra is also nilpotent. In this paper, by choosing bases, we can identify V and V^* with $\mathbb{R}^N$ with the canonical basis $\{e_1, e_2, \ldots, e_N\}$ (i.e. $V \simeq V^* \simeq \mathbb{R}^N$), so that the pairing between V and V^* is given by the scalar product. Then formula (12) can be rewritten in the form

$$[u, w]_{(F,v)} = \langle w|v\rangle F^T u - \langle u|v\rangle F^T w \quad \text{for } u, w \in \mathbb{R}^N, \tag{13}$$

where $\langle\cdot|\cdot\rangle$ is the scalar product in $\mathbb{R}^N$.

Remark 2 The Lie bracket (13) for a pair (F, v) is related to the generalised $ax + b$-group structure, see [2]. Assuming

$$u = \begin{pmatrix} s_1 \\ t_1 \end{pmatrix}, \quad w = \begin{pmatrix} s_2 \\ t_2 \end{pmatrix},$$

where $s_1, s_2 \in \mathbb{R}^{N-1}$, $t_1, t_2 \in \mathbb{R}$ and putting

$$v = e_N, \quad F = \begin{pmatrix} -G^T & 0 \\ 0 & 0 \end{pmatrix}, \text{ where } G \in Mat_{N-1}(\mathbb{R}),$$

Lie bracket has a form

$$[u, w]_{(F,v)} = t_1 G s_2 - t_2 G s_1,$$

which can be written as

$$\left[\begin{pmatrix} s_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} s_2 \\ t_2 \end{pmatrix}\right] = \begin{pmatrix} t_1 G s_2 - t_2 G s_1 \\ 0 \end{pmatrix}.$$

Let notice that if we consider endomorphism in the form $\tilde{F} = \left(\begin{array}{c|c} -G^T & 0 \\ \hline p^T & \lambda_N \end{array}\right)$, where $p \in \mathbb{R}^{N-1}$, $\lambda_N \in \mathbb{R}$ then we get the same Lie bracket, i.e., $[u, w]_{(F,e_N)} = [u, w]_{(\tilde{F},e_N)}$.

Remark 3 For a Lie algebra $\mathfrak{g}$ with the basis $\{e_1, e_2, \ldots, e_N\}$, given by commutator relations $[e_i, e_j] = \sum_{k=1}^{N} c_{ij}^k e_k$, we can assign N-pairs $(F_1, e_N), \ldots, (F_{N-i+1}, e_i), \ldots, (F_N, e_1)$, where

$$F_1 = \left(\begin{array}{cccc|c} c_{1N}^1 & c_{1N}^2 & \ldots & c_{1N}^{N-1} & 0 \\ c_{2N}^1 & c_{2N}^2 & \ldots & c_{2N}^{N-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{N-1\,N}^1 & c_{N-1\,N}^2 & \ldots & c_{N-1\,N}^{N-1} & 0 \\ \hline 0 & 0 & \ldots & 0 & 0 \end{array}\right), \quad F_N = \left(\begin{array}{c|cccc} 0 & 0 & \ldots & 0 \\ \hline 0 & -c_{1\,2}^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -c_{1\,N}^N \end{array}\right),$$

$$F_{N-i+1} = \left(\begin{array}{cccc|c|cccc} c_{1\,i}^1 & c_{1\,i}^2 & \ldots & c_{1\,i}^{i-1} & 0 & c_{1\,i}^{i+1} & \ldots & c_{1\,i}^N \\ c_{2\,i}^1 & c_{2\,i}^2 & \ldots & c_{2\,i}^{i-1} & 0 & c_{2\,i}^{i+1} & \ldots & c_{2\,i}^N \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{i-1\,i}^1 & c_{i-1\,i}^2 & \ldots & c_{i-1\,i}^{i-1} & 0 & c_{i-1\,i}^{i+1} & \ldots & c_{i-1\,i}^N \\ \hline 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & \ldots & 0 & 0 & -c_{i\,i+1}^{i+1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & -c_{i\,N}^N \end{array}\right).$$

The above construction is described in detail in the work [10]. This correspondence is not canonical (see Remark 2), one can choose linear mappings and their eigenvectors differently.

Let an eigenvector of a mapping F be chosen as the last vector of the basis, $v = e_N$. Then the operator F in basis B has the form

$$F = \begin{pmatrix} a_{11} & \dots & a_{1N-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{N-11} & \dots & a_{N-1N-1} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}. \tag{14}$$

Note that for each pair (F, v) we can associate two vector fields $X, Y \in \Gamma(T\mathbb{R}^N)$. In the case of the mapping F given by (14) and the eigenvector $v = e_N$, we have the connection

$$X_F = \sum_{i,j=1}^{N-1} a_{ij} x_j \frac{\partial}{\partial x_i}, \tag{15}$$

$$Y_{e_N} = \frac{\partial}{\partial x_N}, \tag{16}$$

where $x = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^N$, see [3]. The formulas (15) and (16) can be written as $X_F = \langle Fx|\nabla\rangle$, $Y_{e_N} = \langle e_N|\nabla\rangle$. This allows us to connect the vector fields X_F, Y_v with any pair (F, v) by

$$X_F = \langle Fx|\nabla\rangle, \quad Y_v = \langle v|\nabla\rangle. \tag{17}$$

Theorem 1 *If vector fields $X_F, Y_v \in \Gamma\left(T\mathbb{R}^N\right)$ are given by a formula (17), then the Schouten–Nijenhuis bracket for a bivector $X_F \wedge Y_v \in \Gamma\left(\bigwedge^2 T\mathbb{R}^N\right)$ is equal to zero, namely*

$$[X_F \wedge Y_v, X_F \wedge Y_v] = 0. \tag{18}$$

Proof Applying the properties (6) and (7) of the Schouten–Nijenhuis bracket we have

$$[X_F \wedge Y_v, X_F \wedge Y_v] = 2[X_F, Y_v] \wedge X_F \wedge Y_v.$$

The vector fields X_F, Y_v commute, so we get the result. □

According to the property (18), $X_F \wedge Y_v$ is a Poisson tensor. There is a well known one-to-one correspondence between the structure of the Lie algebra and the linear Poisson structure on the dual space of this algebra.

