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Geometry of lines asymptotic to a pencil of conics

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Abstract. In a previous article we showed how the line pairs that occur as asymptotes to hyperbolas in a pencil of affine conics can be detected from their incidence geometry alone, without reference to the pencil. In this article we classify the configurations of all such line pairs up to affine equivalence. Many of our methods work over an arbitrary field $\Bbbk$ of characteristic $\neq 2$, and we use tools from the theory of algebraic curves and projective duality to obtain a complete classification of the collections of line pairs that are asymptotic to a pencil of conics if $\Bbbk$ is real closed or algebraically closed, while we obtain a partial classification if $\Bbbk$ is a finite field. A classification for other fields remains an open question. Ultimately this is a question regarding affine equivalence within a system of certain rational quartic curves.

Mathematics Subject Classification. Primary 52C30, 14N20, 14H45.

Keywords. Bisector field, Dual curve, Quartic curve, Pencil of conics.

1. Introduction

Consider a pencil of affine conics, a collection of affine conics in the plane parameterized by the projective line. As in Figure 1, the asymptotes of the hyperbolas in the pencil form an arrangement of line pairs in the plane. Pairs of parallel lines can appear in the pencil too, such as in Figure 1(c), and when this is the case other pairs of parallel lines sharing the same midline can be viewed as asymptotic to the pencil also. The orderliness of the line pairs in Figure 1 suggests the question: Is there anything special about the line pair arrangements that are asymptotic to a pencil of conics?

In [9], it is shown that the answer is yes—that these line pairs are detectable from the incidence geometry of the lines themselves, independent of any mention of the pencil. The line pairs are distinguished by virtue of satisfying a rather demanding condition: each line in each pair simultaneously bisects all the other pairs with the same midpoint, where a line ℓ bisects a pair with midpoint p if the points where ℓ crosses the pair have p as their midpoint.

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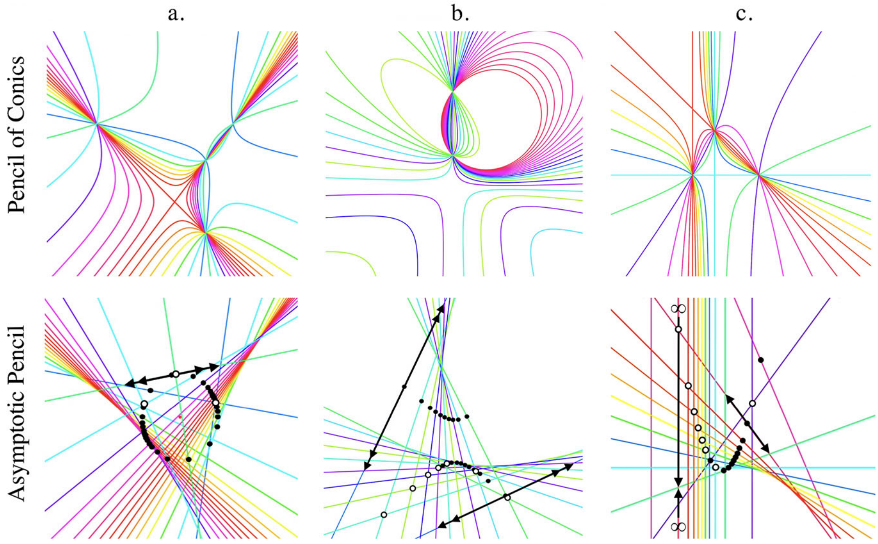

Figure 1. Selected conics from three pencils of conics (top row) and the line pairs asymptotic to the conics in the pencil (bottom row). The black points are the midpoints of the intersection points where the line crosses all the other pairs. The black arrows in the bottom row help illustrate the bisection property

This curious self-bisection property appears also in other contexts. For example, studying parallelograms inscribed in line arrangements, such as in [12–14, 17, 18] where rectangles are considered, leads eventually to the limit case of degenerate parallelograms inscribed in quadrilaterals. If we pair the degenerate parallelograms when their centers are reflections of each other across the centroid of the quadrilateral, the lines through the four collinear vertices of the degenerate parallelograms exhibit the same incidence geometry as the asymptotes of a pencil of conics: each line in each pair simultaneously bisects all the other pairs with the same midpoint.[1] See Figure 2. These lines in fact are precisely the bisectors of the quadrilateral, as is proved in [10].

It follows from results in the present paper that there is another source of this self-bisection property for line pair arrangements, that of the lines tangent to some curves of classical interest such as the Steiner deltoid or more

[1]Ultimately, this is explainable by the fact that the line pairs *are* the asymptotes from a pencil of conics, namely the pencil of conics through the vertices of the quadrilateral. See [10]. But making this connection amounts first to showing that the degenerate parallelograms lie on bisectors of the quadrilateral and hence is really more of a consequence than an explanation.

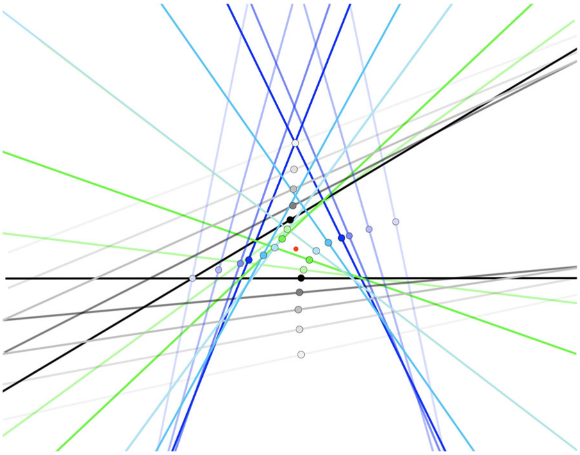

Figure 2. Selected pairs of bisectors of the quadrilateral whose pairs of opposite sides are black and dark blue. The marked points trace out a hyperbola and are the midpoints of bisectors of the same color. Each bisector bisects all other pairs with respect to the same midpoint

generally the real part of a rational tricuspidal quartic. The pairing in this case is such that two tangent lines to the deltoid are a pair if they meet on the circle inscribed in the deltoid. See Figure 3(b). The lines tangent to a deltoid appear in numerous contexts, such as the envelope of Wallis-Simson lines [1]; construction of the deltoid as a hypocycloid [8]; Böröczky's configurations of lines that has an optimal number of triple points [3,6]; and the Poincaré sphere of polarization ellipses in the theory of optics [11]. However this self-bisection property of the given pairing does not seem to have been noted in any of these contexts before.

In the examples mentioned above, the collection of line pairs often has another strong property: no more line pairs can be added to the collection without destroying the bisection property. Following [9,10], we call such a maximal arrangement of line pairs a bisector field. In a future article, we use bisector fields as a tool to describe the moduli space of quadrilaterals, the idea being that these fields contain "entangled" classes of quadrilaterals all yoked together by a shared nine-point conic.

In any case, these examples and applications raise the question of whether it is possible to classify all possible bisector fields, since doing so would classify all possible line arrangements that are asymptotes for a pencil of conics. Since midpoints are preserved by affine transformations, an affine transformation carries a bisector field to a bisector field, and so it is natural to ask for such a

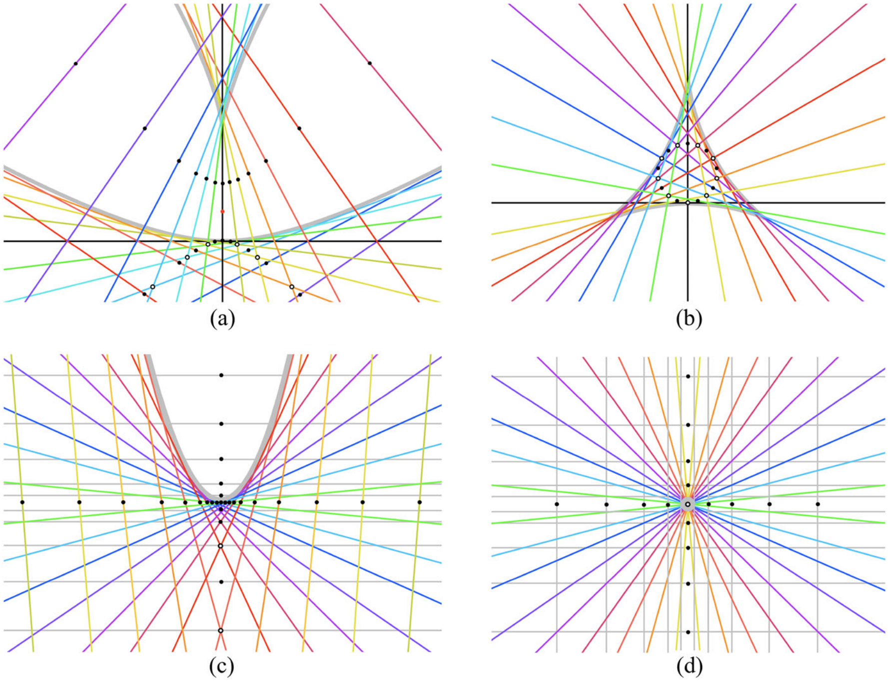

Figure 3. It is proved in Section 8 that over the field of real numbers, each bisector field is affinely equivalent to one of the four bisector fields represented here. (Only selected bisectors are shown; the bisector field itself would cover the entire plane.). Two lines of the same color indicate a bisector pair. The midpoints of the bisectors are marked with black points. White points indicate an intersection of two bisectors from the same pair that is not also a midpoint for any of the bisectors visible in the figures.

classification up to affine equivalence. In this article, we give such a classification of bisector fields up to affine equivalence, which for the real plane turns out to be surprisingly short list of just four line arrangements; see Figure 3.

Most of our methods are algebro-geometric and extend without extra effort to an arbitrary field $\Bbbk$ of characteristic $\neq 2$, and so we work often at this level of generality. Occasionally the field does matter, and we point out when this is the case and pose some questions that we could not resolve for certain fields.

We define a boundary of a bisector field $\mathbb{B}$ and show it is a rational quartic curve, a parabola or a point. We prove in the first two cases that all the lines tangent to the boundary of $\mathbb{B}$ are bisectors in $\mathbb{B}$. In the first case, that in which the boundary is a quartic, the set of tangent lines is the entire bisector field, and so the bisector field obeys a kind of holographic principle in which all information in the bisector field is encoded on the boundary. See Figures 3(a) and 3(b). In the second case, that in which the boundary is a parabola as in Figure 3(c), there are additional bisectors that are not tangent to the boundary, but these are easily described as a pencil of parallel lines. The final case is the simplest one, that in which the boundary is a point, all the lines through the point are bisectors and there are two additional pencils, each consisting of parallel lines; see Figure 3(d).

Using these ideas, as well as an arithmetic criterion given in Section 5 for determining affine equivalence of bisector fields, we classify in the last section of the paper the bisector fields over an algebraically closed field and show in this case that up to affine equivalence there are three bisector fields, each of which can be explicitly described. Over a real closed field, we show there are four bisector fields, and over the field of rational numbers there are infinitely many. See Figure 3 for the classification over the field of real numbers. For finite fields, we are only able to give partial results regarding classification. This raises the question, stated at the end of the paper, of how many bisector fields there are for a given choice of the field $\mathbb{k}$.

Terminology. By a *(complete) quadrilateral* $Q = ABA'B'$ we mean a collection of four distinct lines A, B, A', B', the *sides* of Q, and their six points of intersection, some of which are possibly at infinity. The pairs A, A' and B, B' are the pairs of *opposite sides* of Q. All other pairs are pairs of *adjacent sides.* The intersection of two adjacent sides is a *vertex* of Q. The *diagonals* of Q are the two lines through non-adjacent vertices of Q. We require that no pair of adjacent sides of Q consists of parallel lines, and that all four sides do not go through the same point. We do, however, allow the case in which three sides go through a single point. (In [9], adjacent sides of a quadrilateral are allowed to be parallel, i.e., have a vertex at infinity, but we are excluding that case here.)

2. Background: Bisector fields

We formalize the notion of a bisector field and recall several properties of bisectors from [10]. Apart from the discussion in this section, the present paper is mostly independent of the articles [9] and [10].

Let P be a pair of lines in the plane and ℓ also be a line in the plane. Denote by $\mathrm{mid}_{\mathsf{P}}(\ell)$ the midpoint of the two points where ℓ meets P, with the following stipulations. If exactly one of these points is at infinity, then $\mathrm{mid}_{\mathsf{P}}(\ell)$ is defined to be the point at infinity for ℓ, while if both are at infinity or ℓ is one of the lines in P, then $\mathrm{mid}_{\mathsf{P}}(\ell)$ is left undefined. If neither of the two points is at infinity then $\mathrm{mid}_{\mathsf{P}}(\ell)$ is *finite*.

