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RESEARCH PAPER

WELL-POSEDNESS OF THE FRACTIONAL ZENER WAVE EQUATION FOR HETEROGENEOUS VISCOELASTIC MATERIALS

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Abstract

Zener's model for viscoelastic solids replaces Hooke's law $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}$, relating the stress tensor $\boldsymbol{\sigma}$ to the strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$, where $\mathbf{u}$ is the displacement vector, $\mu > 0$ is the shear modulus, and $\lambda \geq 0$ is the first Lamé coefficient, with the constitutive law $(1+\tau D_t)\boldsymbol{\sigma} = (1+\rho D_t)[2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}]$, where $\tau > 0$ is the characteristic relaxation time and $\rho \geq \tau$ is the characteristic retardation time. It is the simplest model that predicts creep/recovery and stress relaxation phenomena. We explore the well-posedness of the fractional version of the model, where the first-order time-derivative D_t in the constitutive law is replaced by the Caputo time-derivative D_t^α, with $\alpha \in (0,1)$, μ, λ belong to $\mathrm{L}^\infty(\Omega)$, μ is bounded below by a positive constant and λ is nonnegative. We show that, when coupled with the equation of motion $\varrho\ddot{\mathbf{u}} = \mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}$, considered in a bounded open Lipschitz domain Ω in $\mathbb{R}^3$ and over a time interval $(0,T]$, where $\varrho \in \mathrm{L}^\infty(\Omega)$ is the density of the material, assumed to be bounded below by a positive constant, and $\mathbf{f}$ is a specified load vector, the resulting model is well-posed in the sense that the associated initial-boundary-value problem, with initial conditions $\mathbf{u}(0,\mathbf{x}) = \mathbf{g}(\mathbf{x})$, $\dot{\mathbf{u}}(0,\mathbf{x}) = \mathbf{h}(\mathbf{x})$, $\boldsymbol{\sigma}(0,\mathbf{x}) = \mathbf{S}(\mathbf{x})$, for $\mathbf{x} \in \Omega$, and a homogeneous Dirichlet boundary condition, possesses a unique weak solution for any choice of $\mathbf{g} \in [\mathrm{H}_0^1(\Omega)]^3$, $\mathbf{h} \in [\mathrm{L}^2(\Omega)]^3$, and $\mathbf{S} = \mathbf{S}^{\mathrm{T}} \in [\mathrm{L}^2(\Omega)]^{3\times 3}$, and any load vector $\mathbf{f} \in \mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^3)$, and that this unique weak solution depends continuously on the initial data and the load vector.

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1. **Statement of the model**

Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded, open, simply-connected Lipschitz domain, with boundary $\partial\Omega$, occupied by a viscoelastic material, and let $T > 0$. Consider the equation of motion

$$\varrho \ddot{\mathbf{u}} = \operatorname{Div} \boldsymbol{\sigma} + \mathbf{f} \qquad \text{in } (0, T] \times \Omega, \tag{1.1}$$

with $\varrho > 0$ signifying the density of the material, $\mathbf{u}$ the displacement vector, $\boldsymbol{\sigma}$ the stress tensor, and $\mathbf{f}$ the load vector, with the material being considered subject to the initial conditions

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}), \quad \dot{\mathbf{u}}(0, \mathbf{x}) = \mathbf{h}(\mathbf{x}), \quad \boldsymbol{\sigma}(0, \mathbf{x}) = \mathbf{S}(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \tag{1.2}$$

and a suitable boundary condition, which for the sake of simplicity of the exposition we shall assume to be the homogeneous Dirichlet boundary condition

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{0} \qquad \text{for all } (t, \mathbf{x}) \in (0, T] \times \partial\Omega. \tag{1.3}$$

The discussion below trivially extends to the case of a mixed homogeneous Dirichlet/nonhomogeneous Neumann boundary condition provided that the Dirichlet part of $\partial\Omega$ has positive two-dimensional surface measure (cf. the concluding remarks at the end of the paper for further comments in this direction). In the case of a classical linear (Hookean) elastic body the stress tensor $\boldsymbol{\sigma}$ is related to the strain tensor (symmetric displacement gradient)

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}})$$

through *Hooke's law* $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}$, where $\mu > 0$ is the *shear modulus* and $\lambda \geq 0$ is the *first Lamé coefficient*. In this case the initial value $\mathbf{S} = \boldsymbol{\sigma}|_{t=0}$ of $\boldsymbol{\sigma}$ is automatically equal to $2\mu\boldsymbol{\varepsilon}(\mathbf{g}) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}$, by Hooke's law, and need not (or, more precisely, should not) be specified independently, as otherwise the resulting initial-boundary-value problem will be over-determined and will have no solution in general. However for Zener's model under consideration here the situation is different: the constitutive law relating the stress tensor $\boldsymbol{\sigma}$ to the strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ involves the time-derivative of order $\alpha \in (0, 1]$ of $\boldsymbol{\sigma}$:

$$(1 + \tau^\alpha D_t^\alpha)\boldsymbol{\sigma} = (1 + \rho^\alpha D_t^\alpha)[2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}],$$

with $\tau > 0$ signifying the *characteristic relaxation time* and $\rho \geq \tau$ is the *characteristic retardation time*, — which then necessitates the specification of an initial datum $\mathbf{S}$ for $\boldsymbol{\sigma}$ (cf. the final paragraph of the paper concerning the case of $\tau = 0$, which we shall exclude from the analysis pursued here as it is already covered by existing results in the literature).