The structure of Lie algebra $(\mathbb{R}^N, [\cdot,\cdot]_{(F,e_N)})$ defines in a natural way the Lie–Poisson structure

$$\{f,g\} = X_F \wedge Y_{e_N}(df,dg) = \langle Fx|\nabla f\rangle\langle e_N|\nabla g\rangle - \langle e_N|\nabla f\rangle\langle Fx|\nabla g\rangle. \quad (19)$$

Calculating the structure constants of the Lie algebra $(\mathbb{R}^N, [\cdot,\cdot]_{(F,e_N)})$, we obtain the following non-zero commutator relations

$$[e_i, e_N]_{(F,e_N)} = a_{i1}e_1 + a_{i2}e_2 + \ldots + a_{iN-1}e_{N-1}, \quad i = 1,2,\ldots,N-1.$$

There is a connection between the matrix elements of the mapping F and the structure constants of the Lie algebra, namely $a_{ij} = c^j_{iN}$. From now on the matrix elements of F will be the structure constants of a given Lie algebra.

It allows to rewrite equations for Casimirs in terms of a pair (F, e_N). In our considerations we assume that $F \neq 0$.

Theorem 2 *Let $(\mathbb{R}^N, [\cdot,\cdot]_{(F,e_N)})$ be a Lie algebra, then Casimirs c_i, $i = 1,2,\ldots,N-2$, of the algebra fulfill the following conditions*

$$\langle Fx|\nabla c_i(x)\rangle = 0, \quad (20)$$
$$\langle e_N|\nabla c_i(x)\rangle = 0 \quad (21)$$

for all $x \in \mathbb{R}^N$.

Proof The condition (21) can be written as $Y_{e_N}c_i = 0$, which means that the function c depends only on the coordinates $x_1, x_2, \ldots, x_{N-1}$. The first condition (20) can be described as $X_F c_i = 0$. From the definition of the Casimir function (11) and from the form of the bracket (19), it follows that these functions must satisfy the condition

$$\langle Fx|\nabla c_i\rangle e_N - \langle e_N|\nabla c_i\rangle Fx = 0.$$

The vectors $e_N, Fx \in \mathbb{R}^N$ are linearly independent, so we obtain (20) and (21).

From the formulas (20) and (21) it follows that ∇c_i are orthogonal to the vectors e_N and Fx. This means that they span an $N-2$-dimensional subspace, $N \geq 2$. In summary, for a pair (F, v) giving a Lie algebra structure, we always have $N-2$ Casimir functions. □

Example 1 Consider a pair (F, v), where

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad v = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which corresponds to solvable Lie algebra $\mathfrak{s}_{3,2}$, [21]. The vector fields have a form

$$X_F = x_1\frac{\partial}{\partial x_1} + (x_1+x_2)\frac{\partial}{\partial x_2}, \quad Y_{e_3} = \frac{\partial}{\partial x_3}. \quad (22)$$

It follows from Theorem 2, that there is only one Casimir function c. Condition (21) implies that Casimir depends on the coordinates x_1, x_2. The equation (20) has the form

$$\langle Fx|\nabla c\rangle = \left\langle \begin{pmatrix} x_1 \\ x_1+x_2 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \\ 0 \end{pmatrix} \right\rangle = x_1 \frac{\partial c}{\partial x_1} + (x_1+x_2)\frac{\partial c}{\partial x_2} = 0$$

and its solution is obviously given by $c(x_1, x_2) = x_1 exp(-x_2/x_1)$.

We consider now the situation where two pairs (F_1, v_1), (F_2, v_2) give a Lie bracket, i.e., a linear combination of Lie brackets

$$[\cdot,\cdot]_{(F_1,v_1)} + \lambda[\cdot,\cdot]_{(F_2,v_2)} \tag{23}$$

is a Lie bracket. It is known that only the Jacobi identity has to be checked. We write the problem in terms of bivectors. We associate vector fields X_{F_1}, Y_{v_1} and X_{F_2}, Y_{v_2} with pairs (F_1, v_1), (F_2, v_2) using (15) and (16). The formula (23) can be written with respect to bivectors as $\pi_\lambda = X_{F_1} \wedge Y_{v_1} + \lambda X_{F_2} \wedge Y_{v_2}$. Using the Schouten–Nijenhuis bracket properties we check when the bivector fulfills the formula (9), i.e. two Poisson structures given by pairs (F_1, v_1), (F_2, v_2) are compatible. First, we calculate the following commutators of the vector fields.

$$[X_{F_1}, Y_{v_2}] = -Y_{F_1 v_2}, \quad [X_{F_1}, X_{F_2}] = -X_{[F_1,F_2]}.$$