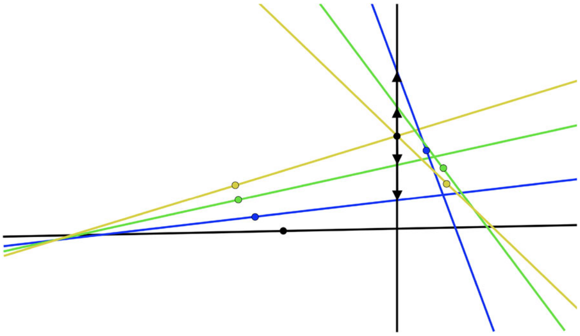

Figure 4. Four pairs of bisectors from a bisector field. The arrows on the black line help visualize the bisection property. The points shown are the midpoints of the bisectors. Any line chosen from a bisector field along with its midpoint will bisect *all* pairs from this same midpoint

If p is a point, the line ℓ *bisects a set* $\mathbb{B}$ *of pairs of lines with midpoint* p if $p = \mathrm{mid}_{\mathsf{P}}(\ell)$ for all pairs P in $\mathbb{B}$ such that $\mathrm{mid}_{\mathsf{P}}(\ell)$ is defined.

Definition 2.1. A set $\mathbb{B}$ of pairs of lines is an *affine bisector arrangement* if each line in each pair in $\mathbb{B}$ bisects $\mathbb{B}$ with a finite midpoint (see Figure 4). An *affine bisector field* is an affine bisector arrangement $\mathbb{B}$ such that not all lines in $\mathbb{B}$ go through the same point and $\mathbb{B}$ cannot be extended to a larger bisector arrangement. The pairs of lines in $\mathbb{B}$ are referred to as $\mathbb{B}$-*pairs.*

See Figure 3 for examples of affine bisector fields. The reason for the adjective "affine" is that each bisector in $\mathbb{B}$ has a finite midpoint, whereas in [9] we consider also bisector fields for which a bisector can have a midpoint at infinity, and in such a case "affine" is omitted from the definition. It is not obvious from the definition that affine bisector fields exist, but a large source of examples is provided by one of the main theorems in [9]: *A bisector field consists of the pairs of lines that occur as the degenerations of conics in a non-trivial pencil of conics* [9, Theorem 6.3]. For example, if $\mathcal{C}$ is the set of conics through four points in general position, then the asymptotes of the hyperbolas in the pencil are line pairs in a bisector field, and the only other line pairs in the bisector field are those that share an axis of symmetry with any degenerate parabolas that occur in the pencil. Therefore, a pencil of conics in the plane is asymptotically a bisector field, and every bisector field arises this way. All this works over any field of characteristic other than 2.

To streamline the present article we will exclude the non-affine bisector fields, which amount to a couple of pathological cases, and drop the adjective "affine" from here on out:

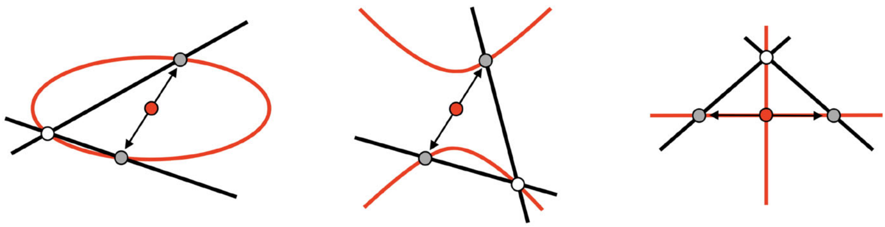

FIGURE 5. Q-pairs of bisectors. (The quadrilateral Q is not pictured.) The bisector locus for Q is the red conic, and the red point is the center of the conic. Q-antipodal pairs of bisectors meet on the bisector locus

Standing assumption: *Throughout the rest of the article, all bisector fields are understood to be affine, as in Definition 2.1.*

A rather different source of bisector fields is relevant for the proofs and theorems in the present paper. By a bisector of a quadrilateral $Q = ABA'B'$, we mean a bisector of the set of pairs $\{\{A, A'\}, \{B, B'\}\}$, that is, a line ℓ that is either a side of Q, parallel to a pair of opposite sides of Q, or crosses all four sides of Q and satisfies $\mathrm{mid}_{AA'}(\ell) = \mathrm{mid}_{BB'}(\ell)$. We denote the set of bisectors of Q by $\mathrm{Bis}(Q)$. Theorem 2.3 below asserts that every bisector field occurs as the set of bisectors of some quadrilateral. However, to state this theorem precisely, it remains to indicate how to pair the bisectors of a quadrilateral Q. For this, we need the following lemma.

Lemma 2.2. [10, Section 2] *For each quadrilateral Q is an inner product (a nondegenerate symmetric bilinear form) under which the slope vectors of opposite sides of Q are orthogonal, as are the slope vectors of the diagonals of Q.*

Two lines are Q-*orthogonal* if their slope vectors are orthogonal under the inner product of the lemma. Because of the possibility of parallel lines in the bisector field, the notion of Q-orthogonality is not quite sufficient to define a pairing on the bisectors of Q. In order to complete the definition, we require a symmetry condition also: A pair of bisectors of Q is Q-*antipodal* if the midpoint of the midpoints of the bisectors is the centroid of Q. The *bisector locus*, the set of midpoints of the bisectors of Q, is a conic (in fact, the nine-point conic for Q) [10, Theorem 5.2 and Corollary 5.4], whose center is the centroid of Q. See Figure 3 for examples of bisector loci. Thus two bisectors are Q-antipodal if and only if their midpoints lie on antipodal points of the bisector locus; see Figure 2. 5

We can now define the pairing: A pair of bisectors is a Q-*pair* if it is Q-antipodal and Q-orthogonal. In almost all cases, a pair of bisectors is Q-antipodal if and only if it is Q-orthogonal. This happens for example if no sides or diagonals of Q are parallel [10, Corollary 6.6]. If ℓ is a bisector of a quadrilateral Q, then there is a unique bisector ℓ' such that ℓ, ℓ' is a Q-pair [10, Corollary 6.7]. Each pair of opposite sides of Q is a Q-pair of bisectors, as is the pair of diagonals of Q; this follows from [10, Proposition 2.5].

Thus we may view $\mathrm{Bis}(Q)$ as a collection of paired lines whose pairing is given by Q-pairing. One of the main theorems in [10] is that bisector pairs are themselves bisected, i.e., the Q-pairs of bisectors of Q not only bisect Q but are in turn bisected by every bisector of Q. It is this symmetry that is the crucial property for a bisector field, and which is a part of the following theorem. By a *quadrilateral in* $\mathbb{B}$ we mean a quadrilateral whose pairs of opposite sides are $\mathbb{B}$-pairs.

Theorem 2.3. [10, Theorem 6.9] *If* $\mathbb{B}$ *is a bisector field (as in Definition 2.1), then for each quadrilateral* Q *in* $\mathbb{B}$*,* $\mathbb{B} = \mathrm{Bis}(Q)$ *and the pairing on* $\mathbb{B}$ *is* Q*-pairing.*

Any two quadrilaterals in the same bisector field have the same centroid, and so we designate this shared centroid as the *center* of the bisector field. That all the quadrilaterals in a bisector field have the same centroid is a consequence of the fact that all these quadrilaterals share the same nine-point conic, and this conic has the centroid of the quadrilateral as its center; see [10, Theorem 5.2].

3. The dual curve of bisectors

For $n \geq 1$, we denote by $\mathbb{P}^n(\Bbbk)$ the projective closure of $\Bbbk^n$. The points in $\mathbb{P}^n(\Bbbk)$ are written in homogeneous coordinates $[x_1 : \ldots : x_{n+1}]$, where $(x_1, \ldots, x_{n+1}) \neq (0, \ldots, 0)$ and $[x_0 : \ldots : x_{n+1}]$ is the line in $\Bbbk^{n+1}$ through the point $(x_1, \ldots, x_{n+1})$. Under point-line duality of the projective plane $\mathbb{P}^2(\Bbbk)$, each line $tX - uY + vZ = 0$ in $\mathbb{P}^2(\Bbbk)$ corresponds to the point $[t : u : v]$ in the dual projective plane. Thus, in describing a bisector field, it is useful to consider the dual points of the bisectors. We do this in the present section and show that the coefficients of the bisectors in a bisector field are encoded onto a singular cubic curve in $\mathbb{P}^2(\Bbbk)$.

As discussed in the last section, to describe a bisector field, it is sufficient to describe the bisectors of any quadrilateral in the bisector field. We begin by associating two polynomials to a quadrilateral, the shape polynomial and the position polynomial. We will show in Corollary 3.4 that these two polynomials completely determine the bisectors of Q. We use the following notational convention throughout the paper.

> Given a line L in the affine plane $\Bbbk^2$, we write an equation for L as $t_L x - u_L y + v_L = 0$, where $t_L, u_L, v_L \in \Bbbk$. In order to have a canonical choice of coefficients for the line, we assume that if $u_L = 0$

then $t_L = 1$ and if $u_L \neq 0$ then $u_L = 1$. The slope of L will often be represented as the element $[t_L : u_L]$ in $\mathbb{P}^1(\Bbbk)$.

Let $Q = ABA'B'$ be a quadrilateral, and consider the polynomial in $\Bbbk[T, U]$ whose zeroes in $\mathbb{P}^1(\Bbbk)$ are all the slopes of the sides of Q:

$$\Theta(T, U) = (u_A T - t_A U)(u_B T - t_B U)(u_{A'} T - t_{A'} U)(u_{B'} T - t_{B'} U).$$

Let Θ_A be Θ with the A term $u_A T - t_A U$ deleted, and let $\Theta_{A'}$ be Θ with the A' term deleted. Define Θ_B and $\Theta_{B'}$ similarly, but by deleting the B and B' terms from $-\Theta$ rather than Θ.

Definition 3.1. The *position polynomial* for Q is

$$\Psi(T, U) = \sum v_L \Theta_L(T, U), \text{ where } L \text{ ranges over the sides of } Q.$$

The *shape polynomial* for Q is the polynomial $\Phi(T, U) \in \Bbbk[T, U]$ for which

$$T\Phi(T, U) = \sum t_L \Theta_L(T, U) \quad \text{and} \quad U\Phi(T, U) = \sum u_L \Theta_L(T, U),$$

Alternatively, $\Phi(T, U) = \alpha T^2 - 2\beta TU + \gamma U^2$, where

$$\begin{aligned}
\alpha &= t_A u_B u_{A'} u_{B'} - u_A t_B u_{A'} u_{B'} + u_A u_B t_{A'} u_{B'} - u_A u_B u_{A'} t_{B'} \\
\beta &= t_A u_B t_{A'} u_{B'} - u_A t_B u_{A'} t_{B'} \\
\gamma &= t_A t_B t_{A'} u_{B'} - t_A t_B u_{A'} t_{B'} + t_A u_B t_{A'} t_{B'} - u_A t_B t_{A'} t_{B'}.
\end{aligned}$$

We write Φ_Q and Ψ_Q for Φ and Ψ when the quadrilateral is not clear from context.

That these two definitions for Φ are equivalent follows from the fact that

$$\alpha = \sum t_L \Theta_L(1, 0), \ \ \gamma = \sum t_L \Theta_L(0, 1), \ \ -2\beta = \sum t_L \Theta_L(1, 1) - \alpha - \gamma,$$

and similarly for the polynomial $\sum_L u_L \Theta_L(T, U)$. Although not needed in what follows, the polynomial Φ can also be viewed as the squared norm of the inner product $\langle -, - \rangle_Q$ from Lemma 2.2. This follows from the explicit description of the inner product given in [10, Section 2].

The shape polynomial Φ depends only on the slopes of the sides of Q, while the position polynomial Ψ depends also on the position of Q. The reason for "shape" is that Φ gives the shape of the bisector locus for Q which, if Q is proper, coincides with the nine-point conic for Q. (The bisector locus is discussed in Section 2.) Namely, let (h, k) be the centroid of Q and let (a, b) be a diagonal point of Q, that is, a point in $\Bbbk^2$ where a pair of opposite sides or diagonals of Q meets. It is shown in [10, Theorem 5.2] that the bisector locus of Q is a conic with center (h, k) given by the equation

$$\Phi(Y - k, X - h) = \Phi(b - k, a - h).$$

Definition 3.2. For a quadrilateral Q, the *dual polynomial* for Q is the homogeneous polynomial $\Psi(T, U) - V\Phi(T, U) \in \Bbbk[T, U, V]$. The *dual curve* of Q is the zero set in $\mathbb{P}^2(\Bbbk)$ of the dual polynomial.

Since Ψ and Φ are homogeneous in the variables T and U and the quadratic form Φ is never the zero polynomial (this follows from the nondegeneracy of the bilinear form in Lemma 2.2 and the fact that Φ is the quadratic form associated to this inner product), the dual polynomial has degree 3 for any choice of quadrilateral Q. Also, since Ψ and Φ are homogeneous, the dual polynomial has at least one singularity, namely the point $[0:0:1]$. The only other singularities are those that appear when Ψ and Φ share a common zero, in which case the cubic is not an irreducible curve. As the next theorem shows, the reason for the terminology of dual curve is that the points on this curve are the dual points for the bisectors of a quadrilateral.