Then we have the following condition (the Jacobi identity)

$$\begin{aligned} 0 &= \frac{1}{2\lambda}[\pi_\lambda, \pi_\lambda] \\ &= -X_{[F_1,F_2]} \wedge Y_{v_1} \wedge Y_{v_2} + X_{F_1} \wedge Y_{F_2 v_1} \wedge Y_{v_2} + X_{F_2} \wedge Y_{F_1 v_2} \wedge Y_{v_1}. \end{aligned} \tag{24}$$

Note, that for two pairs (F_1, e_N), (F_2, e_{N-1}), using formula (19), a Poisson tensor $\pi_\lambda = X_{F_1} \wedge Y_{e_N} + \lambda X_{F_2} \wedge Y_{e_{N-1}}$ defines a Poisson bracket

$$\begin{aligned} \{f, g\} =& \langle F_1 x|\nabla f\rangle\langle e_N|\nabla g\rangle - \langle e_N|\nabla f\rangle\langle F_1 x|\nabla g\rangle \\ &+ \lambda\langle F_2 x|\nabla f\rangle\langle e_{N-1}|\nabla g\rangle - \lambda\langle e_{N-1}|\nabla f\rangle\langle F_2 x|\nabla g\rangle. \end{aligned}$$

Thus, the Casimir functions are calculated from the equation

$$\langle F_1 x|\nabla c\rangle e_N - \langle e_N|\nabla c\rangle F_1 x + \lambda\langle F_2 x|\nabla c\rangle e_{N-1} - \lambda\langle e_{N-1}|\nabla c\rangle F_2 x = 0. \tag{25}$$

The number of conditions for Casimir functions depends on the dimension of the space $V_2 = span\{e_N, e_{N-1}, F_1 x, F_2 x\}$. If $k = \dim V_2$, and m denotes the number of linear independent conditions, then there are $N-m$ Casimirs, where $2 \le m \le k$.

Similarly, if we have s pairs (F_1, e_N), (F_2, e_{N-1}), $\ldots$, (F_s, e_{N-s+1}), then the Poisson bracket given by these pairs is of the form

$$\{f, g\} = \sum_{i=1}^{s} \lambda_i \left(\langle F_i x | \nabla f\rangle \langle e_{N-i+1} | \nabla g\rangle - \langle e_{N-i+1} | \nabla f\rangle \langle F_i x | \nabla g\rangle\right),$$

where $\lambda_i = const$, $i = 1, 2, \ldots, s$, and the conditions for Casimir functions are

$$\sum_{i=1}^{s} \left(\langle F_i x | \nabla c\rangle e_{N-i+1} - \langle e_{N-i+1} | \nabla c\rangle F_i x\right) = 0.$$

If $V_s = span\{e_N, e_{N-1}, \ldots, e_{N-s+1}, F_1 x, F_2 x, \ldots, F_s x\}$, $k = \dim V_s$ and m–the number of linear independent conditions, then there are $N-m$ Casimirs, where $s \leq m \leq k$.

As shown in [10], the Casimir equation can be written as

$$\nabla c_i \wedge \star \sum_{j=1}^{N} \left(F_j x \wedge v_j\right) = 0, \tag{26}$$

where $\star : \bigwedge^{2} \mathbb{R}^N \to \bigwedge^{N-2} \mathbb{R}^N$ is the Hodge star operator. However, it may be the case that the tensor $\star \sum_{j=1}^{N} \left(F_j x \wedge v_j\right)$ is a decomposable tensor which splits into 1-vectors. Then ∇c_i will be parallel to some vector in the decomposition. Recall that tensor $t \in \bigwedge^{N} V$ is decomposable if there are vectors $w_i \in V, i = 1, 2, \ldots, N$, such that $t = w_1 \wedge \ldots \wedge w_N$. The number of Casimirs is related to the number of vectors in the decomposition of this tensor. In the following examples we will demonstrate such situation.

Example 2 Consider two pairs (F_1, v_1), (F_2, v_2), where

$$F_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_1 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$F_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_2 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

which correspond to $\mathfrak{so}(3)$. The space V_2 is spanned by vectors e_3, e_2, F_1x, and F_2x, $k = 3$. Using the formula (25)

$$\left\langle \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \\ \frac{\partial c}{\partial x_3} \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \\ \frac{\partial c}{\partial x_3} \end{pmatrix} \right\rangle \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$$
$$+ \left\langle \begin{pmatrix} -x_3 \\ 0 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \\ \frac{\partial c}{\partial x_3} \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \\ \frac{\partial c}{\partial x_3} \end{pmatrix} \right\rangle \begin{pmatrix} -x_3 \\ 0 \\ 0 \end{pmatrix} = 0$$

conditions for Casimir functions are the following

$$\begin{aligned} x_2 \frac{\partial c}{\partial x_1} - x_1 \frac{\partial c}{\partial x_2} &= 0, \\ -x_3 \frac{\partial c}{\partial x_1} + x_1 \frac{\partial c}{\partial x_3} &= 0, \\ x_3 \frac{\partial c}{\partial x_2} - x_2 \frac{\partial c}{\partial x_3} &= 0. \end{aligned}$$

There are 2 linear independent conditions, so $m = 2$. There is $N - m = 1$ Casimir, namely $c(x) = x_1^2 + x_2^2 + x_3^2$. The vector $(x_1, x_2, x_3)^T$ is known to be proportional to the vector ∇c, which can be written in terms of the pairs (F_1, e_3), (F_2, e_2) by the formula $\star(F_1x \wedge e_3 + F_2x \wedge e_2) = -x_1e_1 - x_2e_2 - x_3e_3$.

Moreover, in case of 3-dimensional Lie algebras, each of them has one Casimir function (for details, see Table 1).

Example 3 Consider two pairs (F_1, v_1), (F_2, v_2), where

$$F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad v_1 = e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$
$$F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad v_2 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

which correspond to solvable Lie algebra $\mathfrak{s}_{4,7}$, see [21]. There are three conditions for Casimir functions

$$\begin{aligned} -x_3\frac{\partial c}{\partial x_2} + x_2\frac{\partial c}{\partial x_3} &= 0,\\ -x_1\frac{\partial c}{\partial x_3} - x_3\frac{\partial c}{\partial x_4} &= 0,\\ x_1\frac{\partial c}{\partial x_2} + x_2\frac{\partial c}{\partial x_4} &= 0, \end{aligned}$$

among which only two are linear independent. Similarly as in the previous example $N - m = 2$, so there are two Casimir functions, namely

$$c_1(x) = x_1, \quad c_2(x) = -2x_1x_4 + x_2^2 + x_3^2.$$

We can consider the above calculations in another way. Let us compute

$$\begin{aligned} \star(F_1x \wedge e_4 + F_2x \wedge e_3) =&\star(-x_3e_2 \wedge e_4 + x_2e_3 \wedge e_4 - x_1e_2 \wedge e_3) && (27)\\ =&e_1 \wedge (f(x)e_1 + x_2e_2 + x_3e_3 - x_1e_4), && (28) \end{aligned}$$

where f is an arbitrary function. This gives us that ∇c_i, $i = 1, 2$, is parallel to e_1 or to $f(x)e_1 + x_2e_2 + x_3e_3 - x_1e_4$ for $f(x) = -x_4$, respectively.

The tensor $t \in \bigwedge^2 V$, where $dim V = 4$, written in the form $t = \sum_{1\le i<j\le 4} t_{ij}e_i \wedge e_j$ is decomposable in $V \wedge V$ if satisfies quadratic Plücker relation

$$t_{12}t_{34} + t_{14}t_{23} - t_{13}t_{24} = 0, \tag{29}$$

see for example [15]. In case of (27) the property (29) fulfills. Let consider Lie algebra $\mathfrak{s}_{4,10}$, which correspond

$$F_1 = \begin{pmatrix} 2&0&0&0\\ 0&1&0&0\\ 0&1&1&0\\ 0&0&0&0 \end{pmatrix}, \quad v_1 = e_4 = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix},$$
$$F_2 = \begin{pmatrix} 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad v_2 = e_3 = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}.$$

The tensor

$$\star(F_1x \wedge e_4 + F_2x \wedge e_3) = (x_2 + x_3)e_1 \wedge e_2 - x_2e_1 \wedge e_3 - x_1e_1 \wedge e_4 + 2x_1e_2 \wedge e_3$$

does not satisfy (29), so it is indecomposable. It is known that this algebra does not have any Casimir functions.

In case of four-dimensional Lie algebras, we have only two possibilities: splitting into two 1-vectors or no such splitting. Thus in this dimension, Lie algebras have two Casimir functions or do not have them at all (for details, see Table 2).

We can now formulate the main theorems of the paper, which determine the number and form of Casimir functions for a given algebra. First, however, we will introduce the notion of a partially decomposable tensor.

Definition 1 A non-zero tensor $t \in \bigwedge^{N} V$ is s-partially decomposable if there exist $w_i, i = 1, 2, \ldots, s$, vectors and $N - s$-tensor $u \in \bigwedge^{N-s} V$ such that

$$t = w_1 \wedge w_2 \wedge \ldots \wedge w_s \wedge u.$$

Finally, the following theorem holds

Theorem 3 *Let pairs (F_j, v_j), $j = 1, \ldots, N$, give any Lie algebra $\mathfrak{g}$. Functions c_i, $i = 1, \ldots, s$, are functionally independent Casimir functions for $\mathfrak{g}$ if and only if $\star \sum_{j=1}^{N} \left(F_j x \wedge v_j\right) \in \bigwedge^{N-2} \mathbb{R}^N$ is s-partially decomposable, i.e. if there exist $w_i \in \mathbb{R}^N$, $i = 1, 2, \ldots, s$, $u \in \bigwedge^{N-s-2} \mathbb{R}^N$ such that*

$$\star \sum_{j=1}^{N} \left(F_j x \wedge v_j\right) = w_1 \wedge w_2 \wedge \ldots \wedge w_s \wedge u. \tag{30}$$

Furthermore, $\nabla c_i \sim w_i$.

Proof As we well know, a non-zero tensor $t \in \bigwedge^{N} V$ is decomposable in $\bigwedge^{N} V$ if and only if there exist the set of vectors $w_1, w_2, \ldots, w_N$, such that $t \wedge w_j = 0$ for $j = 1, 2, \ldots, N$, see for example [8]. If $c_1, c_2, \ldots, c_s$, are Casimir functions for the Lie algebra $\mathfrak{g}$, then fulfill formula (26). It means that the tensor $\star \sum_{j=1}^{N} \left(F_j x \wedge v_j\right)$ has to be s-partially decomposable

$$\star \sum_{j=1}^{N} \left(F_j x \wedge v_j\right) = \nabla c_1 \wedge \nabla c_2 \wedge \ldots \wedge \nabla c_s \wedge u,$$

where $u \in \bigwedge^{N-s-2} \mathbb{R}^N$. On the other hand, from the decomposition (30) and (26) we see that $\nabla c_i \sim w_i$ and $c_1, c_2, \ldots, c_s$ are Casimir functions for $\mathfrak{g}$. $\square$

Remark 4 If we have a single pair (F, e_N), then obviously the tensor $Fx \wedge e_N$ is decomposable, so consequently the tensor $\star(Fx \wedge e_N) \in \bigwedge^{N-2} \mathbb{R}^N$ is decomposable. Therefore, algebra with this pair must always have $N - 2$ Casimir functions.