Theorem 3.3. *A line $tX - uY + v = 0$ bisects a quadrilateral Q if and only if $[t : u : v]$ is on the dual curve of Q.*

Proof. Write $Q = ABA'B'$. Suppose first the line $\ell : tX - uY + v = 0$ is not parallel to any of the sides of Q. For each line $L \in \{A, B, A', B'\}$, set

$$x_L = \frac{uv_L - u_Lv}{tu_L - t_Lu} \quad \text{and} \quad y_L = \frac{tv_L - t_Lv}{tu_L - t_Lu}.$$

Since ℓ is not parallel to L, the line ℓ crosses L at the point (x_L, y_L). The line ℓ is a bisector of Q if and only if the midpoint of the points where ℓ crosses A and A' is also the midpoint of the points where ℓ crosses B and B'; if and only if

$$x_A - x_B + x_{A'} - x_{B'} = 0 \text{ and } y_A - y_B + y_{A'} - y_{B'} = 0;$$

if and only if

$$\sum(uv_L - vu_L)\Theta_L(t, u) = 0 \quad \text{and} \quad \sum(v_Lt - t_Lv)\Theta_L(t, u) = 0;$$

if and only if

$$\text{(a)}\ \ u\sum v_L\Theta_L(t, u) - v\sum u_L\Theta_L(t, u) = 0 \quad \text{and}$$
$$\text{(b)}\ \ t\sum v_L\Theta_L(t, u) - v\sum t_L\Theta_L(t, u) = 0.$$

By Definition 3.1,

$$u\Phi(t, u) = \sum u_L\Theta_L(t, u) \text{ and } t\Phi(t, u) = \sum t_L\Theta_L(t, u).$$

Thus the equations in (a) and (b) become

$$u\sum v_L\Theta_L(t, u) - vu\Phi(t, u) = 0 \quad \text{and} \quad t\sum v_L\Theta_L(t, u) - vt\Phi(t, u) = 0.$$

Since u and t are not both 0, the equations in (a) and (b) are valid (and hence ℓ is a bisector) if and only if

$$\sum v_L\Theta_L(t, u) - v\Phi(t, u) = 0.$$

Since $\Psi(t, u) = \sum v_L\Theta_L(t, u)$, the theorem is proved under the assumption that ℓ is not parallel to any of the sides of Q.

Now suppose ℓ is parallel to a side M of Q, so that ℓ is defined by the equation $t_M x - u_M y + v = 0$ for some $v \in \Bbbk$. We claim ℓ is a bisector of Q if and only if $[t_M : u_M : v]$ is on the dual curve of Q. Using Definition 3.1 and the definition of the polynomials Θ_L,

$$\begin{aligned} u_M\Phi(t_M,u_M) = \sum u_L\Theta_L(t_M,u_M) \;&=\; u_M\Theta_M(t_M,u_M) \\ t_M\Phi(t_M,u_M) = \sum t_L\Theta_L(t_M,u_M) \;&=\; t_M\Theta_M(t_M,u_M). \end{aligned}$$

Since either $t_M \neq 0$ or $u_M \neq 0$, we conclude $\Phi(t_M,u_M) = \Theta_M(t_M,u_M)$.

Now $[t_M : u_M : v]$ is on the dual curve of Q if and only if $v\Phi(t_M,u_M) = \Psi(t_M,u_M)$; if and only if $v\Theta_M(t_M,u_M) = v_M\Theta_M(t_M,u_M)$; if and only if $v = v_M$ or $\Theta_M(t_M,u_M) = 0$; if and only if $\ell = M$ (since ℓ is parallel to M) or M is parallel to another side of Q; if and only if ℓ is a bisector of Q. □

As discussed in Section 2 (and proved in [10, Corollary 6.10]), two quadrilaterals that have the same bisectors have the same bisector field. The next corollary shows that the dual polynomial can also be used to determine if two quadrilaterals have the same bisector field.

Corollary 3.4. *Two quadrilaterals Q and Q' have the same bisectors (and hence the same bisector fields) if and only if Q and Q' have the same dual polynomial; if and only if $\Phi_Q = \lambda\Phi_{Q'}$ and $\Psi_Q = \lambda\Psi_{Q'}$ for some $0 \neq \lambda \in \Bbbk$.*

Proof. For each $i = 1,2$, let $\overline{Q_i}$ be the extension of Q_i to the affine plane over the algebraic closure $\overline{\Bbbk}$ of $\Bbbk$. Then Q_i and $\overline{Q_i}$ share the same shape, position and dual polynomials since these are defined in terms of the coefficient data of the lines that comprise the sides of Q_i. Since $\overline{\Bbbk}$ is infinite, Theorem 3.3 implies $\overline{Q_1}$ and $\overline{Q_2}$ share the same bisectors if and only if the dual curves of $\overline{Q_1}$ and $\overline{Q_2}$ are the same; if and only if the dual polynomial of Q_1 is a scalar multiple (over $\overline{\Bbbk}$) of the dual polynomial of Q_2; if and only if there is $0 \neq \lambda\lambda \in \overline{\Bbbk}$ such that $\Phi_{Q_1} = \lambda\Phi_{Q_2}$ and $\Psi_{Q_1} = \lambda\Psi_{Q_2}$. In this case, $\lambda \in \Bbbk$ since the coefficients of Φ_{Q_1} and Ψ_{Q_2} are in $\Bbbk$. Thus to prove the corollary it suffices to prove that Q_1 and Q_2 have the same bisectors if and only if $\overline{Q_1}$ and $\overline{Q_2}$ do.

Suppose Q_1 and Q_2 have the same bisectors. For this part of the proof we rely on the discussion of pairing from Section 2. By [10, Corollary 6.10], Q_1-pairing and Q_2-pairing are the same for these bisectors. Since $\overline{Q_i}$-orthogonality is defined by the same inner product as Q_i-orthogonality (it is defined in terms of the coefficients of the lines that are the sides of Q_i; see [10, Section 2]), and Q_i and $\overline{Q_i}$ have the same centroid, it follows that the extensions $\overline{\ell}, \overline{\ell'}$ of two bisectors ℓ, ℓ' of Q_i form a $\overline{Q_i}$-pair if and only if ℓ, ℓ' form a Q_i-pair. Therefore, since the sides of $\overline{Q_1}$, being extended from bisectors of Q_1, and hence bisectors of Q_2, are bisectors of $\overline{Q_2}$, it follows that the quadrilateral $\overline{Q_1}$ is in the bisector field of $\overline{Q_2}$, which, as discussed in Section 2, implies $\overline{Q_1}$ and $\overline{Q_2}$ have the same bisectors. Conversely, if $\overline{Q_1}$ and $\overline{Q_2}$ have the same bisectors, then, as we have already proved, Q_1 and Q_2 have the same dual polynomials up to scalar multiple, and hence Q_1 and Q_2 have the same bisectors by Theorem 3.3. □

4. Cubic, quadratic and linear bisector fields

Since a bisector field is the set of bisectors of a quadrilateral, Corollary 3.4 allows us to shift our focus from quadrilaterals to bisector fields and attach to bisector fields the polynomial data developed in the last section for quadrilaterals.

Definition 4.1. For each bisector field $\mathbb{B}$, fix a quadrilateral Q such that $\mathbb{B} = \mathrm{Bis}(Q)$. The *shape polynomial for* $\mathbb{B}$ is the shape polynomial Φ for Q, the *position polynomial for* $\mathbb{B}$ is the position polynomial Ψ for Q, the *dual polynomial for* $\mathbb{B}$ is the dual polynomial for Q, and the *dual curve of* $\mathbb{B}$ is the dual curve of Q.

For the polynomials in Definition 4.1, different quadrilaterals in $\mathbb{B}$ can yield different polynomials but these polynomials are by Corollary 3.4 scalar multiples of those for Q. Ultimately, we are interested in zeroes of these polynomials (such as in Theorem 3.3) so uniqueness up to scalar multiple is all that is needed.

The polynomials Ψ and Φ may share common factors (see Lemma 4.4) and hence the dual polynomial from Theorem 3.3 may not be irreducible. We remove these factors in the formulation of the reduced dual polynomial, defined next. This irreducible polynomial is needed later to distinguish between moving bisectors and bisectors in a pencil of parallel lines. As discussed in the last section, Φ is never the zero polynomial. However, it can happen that $\Psi = 0$ (see Lemma 5.2), and this case matters in the next definition.

Definition 4.2. Let $\mathbb{B}$ be a bisector field with shape and position polynomials Φ and Ψ. If $\Psi \neq 0$, let ϕ and ψ be homogeneous relatively prime polynomials in $\Bbbk[T,U]$ such that $\Phi\psi = \Psi\phi$ and ϕ is monic in T; otherwise, if $\Psi = 0$, set $\psi(T,U) = 0$ and $\phi(T,U) = 1$. We define the *reduced dual polynomial* for $\mathbb{B}$ as

$$F(T,U,V) = \psi(T,U) - V\phi(T,U).$$

The *reduced dual curve* of $\mathbb{B}$ is the zero set of F in $\mathbb{P}^2(\Bbbk)$.

The reduced dual polynomial F is a factor of the dual polynomial. The degree of F, which is at most three, will be important in the next sections for describing the nature of the bisector field.

Definition 4.3. Let $\mathbb{B}$ be a bisector field, and let F be the reduced dual polynomial for $\mathbb{B}$. Then $\mathbb{B}$ is *cubic* if $\deg F = 3$; *quadratic* if $\deg F = 2$; and *linear* if $\deg F = 1$.

A geometric interpretation of these classes of bisector fields is given in Theorem 4.6 and Corollary 4.7.

We describe the properties of a special class of bisectors in the next lemma. Recall that a pencil of lines is the set of lines through a point. This point can be at infinity, and so the set of all lines having a fixed slope is a pencil of parallel lines through a point at infinity.

Lemma 4.4. *Let $\mathbb{B} = \mathrm{Bis}(Q)$ be a bisector field, where Q is a quadrilateral. The following are equivalent for a bisector ℓ of $\mathbb{B}$.*

(1) *ℓ is parallel to another bisector in $\mathbb{B}$.*
(2) *ℓ is parallel to, and distinct from, a side or diagonal of Q.*
(3) *ℓ is parallel to two sides or diagonals of Q.*
(4) *ℓ belongs to a pencil of parallel bisectors in $\mathbb{B}$.*
(5) *$\Phi(t_\ell, u_\ell) = 0$.*
(6) *$\Phi(t_\ell, u_\ell) = \Psi(t_\ell, u_\ell) = 0$.*

Proof. The equivalence of (3), (4) and (5) follows from [10, Remark 2.6 and Lemma 5.3]. That (2) implies (1) is clear since sides and diagonals of Q are bisectors. To see that (1) implies (5), assume the bisector ℓ is parallel to a different bisector ℓ'. In this case, $[t_\ell : u_\ell] = [t_{\ell'} : u_{\ell'}]$ and $v_\ell \neq v_{\ell'}$, and so Theorem 3.3 implies $\Phi(t_\ell, u_\ell) = 0$. That (3) implies (2) is clear, so (1)–(5) are equivalent. To see that (5) implies (6),

use Theorem 3.3. That (6) implies (5) is clear. □

Definition 4.5. A bisector in $\mathbb{B}$ of a quadrilateral Q is *null* if it satisfies the equivalent conditions of Lemma 4.4.

Since the inner product in Lemma 2.2 has Φ as its squared norm, the null bisectors of Q are the bisectors that are Q-orthogonal to themselves.

Theorem 4.6. *A bisector field is linear if and only if it contains exactly two pencils of parallel lines; it is quadratic if and only if it contains exactly one pencil of parallel lines; and it is cubic if and only if it contains no pencils of parallel lines.*

Proof. Let $\mathbb{B}$ be a bisector field, and let F be the reduced dual polynomial of $\mathbb{B}$. By Theorem 3.3 and Lemma 4.4, if $[t : u] \in \mathbb{P}^1(\mathbb{k})$, then there is a pencil of parallel lines in $\mathbb{B}$ of slope $[t : u]$ if and only if $\Phi(t, u) = \Psi(t, u) = 0$. Thus to prove the theorem, it suffices to show that the number of mutual zeroes of Φ and Ψ in $\mathbb{P}^1(\mathbb{k})$ is $3 - \deg F$. By the choice of ϕ, $2 - \deg \phi$ is the number of mutual zeroes of Φ and Ψ in $\mathbb{P}^1(\mathbb{k})$. Thus, since $\deg F = 1 + \deg \phi$, we have that $3 - \deg F = 2 - \deg \phi$ is the number of mutual zeroes of Φ and Ψ, which proves the theorem. □

Theorem 4.6 implies that in Figure 3, the bisector fields in (a) and (b) are cubic bisector fields; the bisector field in (c) is quadratic; and the bisector field in (d) is linear.