Table 1 Linear mappings, their eigenvectors, tensors and invariants (Casimirs) for three dimensional Lie algebras

Algebra	F_j	v_j	$\star\sum_{j=1}^{N}\left(F_j x \wedge v_j\right)$	$\nabla c_i \sim w_i$	Casimirs
$\mathfrak{n}_{3,1}$	$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$x_1 e_1$	$\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$	$c_1 = x_1$
$\mathfrak{s}_{3,1}$	$F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$a x_2 e_1 - x_1 e_2$	$\begin{pmatrix} a x_2 \\ -x_1 \\ 0 \end{pmatrix}$	$c_1 = \frac{x_1^a}{x_2}$
$\mathfrak{s}_{3,2}$	$F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$(x_1 + x_2) e_1 - x_1 e_2$	$\begin{pmatrix} x_1 + x_2 \\ -x_1 \\ 0 \end{pmatrix}$	$c_1 = x_1 e^{-\frac{x_2}{x_1}}$
$\mathfrak{s}_{3,3}$	$F_1 = \begin{pmatrix} a & -1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$(x_1 + a x_2) e_1 + (x_2 - a x_1) e_2$	$\begin{pmatrix} x_1 + a x_2 \\ x_2 - a x_1 \\ 0 \end{pmatrix}$	$c_1 = (x_1^2 + x_2^2) e^{2a \arctan \frac{x_1}{x_2}}$
$\mathfrak{sl}_{2,\mathcal{R}}$	$F_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$2 x_3 e_1 + x_2 e_2 + 2 x_1 e_3$	$\begin{pmatrix} 2x_3 \\ x_2 \\ 2x_1 \end{pmatrix}$	$c_1 = 4 x_1 x_3 + x_2^2$
	$F_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$			
$\mathfrak{so}_{3,\mathcal{R}}$	$F_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$x_1 e_1 + x_2 e_2 + x_3 e_3$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$	$c_1 = x_1^2 + x_2^2 + x_3^2$
	$F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$			

Table 2 Linear mappings, their eigenvectors, tensors and invariants (Casimirs) for four dimensional Lie algebras

Algebra	F_j	v_j	$\star\sum_{j=1}^{N}\left(F_j x\wedge v_j\right)$	$\nabla c_i \sim w_i$	Casimirs
$\mathfrak{n}_{4,1}$	$F_1=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$	$v_1=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$e_1\wedge(f(x)e_1+x_2e_2-x_1e_3)$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-x_3\\x_2\\-x_1\\0\end{pmatrix}$ for $f(x)=-x_3$	$c_1=x_1$ $c_2=2x_1x_3-x_2^2$
$\mathfrak{s}_{4,1}$	$F_1=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$	$v_1=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$e_1\wedge(f(x)e_1+x_3e_2-x_1e_3)$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-\frac{x_3x_2}{x_1}\\x_3\\-x_1\\0\end{pmatrix}$ for $f(x)=-\frac{x_3x_2}{x_1}$	$c_1=x_1$ $c_2=x_3e^{-\frac{x_2}{x_1}}$
$\mathfrak{s}_{4,2}$	$F_1=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$	$v_1=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\left(\left(-\frac{x_2+x_3}{x_1}+(x_1+x_2)f(x)\right)e_1\right.$ $\left.-x_1f(x)e_2+e_3\right)\wedge((x_1+x_2)e_1-x_1e_2)$	$\begin{pmatrix}\frac{x_2^2-x_1x_3}{x_1^2}\\-\frac{x_2}{x_1}\\1\\0\end{pmatrix},\begin{pmatrix}x_1+x_2\\-x_1\\0\\0\end{pmatrix}$ for $f(x)=\frac{x_2}{x_1^2}$	$c_1=\frac{2x_1x_3-x_2^2}{x_1^2}$ $c_2=x_1e^{-\frac{x_2}{x_1}}$
$\mathfrak{s}_{4,3}$	$F_1=\begin{pmatrix}1&0&0&0\\0&a&0&0\\0&0&b&0\\0&0&0&0\end{pmatrix}$	$v_1=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$(ax_2e_1-x_1e_2)\wedge(ax_2f(x)e_1$ $\left.+\left(\frac{bx_3}{ax_2}-x_1f(x)\right)e_2-e_3\right)$	$\begin{pmatrix}ax_1\\-x_1\\0\\0\end{pmatrix},\begin{pmatrix}\frac{bx_3}{x_1x_2}\\0\\-1\\0\end{pmatrix}$ for $f(x)=\frac{bx_3}{ax_1x_2}$	$c_1=\frac{x_1^a}{x_2}$ $c_2=\frac{x_1^b}{x_3}$