Corollary 4.7. *Let $\mathbb{B}$ be a bisector field.*

1. (1) *$\mathbb{B}$ is cubic if and only if $\mathbb{B}$ is the bisector field of a quadrilateral with no parallel sides or diagonals.*
2. (2) *$\mathbb{B}$ is quadratic if and only if $\mathbb{B}$ is the bisector field of a trapezoid that is not a parallelogram.*
3. (3) *$\mathbb{B}$ is linear if and only if $\mathbb{B}$ is the bisector field of a parallelogram.*

Proof. By Lemma 4.4 and Theorem 4.6, $\mathbb{B}$ is cubic if and only if no bisector is null, and so (1) follows from Lemma 4.4. For (2), suppose $\mathbb{B}$ is quadratic. By Theorem 4.6, there is a $\mathbb{B}$-pair of distinct parallel bisectors A and A' in $\mathbb{B}$. Let B, B' be a $\mathbb{B}$-pair of non-null bisectors. Since B and B' are not parallel to each other or any other bisectors, $Q = ABA'B'$ is a trapezoid that is not a parallelogram, and by Theorem 2.3, $\mathbb{B} = \mathrm{Bis}(Q)$. For the converse, Lemma 4.4 implies that $\mathbb{B}$ contains exactly one pencil of parallel lines and so $\mathbb{B}$ is quadratic by Theorem 4.6. The proof of statement (3) is similar to that of (2). $\square$

By Lemma 4.4, each null bisector in a bisector field is in a pencil of parallel lines in that bisector field. Thus the set of null bisectors is either empty, a single pencil of parallel lines or a union of two pencils of parallel lines. Non-null bisectors have a more interesting description, and these will be described in Sections 6 and 7 with the notion of moving bisectors.

5. Affine equivalence

Since affine transformations preserve midpoints and parallel lines, the image of a bisector field under such a transformation is again a bisector field. Two bisector fields $\mathbb{B}_1$ and $\mathbb{B}_2$ are *affinely equivalent* if there is an affine transformation of the plane that carries the lines in $\mathbb{B}_1$ onto the lines in $\mathbb{B}_2$. The pairing on the lines in $\mathbb{B}_1$ becomes a pairing on the lines in $\mathbb{B}_2$, and as discussed in Section 2 this is the only possible pairing on $\mathbb{B}_2$ for which $\mathbb{B}_2$ is a bisector field. We will use affine transformations to reduce to bisector fields of a form in which the shape and position polynomials Φ and Ψ become more tractable. This is helpful because the coefficients of a typical polynomial in our context need to be treated as variables in the course of proofs and so these equations can involve large expressions. For example, in the general case, with the coefficients taken as variables, the position polynomial has fifteen variables, degree seven and too many monomial terms to list reasonably. Lemma 5.2 shows that finding the right transformation for the bisector field greatly simplifies these expressions.

Definition 5.1. A bisector field $\mathbb{B}$ is in *standard form* if the axes $X = 0$ and $Y = 0$ are a $\mathbb{B}$-pair of bisectors in $\mathbb{B}$.

Every bisector field $\mathbb{B}$ is affinely equivalent to a bisector field in standard form.

This is because there is a $\mathbb{B}$-pair of lines ℓ and ℓ' in $\mathbb{B}$ that are not parallel, and so after a translation of the plane, we may assume ℓ and ℓ' meet at the origin. An invertible linear transformation carries ℓ onto the line $Y=0$ and ℓ' onto $X=0$, and so the image of $\mathbb{B}$ under this transformation is a bisector field in standard form. We will use the following lemma often in what follows.

Lemma 5.2. *Let $\mathbb{B}$ be a bisector field in standard form with center (h,k). Then $t_\ell t_{\ell'}$ has the same value μ for all $\mathbb{B}$-pairs of bisectors ℓ,ℓ' distinct from the X and Y-axes, and*

$$\Phi(T,U)=T^2-\mu U^2 \quad \textit{and} \quad \Psi(T,U)=4TU(kT+\mu hU).$$

Proof. Let A be the line $Y=0$ and A' the line $X=0$, and let B,B' be a $\mathbb{B}$-pair of distinct bisectors in $\mathbb{B}$, neither of which is parallel to A or A'. Then

$$t_A=u_{A'}=v_A=v_{A'}=0 \text{ and } u_A=t_{A'}=u_B=u_{B'}=1.$$

Let $\mu=t_Bt_{B'}$. The vertices of the quadrilateral $Q=ABA'B'$ are

$$\begin{aligned} A\cdot B&=\left(-\frac{t_{B'}v_B}{\mu},0\right),\ B\cdot A'=(0,v_B),\ A'\cdot B'=(0,v_{B'}),\\ B'\cdot A&=\left(-\frac{t_Bv_{B'}}{\mu},0\right). \end{aligned} \tag{1}$$

Thus the centroid (h,k) of Q, which is the center of $\mathbb{B}$, is given by the equations

$$-4h\mu=t_{B'}v_B+t_Bv_{B'} \text{ and } 4k=v_B+v_{B'}.$$

Substituting all this data into the equations for Ψ and Φ from Section 3 yields

$$\Phi(T,U)=T^2-\mu U^2 \text{ and } \Psi(T,U)=4TU(kT+\mu hU).$$

In particular, a $\mathbb{B}$-pair ℓ,ℓ' of bisectors in $\mathbb{B}$, since these bisectors are Q-orthogonal and $\Phi(T,U)=T^2-\mu U^2$, satisfies $t_\ell t_{\ell'}-\mu u_\ell u_{\ell'}=0$ (see [10, Definition 2.3]). If ℓ and ℓ' are not parallel to A or A', we have $u_\ell=u_{\ell'}=1$ and hence $\mu=t_\ell t_{\ell'}$, and so the value μ is independent of the choice of ℓ and ℓ'. □

The quantity μ in Lemma 5.2 is independent of the choice of $\mathbb{B}$-pair, as long as the pair is not the pair of axes. Because of this, we can make the following definition.

Definition 5.3. The *coefficient* of a bisector field $\mathbb{B}$ in standard form is the product $\mu=t_\ell t_{\ell'}$ of the slopes t_ℓ and t'_ℓ of any $\mathbb{B}$-pair ℓ,ℓ' of bisectors in $\mathbb{B}$ not parallel to $X=0$ or $Y=0$.

The coefficient of the bisector field in Figure 3(a) is -1 since all the $\mathbb{B}$-pairs in this bisector field are perpendicular. The coefficient of the bisector field in Figure 3(b) is 1 since the lines in the $\mathbb{B}$-pairs are reflected about the line $Y = X$. The next theorem is important for the classifications of bisector field in Section 8.

Theorem 5.4. *Two bisector fields in standard form are the same if and only if they have the same center and the same coefficient.*

Proof. By Corollary 3.4, two bisector fields are the same if and only if they have the same shape and position polynomials up to scalar multiple. This fact and Lemma 5.2 imply the theorem. $\square$

The theorem shows the center and coefficient uniquely determine a bisector field in standard form, and so when seeking to determine whether two bisector fields are affinely equivalent it is natural to do so using the data of the center and coefficient, as in the next lemma, which will be important in Section 8 as well as in the rest of this section.

Lemma 5.5. *Let $\mathbb{B}$ be a bisector field in standard form with coefficient μ_1, and let $0 \neq \mu_2 \in \Bbbk$ such that $\mu_1\mu_2$ is a square in $\Bbbk$.*

If ℓ, ℓ' is a $\mathbb{B}$-pair of bisectors that are not parallel, then there is an affine transformation of the plane that sends the lines ℓ and ℓ' to the lines $X = 0$ and $Y = 0$ and sends $\mathbb{B}$ to a bisector field in standard form with coefficient μ_2.

Proof. Let $\mathbb{B}_1 = \mathbb{B}$, and let μ_1 be the coefficient of $\mathbb{B}_1$. Write the slope of ℓ as $[t : u]$ and that of ℓ' as $[t' : u']$. Then $tt' - \mu_1 uu' = 0$ since ℓ, ℓ' is a $\mathbb{B}_1$-pair and μ_1 is the coefficient of $\mathbb{B}_1$. Also, since ℓ and ℓ' are not parallel, ℓ is not null (Lemma 4.4), and so $\Phi(t, u) = t^2 - \mu_1 u^2 \neq 0$ by Lemma 4.4. By assumption, there is $\theta \in \Bbbk$ such that $1 = \theta^2\mu_1\mu_2$. Let $\mathcal{L}$ be the linear transformation of the plane given by

$$\mathcal{L}(x, y) = (\mu_1\theta(tx - uy), -\mu_1 ux + ty) \text{ for all } x, y \in k.$$

Since the determinant of this linear transformation is $\mu_1\theta(t^2 - \mu_1 u^2)$ and no factor in this product is 0, the transformation $\mathcal{L}$ is invertible. Thus the image of $\mathbb{B}_1$ under $\mathcal{L}$ is a bisector field whose pairing is induced by that of $\mathbb{B}_1$.

To determine the slopes of the lines $\mathcal{L}(\ell)$ and $\mathcal{L}(\ell')$, observe that since $tt' - \mu_1 uu' = 0$,

$$\mathcal{L}(u, t) = (0, t^2 - \mu_1 u^2) \text{ and } \mathcal{L}(u', t') = (\mu_1\theta(tu' - ut'), 0).$$

Thus $\mathcal{L}(\ell)$ is parallel to $X = 0$ and $\mathcal{L}(\ell')$ is parallel to $Y = 0$. Let $\mathcal{A}$ be the affine transformation that is the composition of $\mathcal{L}$ with the translation that carries the intersection of $\mathcal{L}(\ell)$ and $\mathcal{L}(\ell')$ to the origin and hence carries these two lines to the lines $X = 0$ and $Y = 0$. Let $\mathbb{B}_2$ be the image of $\mathbb{B}_1$ under $\mathcal{A}$.

Since ℓ, ℓ' is a $\mathbb{B}_1$-pair, the lines $X = 0$ and $Y = 0$ form a $\mathbb{B}_2$-pair of bisectors in $\mathbb{B}_2$. Thus $\mathbb{B}_2$ is a bisector field in standard form.

It remains to show $\mathbb{B}_2$ has coefficient μ_2. Let ℓ_1 and ℓ_2 be a $\mathbb{B}_1$-pair of bisectors in $\mathbb{B}_1$, and let $[t_1 : u_1]$ and $[t_2 : u_2]$ be the slopes of these two lines. For each $i = 1, 2$, let

$$\upsilon_i = \mu_1\theta(tu_i - ut_i) \text{ and } \tau_i = -uu_i\mu_1 + tt_i.$$

Then $\mathcal{L}(u_i, t_i) = (\upsilon_i, \tau_i)$, and so $\mathcal{L}(\ell_i)$ has slope $[\tau_i : \upsilon_i]$. Now $t_1t_2 - \mu_1u_1u_2 = 0$ since ℓ_1, ℓ_2 is a $\mathbb{B}_1$-pair of bisectors in $\mathbb{B}_1$. Using this fact, along with fact that $\mu_1\mu_2\theta^2 = 1$, a calculation shows

$$\tau_1\tau_2 - \mu_2\upsilon_1\upsilon_2 = (\mu_1u^2 - t^2)(\mu_1u_1u_2 - t_1t_2) = 0.$$

Since this is the case for all $\mathbb{B}_1$-pairs ℓ_1, ℓ_2, it follows that the coefficient of $\mathbb{B}_2$ is μ_2. □

The following theorem gives a simple criterion for when a bisector field in standard form can be transformed into one with a specified coefficient.

Theorem 5.6. *Let $\mathbb{B}_1$ be a bisector field in standard form with coefficient μ_1, and let $0 \neq \mu_2 \in \mathbb{k}$. Then $\mu_1\mu_2$ is a square in $\mathbb{k}$ if and only if there is a bisector field in standard form with coefficient μ_2 that is affinely equivalent to $\mathbb{B}_1$.*

Proof. Suppose

$$g : \mathbb{k}^2 \to \mathbb{k}^2 : (x, y) \mapsto (ax + by + e, cx + dy + f)$$

is an affine transformation that sends $\mathbb{B}_1$ to a bisector field $\mathbb{B}_2$ in standard form with coefficient μ_2. Then g sends the line $Y = 0$ to the line whose slope is $[c : a]$ and the line $X = 0$ to the line whose slope is $[d : b]$. Since the X and Y-axes are a $\mathbb{B}_1$-pair of bisectors in $\mathbb{B}_1$, the images of these two lines form a $\mathbb{B}_2$-pair of bisectors in $\mathbb{B}_2$. Thus $cd - \mu_2ab = 0$. The line that maps onto the line $X = 0$ is $aX + bY + e = 0$ and has slope $[a : -b]$. Similarly, the line that maps onto $Y = 0$ is $cX + dY + f = 0$ and has slope $[c : -d]$. These lines form a $\mathbb{B}_1$-pair of bisectors, so $ac - \mu_1bd = 0$. With the aim of showing $\mu_1\mu_2$ is a square in $\mathbb{k}$, we consider solutions a, b, c, d to the equations $cd - \mu_2ab = 0$ and $ac - \mu_1bd = 0$. We will use that fact that $ad - bc \neq 0$ since g is invertible.

First suppose $b = c = 0$. In this case, $a \neq 0$ and $d \neq 0$ and

$$g(x, y) = (ax + e, dy + f) \text{ for all } (x, y) \in \mathbb{k}^2.$$

Let B, B' be a $\mathbb{B}_1$-pair of bisectors in $\mathbb{B}_1$ that are not parallel to $X = 0$ or $Y = 0$. Then $u_B = u_{B'} = 1$ and the image of B under g has slope dt_B/a while the image of B' has slope $dt_{B'}/a$. This implies $d^2t_Bt_{B'} = a^2\mu_2$ since the images of B and B' form a $\mathbb{B}_2$-pair. Thus

$$d^2\mu_1 - a^2\mu_2 = d^2t_Bt_{B'} - a^2\mu_2 = 0.$$

Since $d \neq 0$, this implies μ_1/μ_2 is a square in $\Bbbk$, and so $\mu_1\mu_2$ is a square in $\Bbbk$ since $\mu_2 \in \Bbbk$. A similar argument shows that if $a = d = 0$, then $\mu_1\mu_2$ is a square in $\Bbbk$.

Finally, suppose at least one of b and c is nonzero and at least one of a and d is nonzero. In this case, the equations $cd - \mu_2 ba = 0$ and $ac - \mu_1 bd = 0$ imply there is an element θ of the algebraic closure of $\Bbbk$ such that $\mu_1\mu_2\theta^2 = 1$, $a = \mu_1\theta d$ and $b = \theta c$. If $d = 0$, then $c \neq 0$ since $ad - bc \neq 0$. In this case, since $\theta c = b \in \Bbbk$, we have $\theta \in \Bbbk$ and hence $\mu_1\mu_2 = \theta^{-1} \in \Bbbk$. Similarly, if $c = 0$, then $d \neq 0$ and $\mu_1\mu_2 = \theta^{-1} \in \Bbbk$. The converse of the theorem follows from Lemma 5.5. □

One motivation for the next definition is that the existence of a bisector through the center of the bisector field allows for a significant simplification in calculations. This is because, as we will see later, it is possible to take such a bisector field and transform it into one whose center is on the Y-axis, thus introducing an extra zero into the calculations.

Definition 5.7. A bisector field is *well centered* if there is a bisector passing through its center.

A linear or quadratic bisector field is always well centered. This is because in both of these cases, the bisector field contains a pencil of parallel lines and hence every point in the plane has a bisector through it. Thus the question of whether a bisector field is well centered is of consequence only for cubic bisector fields. See Example 5.9 for an example of a bisector field that is not well centered.

Since the center of a bisector field remains a center under affine transformation, whether a cubic bisector field is well centered is invariant under such transformations, and so we can reduce to the case of a cubic bisector field in standard form. In this case there is a straightforward criterion for when the bisector field is well centered.

Lemma 5.8. *A cubic bisector field in standard form with center (h, k) and coefficient μ is well centered if and only if there is a zero in $\mathbb{P}^1(\Bbbk)$ of the polynomial*

$$hT^3 + 3kT^2U + 3h\mu TU^2 + k\mu U^3. \tag{2}$$

Moreover, every bisector field over the field $\Bbbk$ is well centered if and only if the polynomial $T^3 + 3T^2 + 3\mu T + \mu$ has a zero in $\Bbbk$ for every $\mu \in \Bbbk$.

Proof. By Theorem 3.3 and Lemma 5.2,

a cubic bisector field $\mathbb{B}$ is well centered if and only if there is a zero in $\mathbb{P}^1(\Bbbk)$ of the polynomial

$$hT(T^2 - \mu U^2) - kU(T^2 - \mu U^2) + 4TU(kT + \mu hU).$$

This polynomial is the same as that in (2).

To prove the second assertion, let $\mathbb{B}$ be a bisector field. We can assume $\mathbb{B}$ is a cubic bisector field in standard form with center (h, k), since linear and quadratic bisector fields are always well centered. If $h = 0$ or $k = 0$, then $\mathbb{B}$ is well centered since the axes are in $\mathbb{B}$. Otherwise, if $h \neq 0$ and $k \neq 0$, then after a linear transformation that scales each coordinate we can assume $(h, k) = (1, 1)$. (A bisector field in standard form remains in standard form when scaled.) By the first claim, $\mathbb{B}$ is well centered if and only if $T^3 + 3T^2 + 3\mu T + \mu$ has a zero. $\square$

Thus over an algebraically closed field every cubic bisector field is well centered. This holds true over a real closed field $\Bbbk$ also since every cubic over $\Bbbk$ has a root in $\Bbbk$. As another example, if $\Bbbk$ has characteristic 3, then every bisector field is well centered if and only if $\Bbbk$ is closed under cube roots.

Example 5.9. Not all bisector fields are well-centered. Let $\Bbbk$ be the field with 7 elements, and let $\mathbb{B}$ be the unique (cubic) bisector field in standard form with $h = k = 1$ and $\mu = 2$. The polynomial in Lemma 5.8

has no zero in $\Bbbk$, and so $\mathbb{B}$ is not well centered. Similarly, if $\Bbbk$ is the field of rational numbers and $\mathbb{B}$ is the unique bisector field in standard form with $h = k = \mu = 1$, then the polynomial in Lemma 5.8 has one real zero in $\mathbb{P}^1(\mathbb{R})$ but no zeroes in $\mathbb{P}^1(\mathbb{Q})$, and so $\mathbb{B}$ is not well centered.

The next theorem shows that for well centered bisector fields, the geometric problem of classification of bisector fields up to affine equivalence is also an arithmetical one. The difference between this theorem and Theorem 5.6 is that in the latter we could only assert affine equivalence with *some* bisector field having coefficient μ_2, whereas here we get equivalence with *every* such bisector field.

Theorem 5.10. *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be well-centered cubic bisector fields in standard form with coefficients μ_1 and μ_2, respectively. Then $\mathbb{B}_1$ is affinely equivalent to $\mathbb{B}_2$ if and only if $\mu_1\mu_2$ is a square in $\Bbbk$.*

Proof. If $\mathbb{B}_1$ is affinely equivalent to $\mathbb{B}_2$, then $\mu_1\mu_2$ is a square by Theorem 5.6. Conversely, suppose $\mu_1\mu_2$ is a square in $\Bbbk$. Since $\mathbb{B}_1$ is well-centered, there is a line A through the center of $\mathbb{B}_1$. Let A' be the bisector in $\mathbb{B}_1$ such that A, A' is a $\mathbb{B}_1$-pair. Since $\mathbb{B}_1$ is a cubic bisector field, $\mathbb{B}_1$ has no null bisectors, and so A and A' are not parallel. By Lemma 5.5, $\mathbb{B}_1$ is affinely equivalent to a bisector field in standard form having the same coefficient as $\mathbb{B}_1$ and whose center (h, k) lies on the Y-axis. Thus $h = 0$, and since $\mathbb{B}_1$ is a cubic bisector field, Lemma 5.2 implies the center of $\mathbb{B}_1$ is not the origin since otherwise the reduced dual curve of $\mathbb{B}_1$ will have degree 1, contrary to assumption. Thus $k \neq 0$. Since uniformly scaling the plane does not change the coefficient of a bisector field, $\mathbb{B}_1$ is affinely equivalent to the cubic bisector field in standard form with center $(0, 1/2)$ and coefficient μ_1. Applying this same reasoning to

$\mathbb{B}_2$, we can assume without loss of generality that both $\mathbb{B}_1$ and $\mathbb{B}_2$ have center $(0, 1/2)$.

The fact that $\mu_1\mu_2$ is a square in $\Bbbk$ implies there is $\theta \in \Bbbk$ such that $\mu_2 = \theta^2\mu_1$. Define an affine transformation by

$$g : \Bbbk^2 \to \Bbbk^2 : (x, y) \mapsto (\theta^{-1}x, y).$$

Then $g(0, 1/2) = (0, 1/2)$. Let B, B' be a $\mathbb{B}_1$-pair of bisectors in $\mathbb{B}_1$ that are not parallel to $X = 0$ or $Y = 0$. Then $g(B), g(B')$ is a $g(\mathbb{B}_1)$-pair of bisectors in the bisector field $g(\mathbb{B}_1)$. The definition of the map g shows neither line in this pair is parallel to $X = 0$ or $Y = 0$, so since B, B' is a $\mathbb{B}_1$-pair and neither B nor B' is parallel to the X or Y-axes, we have $\mu_1 = t_B t_{B'}$. Now $g(B)$ has slope θt_B and $g(B')$ has slope $\theta t_{B'}$. Thus the coefficient of the bisector field $g(\mathbb{B}_1)$ that is the image of $\mathbb{B}_1$ is

$$t_{g(B)}t_{g(B')} = \theta^2 t_B t_{B'} = \theta^2\mu_1 = \mu_2,$$

and so the image of $\mathbb{B}_1$ has center $(0, 1/2)$ and coefficient μ_2. Since this image is in standard form (as g sends the X and Y-axes to themselves), we conclude from Theorem 5.4 that the image of $\mathbb{B}_1$ is $\mathbb{B}_2$. $\square$

6. The boundary of a nonlinear bisector field

Let $\mathbb{B}$ be a bisector field, and let ϕ and ψ be as in Definition 4.2. Theorem 3.3 implies the lines of the form

$$t\phi(t, u)X - u\phi(t, u)Y + \psi(t, u) = 0, \quad \text{where } [t : u] \in \mathbb{P}^1(\Bbbk) \text{ and } \phi(t, u) \neq 0 \tag{3}$$

are bisectors in $\mathbb{B}$. These lines form a "moving line" in the sense of [2]. The terminology is motivated by the fact that the map

$$[t : u] \mapsto t\phi(t, u)X - u\phi(t, u)Y + \psi(t, u)$$

from $\mathbb{P}^1(\Bbbk)$ to the set of lines in $\Bbbk^2$ produces a flow of bisectors in $\mathbb{B}$.

Definition 6.1. The *moving bisectors* in a bisector field are the bisectors in (3).

The dual points of the moving bisectors are the points on the reduced dual curve from Definition 4.2:

Proposition 6.2. *A line $\ell : tX - uY + v$ is a moving bisector in a bisector field $\mathbb{B}$ if and only if $[t : u : v]$ is a zero of the reduced dual polynomial for $\mathbb{B}$,*

$$F(T, U, V) = \psi(T, U) - V\phi(T, U).$$

Proof. The line ℓ is a moving bisector if and only if $v\phi(t, u) = \psi(t, u)$ and $\phi(t, u) \neq 0$. The condition $\phi(t, u) \neq 0$ is redundant since by the choice of ϕ and ψ, these polynomials cannot share a zero in $\mathbb{P}^1(\Bbbk)$. Thus ℓ is a moving bisector if and only if $F(t, u, v) = 0$. $\square$

By Theorem 4.6, a bisector field is cubic if and only if every bisector is a moving bisector. For each of the bisector fields in Figure 3, the moving bisectors are the midlines of the pairs of parallel lines and the lines that are not parallel to any other lines.

We define a boundary of a bisector field in terms of the (homogeneous) discriminant of its moving line of bisectors. As the presence of the discriminant suggests, the boundary will be the envelope of the moving bisectors. We work out this connection with the envelope in the next section.

Definition 6.3. The *boundary* of a nonlinear bisector field $\mathbb{B}$ is the curve defined by the homogeneous discriminant $\Delta(X, Y)$ of the moving bisectors of $\mathbb{B}$, i.e., the homogeneous discriminant of

$$T\phi(T,U)X - U\phi(T,U)Y + \psi(T,U)$$

viewed as a polynomial in variables T and U with coefficients in $\Bbbk[X, Y]$. If $\mathbb{B}$ is linear, the *boundary* of $\mathbb{B}$ is the center of $\mathbb{B}$.