Table 2 continued

Algebra	F_j	v_j	$\star \sum_{j=1}^{N} \left(F_j x \wedge v_j\right)$	$\nabla c_i \sim w_i$	Casimirs
$\mathfrak{s}_{4,4}$	$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{gathered} ((x_1 + x_2) f(x) e_1 \\ + \left(-x_1 f(x) + \frac{a x_3}{x_1 + x_2}\right) e_2 + e_3\right) \\ \wedge ((x_1 + x_2) e_1 - x_1 e_2) \end{gathered}$	$\begin{pmatrix} -\frac{a x_3}{x_1} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 + x_2 \\ -x_1 \\ 0 \\ 0 \end{pmatrix}$ for $f(x) = -\frac{a x_3}{x_1 (x_1 + x_2)}$	$c_1 = \frac{x_1^a}{x_3}$ $c_2 = x_1 e^{-\frac{x_2}{x_1}}$
$\mathfrak{s}_{4,5}$	$F_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & -1 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{gathered} \frac{1}{1 - f(x) g(x)} \left[\left(x_3 + b x_2 - f(x) \frac{x_2 + b x_3}{a x_1}\right) e_1 \right. \\ + a x_1 e_2 + f(x) e_3) \\ \wedge \left(\left(-\frac{x_2 + b x_3}{a x_1} + g(x)(x_3 - b x_2)\right) e_1 \right. \\ \left. + a x_1 g(x) e_2 + e_3)\right] \end{gathered}$	$\begin{pmatrix} -b \frac{x_2^2 + x_3^2}{x_2} \\ a x_1 \\ a \frac{x_1 x_3}{x_2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{x_2^2 + x_3^2}{a x_1 x_2} \\ -\frac{x_3}{x_2} \\ 1 \\ 0 \end{pmatrix}$ for $\begin{gathered} f(x) = a \frac{x_1 x_3}{x_2} \\ g(x) = -\frac{x_3}{a x_1 x_2} \end{gathered}$	$c_1 = \frac{\left(x_2^2 + x_3^2\right)^a}{x_1^{2b}}$ $c_2 = x_1 e^{a \arctan \frac{x_2}{x_3}}$
$\mathfrak{s}_{4,6}$	$F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$e_1 \wedge (f(x) e_1 - x_3 e_2 - x_2 e_3 - x_1 e_4)$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_4 \\ -x_3 \\ -x_2 \\ -x_1 \end{pmatrix}$	$c_1 = x_1$ $c_2 = x_2 x_3 + x_1 x_4$
	$F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$		for $f(x) = -x_4$	
$\mathfrak{s}_{4,7}$	$F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$e_1 \wedge (f(x) e_1 - x_2 e_2 + x_3 e_3 - x_1 e_4)$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_4 \\ x_2 \\ x_3 \\ -x_1 \end{pmatrix}$	$c_1 = x_1$ $c_2 = x_2^2 + x_3^2 - 2 x_1 x_4$
	$F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$		for $f(x) = x_4$	

Table 2 continued

Algebra	F_j	v_j	$\star\sum_{j=1}^{N}\left(F_j x \wedge v_j\right)$	$\nabla c_i \sim w_i$	Casimirs
$\mathfrak{s}_{4,8}$	$F_1 = \begin{pmatrix} 1+a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$ax_3 e_1 \wedge e_2 - x_2 e_1 \wedge e_3$ $- x_1 e_1 \wedge e_4 + (1+a) x_1 e_2 \wedge e_3$	–	None
	$F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$			
$\mathfrak{s}_{4,9}$	$F_1 = \begin{pmatrix} 2a & 0 & 0 & 0 \\ 0 & a & -1 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$(x_2 + ax_3) e_1 \wedge e_2 - (ax_2 - x_3) e_1 \wedge e_3$ $-x_1 e_1 \wedge e_4 + 2ax_1 e_2 \wedge e_3$	–	None
	$F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$			
$\mathfrak{s}_{4,10}$	$F_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$(x_2 + x_3) e_1 \wedge e_2 - x_2 e_1 \wedge e_3$ $- x_1 e_1 \wedge e_4 + 2x_1 e_2 \wedge e_3$	–	None
	$F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$			

Table 2 continued

Algebra	F_j	v_j	$\star\sum_{j=1}^{N}\left(F_j x \wedge v_j\right)$	$\nabla c_i \sim w_i$	Casimirs
$\mathfrak{s}_{4,11}$	$F_1 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$	$-x_2e_1 \wedge e_3 - x_1e_1 \wedge e_4 + x_1e_2 \wedge e_3$	–	None
	$F_2 = \begin{pmatrix} 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}$			
$\mathfrak{s}_{4,12}$	$F_1 = \begin{pmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$	$v_1 = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$	$-x_1e_1 \wedge e_3 + x_2e_1 \wedge e_4 - x_2e_2 \wedge e_3 - x_1e_2 \wedge e_4$	–	None
	$F_2 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$	$v_2 = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}$			

Remark 5 From the physical point of view, the Casimir functions are potentials for the forces $w_1, \ldots, w_s$, which appeared in the decomposition $\star \sum_{j=1}^{N} \left(F_j x \wedge v_j\right)$, and are determined with precision to the function.

Remark 6 If there are $N-2$ smooth Casimir functions $c_1, \ldots, c_{N-2}$, this corresponds to the situation that the Poisson bracket arises from the Nambu bracket by fixing $N-2$ functions as Casimir functions. In this case, the formula has a form

$$\{f, g\}\Omega = u\, df \wedge dg \wedge dc_1 \wedge \ldots \wedge dc_{N-2}, \quad f, g \in C^{\infty}(\mathbb{R}^N),$$

where $\Omega = dx_1 \wedge \ldots \wedge dx_N$ is the standard volume element on $\mathbb{R}^N$, and u is some function on $\mathbb{R}^N$. The case, where there are less smooth Casimir functions, namely $c_1, \ldots, c_s$, $s < N-2$, then the Poisson bracket has a form

$$\{f, g\}\Omega = df \wedge dg \wedge dc_1 \wedge \ldots \wedge dc_s \wedge u$$

(in details studied in [7]). It is connected with $s+2$-linear Nambu bracket in dimension N, higher than $s+2$, see [5].