Theorem 6.4. *The boundary of a quadratic bisector field is a nondegenerate parabola. There is an affine transformation of the plane that carries the boundary onto the parabola $Y = X^2$ and the pencil of null bisectors onto the pencil of lines parallel to the X-axis.*

Proof. Let $\mathbb{B}$ be a quadratic bisector field. It suffices to prove the claim in the second sentence, and for this we may assume $\mathbb{B}$ is in standard form. We show first that $h \neq 0$, $k \neq 0$, $\mu = k^2/h^2$, the slope of every null bisector is $-k/h$ and

$$\phi(T,U) = T - kh^{-1}U \ \text{ and } \ \psi(T,U) = 4kTU.$$

Let A be the line $Y = 0$ and A' the line $X = 0$. By Lemma 5.2,

$$\Phi(T,U) = T^2 - \mu U^2 \text{ and } \Psi(T,U) = 4TU(kT + \mu hU).$$

Now $(h, k) \neq (0, 0)$ since otherwise $\Psi(T,U) = 0$, contrary to the fact that $\mathbb{B}$ is not linear. Let B, B' be a $\mathbb{B}$-pair of distinct bisectors in $\mathbb{B}$ such that B is null. It follows from Lemma 4.4 that B is Q-orthogonal to itself, where Q is a quadrilateral such that $\mathbb{B}$ is the bisector field of Q. Since B is Q-orthogonal to B' also, B is parallel to B'. By Lemma 4.4, neither B nor B' is parallel to A or A' since A is not parallel to A'. Since $t_B = t_{B'}$, the calculation of the centroid of the quadrilateral $Q = ABA'B'$ in the proof of Lemma 5.2 shows

$$-4h\mu = t_B(v_B + v_{B'}) = 4t_Bk,$$

so that $h\mu = -t_Bk$, and hence $t_Bh = -k$ since $\mu = t_Bt_{B'} = t_B^2$. Since $(h, k) \neq (0, 0)$, this implies neither h nor k is 0. Also, the fact that $t_Bh = -k$ implies the slope of B is $-k/h$ and $\mu = t_B^2 = k^2/h^2$. Moreover,

$$\Psi(T,U) = 4TU(kT + \mu hU) = 4kTU(T - t_BU).$$

Since also

$$\Phi(T,U) = T^2 - \mu U^2 = (T - t_B U)(T + t_B U),$$

this implies

$$\psi(T,U) = 4kTU \quad \text{and} \quad \phi(T,U) = T + t_B U = T - kh^{-1}U.$$

The moving line from Definition 6.3 is

$$\begin{aligned}&(TX - UY)(T - kh^{-1}U) + 4kTU \\ &\quad = XT^2 + (4k - kh^{-1}X - Y)TU + kh^{-1}YU^2.\end{aligned}$$

Calculating the homogeneous discriminant of this polynomial viewed as a quadratic polynomial with coefficients in $\Bbbk[X,Y]$, we find the boundary of $\mathbb{B}$ to be the curve defined by

$$\Delta(X,Y) = \big(kh^{-1}X - Y\big)^2 - 8k(kh^{-1}X + Y - 2k).$$

Let

$$f(X,Y) = kh^{-1}X - Y \text{ and } g(X,Y) = -8k(kh^{-1}X + Y - 2k).$$

Then $\Delta(X,Y) = f(X,Y)^2 - g(X,Y)$, so since f and g are linear and $-16k^2h^{-1} \neq 0$, the transformation $\mathcal{A}$ given by

$$\mathcal{A}(x,y) = (f(x,y), g(x,y))$$

is affine and carries the curve Δ onto the nondegenerate parabola $Y = X^2$. Since

$$\Delta_X(h,k) = -8k^2h^{-1} \quad \text{and} \quad \Delta_Y(h,k) = -8k,$$

the line ℓ tangent to Δ at (h,k) has slope $[-k : h]$, which, as established earlier in the proof, is the slope of the null bisectors. The image of (h,k) under $\mathcal{A}$ is $(0,0)$ and the image of ℓ is the X-axis, so the theorem follows. $\square$

The boundary of a cubic bisector field is more complicated than that of a quadratic bisector field:

Lemma 6.5. *Let $\mathbb{B}$ be a cubic bisector field in standard form. Let (h,k) denote the center of $\mathbb{B}$, and let μ be the coefficient of $\mathbb{B}$. The boundary of $\mathbb{B}$ is*

$$\begin{aligned}\Delta(X,Y) = 4\mu(&\mu^2X^4 - 12h\mu^2X^3 - 2\mu X^2Y^2 - 20k\mu X^2Y + 4\mu(12h^2\mu + k^2)X^2 \\ &-20h\mu XY^2 + 88hk\mu XY - 32h\mu(2h^2\mu + k^2)X + Y^4 - 12kY^3 \\ &+4(h^2\mu + 12k^2)Y^2 - 32k(h^2\mu + 2k^2)Y + 64h^2k^2\mu).\end{aligned}$$

Proof. Since $\mathbb{B}$ is in standard form, Lemma 5.2 implies the moving line in Definition 6.3 is

$$XT^3 + (4k - Y)T^2U + \mu(4h - X)TU^2 + \mu YU^3.$$

A calculation shows the homogeneous discriminant $\Delta(X,Y)$ of this cubic, viewed as a polynomial in T and U, is as in the lemma. $\square$

As illustrated by Figures 3(a) and 1(b), the boundary of a cubic bisector field can have singularities, unlike the boundary of a quadratic bisector field.

Theorem 6.6. *Suppose $\Bbbk$ is algebraically closed. If $\mathbb{B}$ is a cubic bisector field, then $\mathbb{B}$ is affinely equivalent to a bisector field whose boundary is the rational quartic curve*

$$X^4 + 2X^2Y^2 + Y^4 + 10X^2Y - 6Y^3 - X^2 + 12Y^2 - 8Y = 0.$$

If char $\Bbbk \neq 3$, *the boundary has three singularities, all of which are ordinary cusps. If* char $\Bbbk = 3$, *there is only one singularity, and this singularity is a higher-order cusp.*

Proof. Since $\Bbbk$ is algebraically closed, $\mathbb{B}$ is well centered in the sense of Section 5. Let A' be the bisector passing through the center (h,k) of $\mathbb{B}$, and let A be the line in $\mathbb{B}$ for which A, A' is a $\mathbb{B}$-pair. Since the bisector field $\mathbb{B}$ is cubic, $\mathbb{B}$ contains only moving bisectors and so A is not parallel to A' by Lemma 4.4. Since $\Bbbk$ is algebraically closed, every element in $\Bbbk$ is a square in $\Bbbk$, and so by Lemma 5.5 we may assume $\mathbb{B}$ is in standard form, A is the line $Y = 0$, A' is the line $X = 0$ and $\mathbb{B}$ has coefficient $\mu = -1$. Since the center (h,k) of $\mathbb{B}$ lies on A', we have $h = 0$. If $k = 0$, then the centroid of Q is the intersection of A and A'. However, using Lemma 5.2, this implies $\Psi = 0$, in which case the reduced dual polynomial F is linear, contrary to the assumption that $\mathbb{B}$ is cubic. Thus $k \neq 0$, and

after rescaling by a factor of $1/(2k)$, we can assume the center (h,k) of $\mathbb{B}$ is $(0,1/2)$.

Substituting $h = 0$, $k = 1/2$ and $\mu = -1$ into the equation for the boundary from Lemma 6.5 yields the curve Δ in the present theorem.

We next find the singularities of the curve Δ. Since $\Bbbk$ is algebraically closed there is $\theta \in \Bbbk$ such that $16\theta^2 = 27$. A calculation shows there are polynomials $f_1, f_2 \in \Bbbk[X,Y]$ for which each term of f_1 and f_2 has degree at least 2 and

$$\begin{aligned}
\Delta_1(X,Y) &:= \Delta\left(X - \frac{4}{3}\theta Y + \theta, Y - \frac{1}{4}\right) &= Xf_1(X,Y) + \frac{27}{4}X^2 + 16Y^3(Y-1)\\
\Delta_2(X,Y) &:= \Delta\left(X + \frac{4}{3}\theta Y - \theta, Y - \frac{1}{4}\right) &= Xf_2(X,Y) + \frac{27}{4}X^2 + 16Y^3(Y-1)\\
\Delta_3(X,Y) &:= \Delta\left(X, Y+2\right) &= X^4 + 2X^2Y^2 + 18X^2Y + 27X^2 + (Y+2)Y^3
\end{aligned}$$

Suppose the characteristic of $\Bbbk$ is not 3. At the origin $p = (0,0)$ the curves $\Delta_1, \Delta_2, \Delta_3$ each have a singularity of order 2 and unique tangent line $X = 0$. Also, with $\mathcal{O}_p$ the local ring at p, we have $\mathcal{O}_p/(\Delta_i, X) = \mathcal{O}_p/(Y^3, X)$ and so the intersection multiplicity of the curve Δ_i and line $X = 0$ at p is 3. Therefore, the origin is an ordinary cusp in the sense of [5, Exercise 3.22, p. 82] for each of the curves Δ_i. Each Δ_i is the image of Δ under an affine transformation, and examining these transformations shows Δ has cusps at $(\theta, -1/4)$, $(-\theta, -1/4)$

and $(0,2)$. Following [5, Corollary 1, Section 8.3] or [7, Lemma 3.24, p. 55], the genus g of a plane curve of degree d satisfies

$$0 \le g \le (1/2)(d-1)(d-2) - \sum_q (1/2)m_q(m_q-1),$$

where q ranges over the points on the curve and m_q is the order of the point q on the curve. Thus the genus g of Δ satisfies $0 \le g \le 3-(1+1+1) = 0$, which proves that the three singularities listed are the only singularities of Δ, these singularities are ordinary cusps and Δ is a rational curve.

If char $\Bbbk = 3$, then $\theta = 0$ and $2 = -(4^{-1}) \mod 3$ and so all three singularities listed above coincide with $(0,2)$, and $\Delta_1 = \Delta_2 = \Delta_3$ (when the term $(-4/3)\theta$ is replaced by 0). Working with Δ_3 and reducing mod 3 yields

$$\Delta_3(X,Y) = X^4 + 2X^2Y^2 + Y^4 + 2Y^3.$$

Thus $Y=0$ is the unique tangent line to the curve Δ_3 at the origin, and the origin has order 3 on this curve. This tangent line has multiplicity 4 at the origin. Consequently, the singularity $(0,2)$ is a higher-order cusp. The genus g of Δ satisfies $0 \le g \le 3-3 = 0$ so Δ is a rational curve having a unique singularity, which is a higher-order cusp. □

Removing the restriction to an algebraically closed field, we have

Corollary 6.7. *If $\mathbb{B}$ is a cubic bisector field, then the boundary of $\mathbb{B}$ is a rational quartic curve that has at most three singularities, each of which is a cusp.*

Proof. Extending to the algebraic closure $\overline{\Bbbk}$ of $\Bbbk$, Corollary 4.7 implies the extension $\overline{\mathbb{B}}$ of $\mathbb{B}$ to the plane $\overline{\Bbbk} \times \overline{\Bbbk}$ is a cubic bisector field over $\overline{\Bbbk}$. Applying Theorem 6.6 to $\overline{\mathbb{B}}$, the boundary of $\overline{\mathbb{B}}$ is a rational quartic curve that has at most three singularities, each of which is a cusp. The boundary of $\mathbb{B}$ is the set of $\Bbbk$-rational points on the boundary of $\overline{\mathbb{B}}$, and so the corollary follows. □

7. The boundary as an envelope

We show in this section that the moving bisectors from Definition 6.1 are precisely the lines tangent to the boundary of the bisector field. The results of the last section allow us to work with bisector fields whose boundaries have tractable explicit equations. Because of this, some of the results in this section, such as Lemma 7.1, can alternatively be verified by brute force using computer algebra software, but we have opted instead for a more conceptual approach that uses duality for plane curves, with the caveat that since the base field $\Bbbk$ may have positive characteristic, the use of duality requires a little more care.

For a polynomial $G(X_1,\ldots,X_n) \in \Bbbk[X_1,\ldots,X_n]$, we denote the partial derivative of G with respect to X_i as G_{X_i}.

If $G \in \Bbbk[X, Y, Z]$ is a nonzero homogeneous polynomial, the zero set defined by G is denoted

$$\mathbf{V}(G) = \{[x : y : z] \in \mathbb{P}^2(\Bbbk) : G(x, y, z) = 0\}.$$

Let $\mathbb{B}$ be a nonlinear bisector field, and let F be its reduced dual polynomial, as in Definition 4.2. Let $D(X, Y, Z)$ be the discriminant of

$$T\phi(T, U)X - U\phi(T, U)Y + \psi(T, U)Z$$

viewed as a polynomial with coefficients in $\Bbbk[X, Y, Z]$. Then $\Delta(X, Y) = D(X, Y, 1)$ and D is the homogenization of Δ. Thus $D(x, y, z) = 0$ if and only if

$$T\phi(T, U)x - U\phi(T, U)y + \psi(T, U)z$$

has a repeated root in $\mathbb{P}^1(\overline{\Bbbk})$. Define rational maps at the nonsingular points of F and D by

$$\begin{aligned} f &: \mathbf{V}(F) \to \mathbb{P}^2(\Bbbk) : [t : u : v] \mapsto [F_T(t, u, v) : -F_U(t, u, v) : F_V(t, u, v)] \\ d &: \mathbf{V}(D) \to \mathbb{P}^2(\Bbbk) : [x : y : z] \mapsto [D_X(x, y, z) : D_Y(x, y, z) : D_Z(x, y, z)]. \end{aligned}$$

Denote by $\mathbf{V}(F)^*$ the Zariski closure of the image of f in $\mathbb{P}^2(\Bbbk)$ and by $\mathbf{V}(D)^*$ the Zariski closure of the image of d in $\mathbb{P}^2(\Bbbk)$. The next lemma verifies that f and d define a duality.