4 Eigenvalue problems for operators and complete and vertical lifts of some vector fields

For the eigenvalue problem given by a pair (F, e_N), we define the complete and vertical lifts from $\mathbb{R}^N$ to $\mathbb{R}^{2N}$. Let $B^C = \{e_1, e_2, \ldots, e_N, f_1, f_2, \ldots, f_N\}$ be a basis in $\mathbb{R}^{2N}$.

Definition 2 Let a pair (F, e_N), where $F \in End(\mathbb{R}^N)$, $e_N \in \ker F$, gives an eigenvalue problem.

1. We say that, a pair (F^C, f_N), where

$$F^C = \left(\begin{array}{c|c} F & 0 \\ \hline 0 & F \end{array}\right),$$

   is a complete lift of a pair (F, e_N) from $\mathbb{R}^N$ to $\mathbb{R}^{2N}$.
2. We say that, a pair (F^V, e_N), where

$$F^V = \left(\begin{array}{c|c} 0 & 0 \\ \hline F & 0 \end{array}\right),$$

   is a vertical lift of a pair (F, e_N) from $\mathbb{R}^N$ to $\mathbb{R}^{2N}$.

Note, that the vector fields $X_{F^C}, Y_{f_N} \in \Gamma(T\mathbb{R}^{2N})$ associated with a pair (F^C, f_N) have the following form

$$
\begin{aligned}
X_{F^C} &= \sum_{i,j=1}^{N-1} a_{ij} x_j \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{N-1} a_{ij} y_j \frac{\partial}{\partial y_i},\\
Y_{f_N} &= \frac{\partial}{\partial y_N},
\end{aligned}
$$

where $(x_1, x_2, \ldots, x_N, y_1, \ldots, y_N)$ coordinates in $\mathbb{R}^{2N}$. In the above formulas we recognise a complete and vertical lifts of the vector fields X_F, Y_{e_N} given by the formulas (15), (16), i.e. $X_{F^C} = X_F^C, Y_{f_N} = Y_{e_N}^V$. By analogy, with the pair (F^V, e_N) we associate the vector fields $X_{F^V}, Y_{e_N} \in \Gamma(T\mathbb{R}^{2N})$

$$
\begin{aligned}
X_{F^V} &= \sum_{i,j=1}^{N-1} a_{ij} x_j \frac{\partial}{\partial y_i},\\
Y_{e_N} &= \frac{\partial}{\partial x_N},
\end{aligned}
$$

which are the vertical $X_{F^V} = X_F^V$ and complete $Y_{e_N} = Y_{e_N}^C$ lifts of the vector fields (15) and (16), respectively.

Remark 7 Notice that if we have the Poisson tensor on the manifold M (in our case it is $\mathbb{R}^N$), i.e., $\pi \in \Gamma\left(\bigwedge^2 TM\right)$, then its complete lift π^C gives Poisson structure on the manifold TM. It is called a fiber–wise linear Poisson structure. This structure is connected with Lie algebroid structure on T^*M. Such construction is well known, (see for example [14, 18]). If the Poisson tensor π is decomposable, i.e., $\pi = X \wedge Y$, then its complete lift is given by

$$
\pi^C = (X \wedge Y)^C = X^C \wedge Y^V + X^V \wedge Y^C.
$$

In works [9, 12] were shown that each component $X^C \wedge Y^V, X^V \wedge Y^C$ also gives Poisson structure linear in fibres, under appropriate assumptions on X, Y. It means, that on the space T^*M we can construct Lie algebroids determined by vector fields X and Y satisfying a suitable commutation relations. On the Lie algebra level it allows to construct from N-dimensional Lie algebras, a family of $2N$-dimensional Lie algebras.

Other situations related to $(F^C, e_N), (F^V, f_N)$ pairs can also be considered. This would correspond to a complete or vertical lift of both components.

From (12) we conclude that pairs $\left(\mathbb{R}^{2N}, [\cdot,\cdot]_{(F^C, f_N)}\right)$ and $\left(\mathbb{R}^{2N}, [\cdot,\cdot]_{(F^V, e_N)}\right)$ are Lie algebras. Theorem 2 says that each of the considered structures has $2N-2$ Casimir functions. We can describe such functions in terms of the Casimirs of the initial pair (F, v).