Lemma 7.1. *If $\Bbbk$ is algebraically closed and $\mathbb{B}$ is a nonlinear bisector field, then*

$$\mathbf{V}(F)^* = \mathbf{V}(D),\ \mathbf{V}(D)^* = \mathbf{V}(F) \text{ and}$$
$$d \circ f = 1_{\mathbf{V}(F)}.$$

Proof. We first prove $\mathbf{V}(F)^* \subseteq \mathbf{V}(D)$. Let $[t : u : v] \in \mathbf{V}(F)$. Then $\psi(t, u) = v\phi(t, u)$ and so $\phi(t, u) \neq 0$ since ϕ and ψ do not share a common zero in $\mathbb{P}^1(\Bbbk)$. Let

$$x = F_T(t, u, v) \qquad y = -F_U(t, u, v) \qquad z = F_V(t, u, v).$$

We show $D(x, y, z) = 0$. To prove this, since $D(x, y, z)$ is the discriminant of

$$P(T, U){:=}T\phi(T, U)x - U\phi(T, U)y + \psi(T, U)z,$$

it suffices to show $[t : u]$ is a repeated root of $P(T, U)$. Let $n = \deg F$. By Euler's Formula, $nF = TF_T + UF_U + VF_V$. Using this and the fact that $v\phi(t, u) = \psi(t, u)$, we have

$$P(t, u) = \phi(t, u)(tx - uy + vz) = n\phi(t, u)F(t, u, v) = 0,$$

and so $[t : u]$ is a root of P, which implies $tx - uy = -vz$ since $\phi(t, u) \neq 0$.

To see that this root is repeated, in the case in which $u \neq 0$ it is enough to show $P_T(t, u) = 0$, and in the case that $t \neq 0$ that $P_U(t, u) = 0$. We show the former and omit the analogous proof for the latter. Observe

$$\begin{aligned}
P_T(t,u) &= xt\phi_T(t,u) + x\phi(t,u) - yu\phi_T(t,u) + z\psi_T(t,u) \\
&= \phi_T(t,u)(xt - yu) + x\phi(t,u) + z\psi_T(t,u) \\
&= \phi_T(t,u)(-zv) + x\phi(t,u) + z\psi_T(t,u) \\
&= z(\psi_T(t,u) - v\phi_T(t,u)) + x\phi(t,u) \\
&= -zF_T(t,u,v) + x\phi(t,u) \ = \ -zx + x\phi(t,u) \ = \ 0
\end{aligned}$$

(The last equality follows from the fact that $z = F_V(t,u,v) = \phi(t,u)$.) This proves that $[t:u]$ is a repeated root of $P(T,U)$, and so $D(x,y,z) = 0$. Thus $\mathbf{V}(F)^* \subseteq \mathbf{V}(D)$. Since the curve D is by Theorem 6.4 and Corollary 6.7 a rational, hence irreducible, curve (it is the projective closure of a rational curve) we obtain $\mathbf{V}(F)^* = \mathbf{V}(D)$.

To prove the remaining assertions that $\mathbf{V}(D)^* = \mathbf{V}(F)$ and $d \circ f = 1_{\mathbf{V}(F)}$, it suffices by [15, Propositions 1.2 and 1.5] to show F has finitely many flexes. We show in fact that F has at most three flexes. If $\mathbb{B}$ is quadratic, then $\mathbf{V}(F)$ is a parabola by Theorem 6.4 and hence has no flexes, so we assume $\mathbb{B}$ is cubic. Since $\Bbbk$ is algebraically closed by assumption, we may assume as in the proof of Theorem 6.6 that $\mathbb{B}$ is in standard form with center $(0, 1/2)$ and coefficient -1. By Lemma 5.2,

$$\Phi(T,U) = T^2 + U^2 \ \text{ and } \ \Psi(T,U) = 2T^2U.$$

Thus $F(T,U,V)$ is a factor of the dual polynomial of $\mathbb{B}$,

$$G(T,U,V) = 2T^2U - V(T^2 + U^2).$$

The Hessian of G is the matrix consisting of the second partials

$$H = \begin{bmatrix} 4U - 2V & 4T & -2T \\ 4T & -2V & -2U \\ -2T & -2U & 0 \end{bmatrix}.$$

The flex points and singularities on the curve G are the points in $\mathbb{P}^2(\Bbbk)$ on the intersection of the curve G and the curve

$$\det(H) = 16U(2T^2 - U^2) + 8V(T^2 + U^2);$$

see [7, Theorem 1.35, p. 14]. Eliminating V shows the points on this intersection are the zeroes of $2U(3T^2 - U^2) = 0$. Two such zeroes are $[0:0:1]$ and $[1:0:0]$. If char $\Bbbk = 3$, then these are the only points on the intersection. If char $\Bbbk \neq 3$ and θ is a root of $3Z^2 - 1$, then $[\theta : 1 : 1/2]$ and $[-\theta : 1 : 1/2]$ are the only other points on this intersection. The only singularity on the curve G is $[0:0:1]$, so all the other points listed here are flex points on the curve G. Thus G, and hence F also, has at most three flexes. Regardless of whether $\mathbb{B}$ is cubic or quadratic, we have established the curve F has finitely many flexes, and so the lemma is proved. $\square$

For the next theorem, which is illustrated by Figures 3(a)–(c), we recall the moving bisectors from Definition 6.1.

Theorem 7.2. *The lines tangent to the boundary of a nonlinear bisector field $\mathbb{B}$ are the moving bisectors in $\mathbb{B}$.*

Proof. We first prove the theorem in the case in which $\Bbbk$ is algebraically closed. Suppose $\ell : tX - uY + v$ is a line tangent to the boundary of $\mathbb{B}$ at the point (x, y). To prove that ℓ is a moving bisector, it suffices by Proposition 6.2 to show $F(t, u, v) = 0$. Now $0 = \Delta(x, y) = D(x, y, 1)$ and so $[x : y : 1] \in \mathbf{V}(D)$. Let f and d be as in Lemma 7.1.

If $[x : y : 1]$ is nonsingular, then $[t : u : v] = d([x : y : 1])$. By Lemma 7.1, $[t : u : v] \in \mathbf{V}(D)^* = \mathbf{V}(F)$, and so $F(t, u, v) = 0$ and ℓ is a moving bisector.

It remains to consider the case in which $[x : y : 1]$ is singular. Since in this case Δ is a singular curve, Theorems 6.4 and 6.6 imply $\mathbb{B}$ is a cubic bisector field and $[x : y : 1]$ is a cusp on D. Hence $\ell : tX - uY + v$ is the unique tangent line to D at $[x : y : 1]$. Since $\Bbbk$ is algebraically closed, we may assume as in the proof of Theorem 6.6 that $\Phi(T, U) = T^2 + U^2$ and $\Psi(T, U) = 2T^2U$. Since $\mathbb{B}$ is cubic, $\Phi = \phi$ and $\Psi = \psi$, and hence the reduced dual polynomial is

$$F(T, U, V) = 2T^2U - V(T^2 + U^2).$$

As shown in the proof of Theorem 6.6, the singularity $[x : y : 1]$ of D must be

$$[0 : 2 : 1], [\theta : -1/4 : 1] \text{ or } [-\theta : -1/4 : 1],$$

where $16\theta^2 = 27$. Using the ideas from the proof of Theorem 6.6, we will find the tangent lines at each of these points and show they are moving bisectors.

The argument in the proof of Theorem 6.6 shows the tangent line to Δ at $(0, 2)$ is the image of the line $X = 0$ under the transformation of the plane $(x, y) \mapsto (x, y + 2)$, and hence is again the line $X = 0$. This line has dual point $[1 : 0 : 0]$, and since $F(1, 0, 0) = 0$, this line is a moving bisector by Proposition 6.2. The tangent line to Δ at $(\theta, -1/4)$ is the image of the line $X = 0$ under the transformation

$$(x, y) \mapsto (x - (4/3)\theta y + \theta, y - (1/4)),$$

which is the line $(3/2)X + 2\theta Y - \theta = 0$. This line has dual point $[3/2 : -2\theta : -\theta]$, which is a zero of F, and hence this tangent line is also a moving bisector by Proposition 6.2. Similarly, the tangent line to Δ at $(-\theta, -1/4)$ is the image of $X = 0$ under the transformation

$$(x, y) \mapsto (x + (4/3)\theta y - \theta, y - (1/4)),$$

which is the line $(3/2)X - 2\theta Y + \theta = 0$. The dual point $[3/2 : 2\theta : \theta]$ of this line is on F, so this line is also a moving bisector. This proves that the lines tangent to the boundary of $\mathbb{B}$ are moving bisectors in $\mathbb{B}$.

Conversely, still under the assumption that $\Bbbk$ is algebraically closed, suppose the line $\ell : tX - uY + v$ is a moving bisector. We show ℓ is tangent to the boundary of $\mathbb{B}$. By Proposition 6.2, $F(t, u, v) = 0$, and so $[t : u : v]$ is a nonsingular point on the curve F since the only singularity of this curve is $[0 : 0 : 1]$. Thus the rational map f from Lemma 7.1 is defined at $[t : u : v]$. Let $[x : y : z] = f([t : u : v])$.

We will show $tX - uY + vZ = 0$ is tangent to the curve D at $[x : y : z]$.

Let $t'X - u'Y + v'Z = 0$ be the line tangent to the curve D at $[x : y : z]$.

If $[x : y : z]$ is nonsingular, then $d([x : y : z]) = [t' : u' : v']$, so Lemma 7.1 and the fact that $f([t : u : v]) = [x : y : z]$ implies

$$[t : u : v] = d(f([t : u : v])) = d([x : y : z]) = [t' : u' : v'].$$

Therefore, the line $tX - uY + v = 0$ is the same as the line $t'X - u'Y + v = 0$, and so in the case in which $[x : y : z]$ is nonsingular, the line $tX - uY + vZ = 0$ is tangent to the curve D at $[x : y : z]$.

Now suppose $[x : y : z] = f([t : u : v])$ is a singular point on the curve D. We show that $tX - uY + vZ = 0$ is tangent to $[x : y : z]$ also in this case. By Theorem 6.4 and Corollary 6.7, $\mathbb{B}$ is cubic since its boundary has a singular point, and so as in the proof of Theorem 6.6, the singularities of D are

$$[0 : 2 : 1], \quad [\theta : -1/4 : 1], \quad [-\theta : -1/4 : 1], \quad \text{where } 16\theta^2 = 27.$$

Thus $[x : y : z]$ is one of these points, and the bisector $tX - uY + v = 0$ goes through this point. As established earlier, there is a unique tangent line to D at each of these points, and as shown at the beginning of this proof, each tangent line to D is a bisector in $\mathbb{B}$. Thus to prove the claim that $tX - uY + v = 0$ is tangent to D at the singularity $[x : y : z]$, it suffices to show each of these three singularities has at most one bisector passing through it.

We prove this for the singularity $[0 : 2 : 1]$ and omit the very similar proofs for the singularities $[\pm\theta : -1/4 : 1]$. Since the bisector field $\mathbb{B}$ is cubic, all the bisectors in $\mathbb{B}$ are moving bisectors, and so by Proposition 6.2, the only bisectors passing through $(0, 2)$ are of the form $t'X - u'Y + v'$, where

$$-2u'\Phi(t', u') + \Psi(t', u') = 0.$$

Since $\Phi(T, U) = T^2 + U^2$ and $\Psi(T, U) = 2T^2U$, it follows that

$$2u'((t')^2 + (u')^2) = 2(t')^2u',$$

so that $u' = 0$. It cannot be that $t' = 0$, so since

$$0 = F(t', u', v') = 2(t')^2u' - v'((t')^2 + (u')^2) = -v'(t')^2,$$

we have $v' = 0$. This implies $t'X - u'Y + v' = 0$ is the line $X = 0$, and so there is only one bisector through the singularity $[0 : 2 : 1]$. This proves that if $f([t : u : v]) = [x : y : z]$ is a singular point on D, the line $tX - uY + vZ = 0$ is tangent to the curve D at $[x : y : z]$, which verifies the theorem if $\Bbbk$ is algebraically closed.

Now suppose $\Bbbk$ is not necessarily algebraically closed. Let $\Upsilon = \psi/\phi \in \Bbbk(T, U, V)$. By what we have established, when extended to the plane $\overline{\Bbbk} \times \overline{\Bbbk}$, the moving bisectors

$$tX - uY + \Upsilon(t, u) = 0,$$

where $[t : u] \in \mathbb{P}^1(\Bbbk)$ and $\phi(t, u) \neq 0$, are tangent to $\mathbf{V}(D)$. Given such a line, then with $(x, y) \in \overline{\Bbbk} \times \overline{\Bbbk}$ the point of tangency with the curve Δ, since $f([t : u : v]) = [x : y : 1]$ and the polynomials that define the map f are in $\Bbbk[T, U, V]$, it follows that (x, y) is a $\Bbbk$-rational point, and so the line $tX - uY + \Upsilon(t, u) = 0$ is tangent to the curve Δ in $\Bbbk^2$.