Theorem 4 *If c_s, $s = 1, 2, \ldots, N-2$, are Casimirs for the Lie algebra given by a pair (F, e_N), then $x_1, x_2, \ldots, x_{N-1}, y_N, c_s(x, y) = \sum_{i=1}^{N} \frac{\partial c_s}{\partial x_i}(x) y_i$ are all Casimir functions for the Lie algebra $(\mathbb{R}^{2N}, [\cdot, \cdot]_{(F^V, e_N)})$.*

Proof The result is obtained by a straightforward calculation of the conditions in Theorem 2. □

Theorem 5 *If c_s, $s = 1, 2, \ldots, N-2$, are Casimirs for the Lie algebra given by a pair (F, e_N), then $c_s(x), c_s(y), x_N$ are Casimir functions for the Lie algebra $(\mathbb{R}^{2N}, [\cdot, \cdot]_{(F^C, f_N)})$.*

Proof. The Casimirs $c_s(x), c_s(y), x_N$ are the result of a direct calculation of conditions (20) and (21). □

In the last theorem there is one missing Casimir which can be calculated from the equation

$$\langle Fx|\nabla_x c\rangle + \langle Fy|\nabla_y c\rangle = 0, \tag{31}$$

where $\nabla c = \left(\nabla_x c, \nabla_y c\right)^T = \left(\frac{\partial c}{\partial x_1}, \ldots, \frac{\partial c}{\partial x_N}, \frac{\partial c}{\partial y_1}, \ldots, \frac{\partial c}{\partial y_N}\right)^T.$

Procedure described in Definition 2 one can use to more pairs (F_i, e_i). The following theorem presents a specific situation of this kind.

Theorem 6 *If pairs (F^C, f_N) and (F^V, e_N) are respectively complete and vertical lifts of a pair (F, e_N) from $\mathbb{R}^N$ to $\mathbb{R}^{2N}$, then*

$$[\cdot, \cdot]_{(F^C, f_N), (F^V, e_N)} = [\cdot, \cdot]_{(F^C, f_N)} + \lambda [\cdot, \cdot]_{(F^V, e_N)}$$

is a Lie bracket on $\mathbb{R}^{2N}$ for any $\lambda \in \mathbb{R}$.

Proof We proceed from the previous considerations that the condition (24) has to be fulfilled. We have

$$[F^C, F^V] = 0, \quad F^C e_N = 0, \quad F^V f_N = 0,$$

so the formula (24) holds. □

Example 4 Consider a pair (F, v) from Example 1. Using complete and vertical lifts of the Poisson vector fields (22)

$$\begin{aligned} X^C_{F^C} &= x_1 \frac{\partial}{\partial x_1} + (x_1 + x_2) \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_1} + (y_1 + y_2) \frac{\partial}{\partial y_2}, \quad Y^V_{f_3} = \frac{\partial}{\partial y_3}, \\ X^V_{F^V} &= x_1 \frac{\partial}{\partial y_1} + (x_1 + x_2) \frac{\partial}{\partial y_2}, \quad Y^C_{e_3} = \frac{\partial}{\partial x_3}, \end{aligned}$$

we obtain the following splitting

$$\begin{tikzcd}
 & \mathfrak{s}_{3,2} \arrow[dl] \arrow[dr] & \\
\mathfrak{n}_{5,1} \oplus \langle f_3 \rangle \arrow[dr] & & \mathfrak{s}_{5,6}^{a=1} \oplus \langle e_3 \rangle, \arrow[dl] \\
 & \mathfrak{s}_{6,143} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \left(F=\begin{pmatrix}1&0&0\\1&1&0\\0&0&0\end{pmatrix}, e_3\right) \arrow[dl] \arrow[dr] & \\
\left(F^V=\left(\begin{array}{ccc|ccc}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\ \hline 1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\end{array}\right), e_3\right) \arrow[dr] & & \left(F^C=\left(\begin{array}{ccc|ccc}1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\ \hline 0&0&0&1&0&0\\0&0&0&1&1&0\\0&0&0&0&0&0\end{array}\right), f_3\right). \arrow[dl] \\
 & (F^V, e_3), (F^C, f_3) &
\end{tikzcd}$$

The Poisson tensor

$$X^V_{F^V} \wedge Y^C_{e_3} = -x_1 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_1} - (x_1 + x_2) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_2}$$

corresponds to nilpotent Lie algebra $\mathfrak{n}_{5,1} \oplus \langle f_3 \rangle$. We use the classification and the notation from the book [21]. Casimir functions are $x_1, x_2, y_3, c_1(x, y) = [(x_1+x_2)y_1 - x_1 y_2]/x_1 exp(-x_2/x_1)$ as Theorem 4 says.

The Poisson tensor

$$\begin{aligned} X^C_{F^C} \wedge Y^V_{f_3} = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_3} + (x_1 + x_2) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_3} \\ + y_1 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_3} + (y_1 + y_2) \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y_3} \end{aligned}$$

corresponds to solvable Lie algebra $\mathfrak{s}_{5,6}^{a=1} \oplus \langle e_3 \rangle$. From Theorem 5 we get the following Casimir functions x_3, $c_1(x)=x_1 exp(-x_2/x_1)$, $c_2(y) = y_1 exp(-y_2/y_1)$ and from equation (31) we get the last invariant $c_3(x, y) = x_1/y_1$.

The last Poisson tensor in the splitting

$$(X_F \wedge Y_{e_3})^C = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_3} + (x_1 + x_2) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_3} - x_1 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_1}$$
$$-(x_1 + x_2) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_3} + (y_1 + y_2) \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y_3}$$

corresponds to solvable Lie algebra $\mathfrak{s}_{6,143}$. Casimir functions for complete lift of the Poisson vector fields, in terms of initial Casimirs c_i, $i = 1, \ldots, s$, are given by

$$c_i, \quad \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s} y_s, \quad i = 1, 2, \ldots, s, \tag{32}$$

see [11, 14]. From (32) and Example 1, Casimir functions have the form $c_1(x, y) = x_1 exp(-x_2/x_1)$, $c_2(x, y) = (-y_2 + (x_1 + x_2)y_1)/x_1 exp(-x_2/x_1)$.

Author contributions A.D. and M.S. wrote the main manuscript text. All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

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