Conversely, suppose $tX - uY + v = 0$ is a line in $\overline{\Bbbk} \times \overline{\Bbbk}$ tangent to the boundary at a point $(x, y) \in \Bbbk^2$. By what we have shown in the first part of the proof, $tX - uY + v$ is a moving bisector in $\overline{\Bbbk} \times \overline{\Bbbk}$. Thus $\phi(t, u) \neq 0$ and $v = \Upsilon(t, u)$, and to complete the proof we need only verify that the coefficients t and u can be chosen from $\Bbbk$. If (x, y) is nonsingular, then since Δ has coefficients in $\Bbbk$ and $f([t : u : v]) = [x : y : 1]$, it follows that the coefficients of $tX - uY + v$ can be chosen from $\Bbbk$. If (x, y) is singular, then over $\overline{\Bbbk}$ this point is a cusp. After a translation, we can assume $(x, y) = (0, 0)$ and hence $v = 0$. Since this singularity is a cusp,

$$\Delta(X, Y) = (tX - uY)^e + g(X, Y),$$

where $e > 1$ and g is a polynomial with $\deg g > e$. Since the coefficients of Δ are in $\Bbbk$ and $t^a u^b \in \Bbbk$ for all $a, b \geq 0$ with $a + b = e$, it follows that

$$tu^{-1} = t^e (t^{e-1} u)^{-1} \in \Bbbk$$

if $u \neq 0$ and similarly $t^{-1}u \in \Bbbk$ if $t \neq 0$. Thus t and u can be chosen in $\Bbbk$, which completes the proof. $\square$

Since every bisector in a cubic bisector field is a moving bisector, Theorem 7.2 implies

Corollary 7.3. *A cubic bisector field is the set of lines tangent to its boundary.* $\square$

8. Classification of bisector fields

Using the results of the previous sections, we classify bisector fields up to affine equivalence. For linear and quadratic bisector fields, we obtain a classification that is independent of the nature of the field $\Bbbk$. The case of a cubic bisector field is more complicated and depends on the choice of field $\Bbbk$. For cubic bisector fields we obtain classifications when $\Bbbk$ is an algebraically closed field or a real closed field. We give some partial results for finite fields, and we leave as an open question the classification of cubic bisector fields over other fields.

Theorem 8.1. *A set of lines in $\Bbbk^2$ is a linear bisector field if and only if it is the union of two pencils of parallel lines and a pencil of non-parallel lines. Up to affine equivalence, there is only one linear bisector field over $\Bbbk$.*

Proof. If $\mathbb{B}$ is a linear bisector field, then $\mathbb{B}$ contains two pencils of parallel lines by Theorem 4.6 and these two pencils contain all the null bisectors. Since $\mathbb{B}$ is linear, $\phi(T, U)$ is a nonzero constant, say $c \in \Bbbk$, and $\psi(T, U) = aT - bU$ for some $a, b \in \Bbbk$ where possibly $a = b = 0$. Thus the reduced dual polynomial $F = \psi - V\phi$ defines a pencil $\mathcal{P}$ of lines

$$ctX - cuY + \psi(t, u) = t(cX - a) + u(cY - b),$$

where $[t : u] \in \mathbb{P}^1(\Bbbk)$. Each such line passes through (a, b) and is a moving bisector in $\mathbb{B}$ by Proposition 6.2. Also, any line $\ell : tX - uY + v = 0$ in $\mathbb{B}$ that is not in the two pencils of parallel lines satisfies $\psi(t, u) = v\phi(t, u)$ by Theorem 3.3 and Lemma 4.4, and so ℓ is in the pencil $\mathcal{P}$. Thus $\mathbb{B}$ is the union of two pencils of parallel lines and another pencil that consists of the moving bisectors.

In light of Theorem 4.6, to prove the converse it suffices to show that a union $\mathbb{B}$ of two pencils of parallel lines and a third pencil whose midpoint is a point p in $\Bbbk^2$ is a bisector field. Let ℓ_1 be the line in the first pencil of parallel lines that passes through p, and let ℓ_2 be the line through p in the second pencil, so that ℓ_1 and ℓ_2 meet at p.

Let A, A' be a pair of distinct parallel lines that share ℓ_1 as a midline, let B, B' be a pair of distinct parallel lines that share ℓ_2 as a midline, and let $Q = ABA'B'$. It is straightforward to see that each line in $\mathbb{B}$ bisects the parallelogram Q and that any line that bisects Q occurs in one of the three pencils that comprise $\mathbb{B}$. Therefore, $\mathbb{B} = \mathrm{Bis}(Q)$, and so $\mathbb{B}$ is a bisector field. Since any parallelogram can be transformed into any other parallelogram in the plane, it follows that any two linear bisector fields are affinely equivalent. □

See Figure 3(d) for an illustration of Theorem 8.1 in the case where $\Bbbk$ is the field of real numbers. The next theorem shows that, as with linear bisector fields, the classification of quadratic bisector fields is independent of the nature of the field $\Bbbk$.

Theorem 8.2. *A set of lines is a quadratic bisector field if and only if it is affinely equivalent to the union of the set of lines tangent to the curve $Y - X^2 = 0$ and the pencil of lines parallel to the line $Y = 0$. Up to affine equivalence there is only one quadratic bisector field.*

Proof. Let $\mathbb{B}$ be a quadratic bisector field. By Theorem 6.4, $\mathbb{B}$ is affinely equivalent to the quadratic bisector field whose boundary is the parabola $Y = X^2$ and is such that the pencil of parallel lines of this bisector field consists of lines parallel to $Y = 0$. By Theorem 7.2, this bisector field consists of lines that are tangent to the curve $Y - X^2 = 0$ or in the pencil of lines parallel to the line $Y = 0$. The converse follows from Theorems 6.4 and 7.2. □

Figure 3(c) shows the only quadratic bisector field up to affine equivalence over the reals. The classification of cubic bisector fields is more complicated than that of linear and quadratic bisector fields, but at least if $\Bbbk$ is algebraically closed, the classification is definitive and essentially already done.

Theorem 8.3. *Suppose $\Bbbk$ is an algebraically closed field. A set of lines in $\Bbbk^2$ is a cubic bisector field if and only if it is affinely equivalent to the set of lines tangent to the curve*

$$X^4 + 2X^2Y^2 + Y^4 + 10X^2Y - 6Y^3 - X^2 + 12Y^2 - 8Y = 0. \tag{4}$$

Proof. Apply Theorem 6.6 and Corollary 7.3. □

We show in the next theorem that over a real closed field, the cubic case splits into two cases, the first being the well-known Steiner deltoid. The envelopes in these two cases, as shown in Figures 3(a) and (b), are fundamental singular curves and occur for example as cross-sections of the bifurcation sets for a hyperbolic umbilic and an elliptical umbilic, respectively, two of Thom's seven elementary catastrophes; see [16, Chapter 4].

Theorem 8.4. *Suppose $\Bbbk$ is a real closed field. If $\mathbb{B}$ is a cubic bisector field, then $\mathbb{B}$ is affinely equivalent to either the set of lines tangent to the deltoid in (4) or the cuspidal curve*

$$X^4 - 2X^2Y^2 + Y^4 - 10X^2Y - 6Y^3 + X^2 + 12Y^2 - 8Y = 0.$$

Proof. Since $\Bbbk$ is real closed, every bisector field $\mathbb{B}$ is well centered, and, as in the proof of Theorem 6.6, every cubic bisector field is affinely equivalent to a bisector field in standard form with center $(0, 1/2)$. By Theorem 5.10, two cubic bisector fields $\mathbb{B}_1$ and $\mathbb{B}_2$ in standard form with coefficients μ_1 and μ_2, respectively, are affinely equivalent if and only if μ_1 and μ_2 have the same sign. Thus every bisector field is affinely equivalent either to the cubic bisector field in standard form with coefficient 1 and center $(0, 1/2)$ or to the cubic bisector field in standard form with coefficient -1 and center $(0, 1/2)$. By Lemma 6.5, the boundary in the former case is the cardioid in the theorem; in the latter case it is the deltoid. The theorem follows now from Corollary 7.3. □

We are only able to give partial results for other types of fields.

Example 8.5. If $\Bbbk$ is *the field of rational numbers, then k has infinitely many bisector fields up to affine equivalence.* This is because Theorem 5.6 implies that if for each prime integer p, $\mathbb{B}_p$ is a bisector field in standard form with coefficient p, then for any two distinct primes p_1 and p_2, the bisector fields $\mathbb{B}_{p_1}$ and $\mathbb{B}_{p_2}$ are not affinely equivalent.

The case of finite fields is more complicated, and we give a few observations about such fields in the following examples.

Example 8.6. If $\Bbbk$ is a finite field (as always, of odd characteristic), then up to affine equivalence there are exactly two well-centered cubic bisector fields. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be well-centered bisector fields over $\Bbbk$. Without loss of generality, $\mathbb{B}_1$ and $\mathbb{B}_2$ are in standard form. Let μ_1 and μ_2 denote the coefficients of $\mathbb{B}_1$ and $\mathbb{B}_2$, respectively. Since $\mathbb{B}_1$ and $\mathbb{B}_2$ are well centered, Theorem 5.10 implies $\mathbb{B}_1$ and $\mathbb{B}_2$ are affinely equivalent if and only if $\mu_1\mu_2$ is a square in $\Bbbk$. Since $\Bbbk$ is a finite field, this is the case if and only if μ_1 and μ_2 are both squares or neither is a square. Thus there are two equivalence classes of well-centered cubic bisector fields.

Example 8.7. Suppose $\Bbbk$ is a finite field, and let $\mathbb{B}_1$ and $\mathbb{B}_2$ be bisector fields in standard form with coefficients μ_1 and μ_2, respectively. If either (a) 3 *is a square in $\Bbbk$ and μ_1 and μ_2 are not squares or* (b) 3 *is not a square and μ_1 and μ_2 are squares, then $\mathbb{B}_1$ and $\mathbb{B}_2$ are well centered and affinely equivalent.* Let (h_i, k_i) be the center of $\mathbb{B}_i$. Lemma 5.8 implies $\mathbb{B}_i$ is well centered if and only if the homogeneous polynomial

$$f_i(T, U) = h_iT^3 + 3k_iT^2U + 3h_i\mu_iTU^2 + k_i\mu_iU^3$$

has a zero in $\mathbb{P}^1(\Bbbk)$; if and only if f_i is reducible over $\Bbbk$. The polynomial $f_i(T, U)$ has a zero in $\mathbb{P}^1(\Bbbk)$ if and only if the cubic $f_i(T, 1)$ or the cubic $f_i(1, U)$ has a zero in $\Bbbk$; if and only if $f_i(T, 1)$ or $f_i(1, U)$ is reducible. We claim first $f_i(T, 1)$ is reducible over $\Bbbk$, and to show this it suffices by a theorem of Dickson [4, Theorem 1] to prove the discriminant of f is not a square in $\Bbbk$. The discriminant of $f_i(T, 1)$ is $-108\mu_i(h^2\mu_i - k^2)^2$ and hence is a square if and only if $3\mu_i$ is a square. Since $\Bbbk$ is a finite field, conditions (a) or (b) guarantee $3\mu_i$ is not a square for $i = 1, 2$. Thus $\mathbb{B}_1$ and $\mathbb{B}_2$ are well centered. Since in case (a) or (b), $\mu_1\mu_2$ is a square, Theorem 5.10 implies B_1 and $\mathbb{B}_2$ are affinely equivalent.

The linear and quadratic bisector fields are classified in Theorems 8.1 and 8.2. Theorems 8.3 and 8.4 classify the cubic bisector fields over algebraically closed fields and real closed fields, and the preceding examples give partial information for a potential classification over finite fields. This raises the following question.

Question 8.8. Given a field $\Bbbk$ of characteristic $\neq 2$, what are the affine equivalence classes of the cubic bisector fields over $\Bbbk$?

Up to affine equivalence we can restrict to cubic bisector fields in standard form. These bisector fields are entirely determined by their center (h, k) and coefficient μ. Question 8.8 therefore can be interpreted as seeking an equivalence relation on triples (h, k, μ) that encodes the affine equivalence of the corresponding bisector fields. In light of Theorem 7.3, another way to put this question is:

Question 8.9. Given a field $\Bbbk$ of characteristic $\neq 2$, what are the affine equivalence classes of the system of curves Δ in Lemma 6.5, where $h, k, \mu \in \Bbbk$ and $\mu \neq 0$.

It is straightforward to see that every cubic bisector field is affinely equivalent to a bisector field in standard form with center $(0, 1/2)$ or $(1, 1)$, and so the focus in this question can be placed on μ.

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