In the case of $\alpha = 1$ the model was proposed by Zener [19] (with $\lambda = 0$). The fractional version of Zener's model was introduced (in one space dimension and, again, with $\lambda = 0$) by Caputo and Mainardi (cf. [6], and eq. (13) in [10]), and has the form (with σ, ϵ and u below now being scalar-valued functions in this, simplified, univariate model)

$$(1 + \tau^\alpha D_t^\alpha)\sigma = E(1 + \rho^\alpha D_t^\alpha)\varepsilon(u), \quad \text{with } \sigma(0,\cdot) = E\left(\tfrac{\rho}{\tau}\right)^\alpha \varepsilon(u(0,\cdot)),$$

where, following Bagley and Torvik [4], $E > 0$ is referred to as the *rubbery modulus*, $E(\rho/\tau)^\alpha$ is called the *glassy modulus*, and $\alpha \in (0,1)$ is the *fractional order of evolution*. As has been noted by Freed and Diethelm [10], this model allows for a finite discontinuity in the stress-strain response at time zero (cf. Remark 3.1 below for further comments on this observation in the context of our well-posedness analysis). Bagley and Torvik [4] have demonstrated that the fractional orders of evolution in stress and strain must be the same, as originally proposed in the work of Caputo and Mainardi [6], in order that a material model of fractional order comply with the second law of thermodynamics; Bagley and Calico [3] have also shown that the differential orders need to be the same for the stress and the strain in order to ensure that sound waves in the material propagate at finite speed. For further motivation from the point of view of continuum thermodynamics for considering fractional-order constitutive laws of this kind we refer to [2], [3], [4], and [14], for example.

As the actual value of the characteristic retardation time ρ $(\geq \tau > 0)$ is of no relevance in the discussion that follows, for the sake of simplicity of the exposition we have fixed $\rho = 1$, resulting in the constitutive law

$$(1 + \tau^\alpha D_t^\alpha)\boldsymbol{\sigma} = (1 + D_t^\alpha)[2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}], \tag{1.4}$$

with $\tau \in (0,1]$ and $\alpha \in (0,1)$. As will be seen in what follows, the relation $(1 =)\rho \geq \tau > 0$ guarantees the well-posedness of the resulting model, in agreement with the discussion in [4], where (1.4) has been considered in the case of constant μ and λ, (particularly eqs. (14) and (22)–(25) therein with $\alpha = \beta$) concerning the relevant thermodynamical conditions to ensure nonnegativity of the internal work and guarantee a nonnegative rate of energy dissipation. The constitutive law (1.4) generalizes the one proposed by Caputo and Mainardi in [6] in that we admit $\lambda \geq 0$, motivated by the fact that formally setting $\alpha = 0$ in (1.4) reduces it to Hooke's constitutive law. As a matter of fact, we shall assume, more generally, that

$$
\begin{aligned}
&\varrho \in \mathrm{L}^\infty(\Omega), && \text{and there exists a positive constant } \varrho_0 \text{ such that} \\
& && \varrho(\mathbf{x}) \geq \varrho_0 \text{ a.e. in } \Omega, \\
&\mu \in \mathrm{L}^\infty(\Omega), && \text{and there exists a positive constant } \mu_0 \text{ such that} \\
& && \mu(\mathbf{x}) \geq \mu_0 \text{ a.e. in } \Omega, \\
& && \lambda \in \mathrm{L}^\infty(\Omega), \qquad \text{and } \lambda(\mathbf{x}) \geq 0 \text{ a.e. in } \Omega,
\end{aligned}
\tag{1.5}
$$

so as to admit spatially heterogeneous viscoelastic materials. With straightforward modifications all of our results extend to the case of Hooke's model corresponding to $\alpha = 0$ and the classical Zener model corresponding to $\alpha = 1$; we shall therefore confine ourselves to the, technically more involved, fractional-order setting, when $\alpha \in (0, 1)$.

Zener's constitutive law aims to overcome some of the shortcomings of the Maxwell and Kelvin–Voigt models: the Maxwell model does not describe creep or recovery, and the Kelvin–Voigt model does not describe stress relaxation. Zener's constitutive law is the simplest model that predicts both phenomena. Our aim here is to explore the well-posedness of the model, focusing in particular on its refinement, where the first time-derivative D_t featuring in the constitutive law is replaced by a fractional-order time-derivative D_t^α, with $\alpha \in (0, 1)$. We emphasize that the equation of motion (1.1), expressing balance of the linear momentum in terms of the Cauchy stress, remains unchanged: it is only the constitutive law relating the stress tensor to the strain tensor, which encodes the specific properties of the material, that is altered here by admitting the fractional range $\alpha \in (0, 1)$.

The fractional derivative D_t^α of order $\alpha \in (0, 1)$ appearing in (1.4) is in the sense of Caputo. It is understood to be acting on 3-component vector-functions and 3×3-matrix-valued functions componentwise. In particular, for a scalar-valued function $f \in \mathrm{AC}([0, T])$,

$$(D_t^\alpha f)(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{f}(s)}{(t-s)^\alpha} \, \mathrm{d}s, \qquad t \in (0, T].$$

The partial differential equation (1.1) coupled with the constitutive law (1.4) is referred to as the *fractional Zener wave equation*. Wave propagation in viscoelastic media governed by the fractional Zener constitutive law in one space dimension was first considered by Caputo and Mainardi [6]. The existence and uniqueness of the fundamental solution of a generalized Cauchy problem for the fractional Zener wave equation were proved in [11], and an explicit expression for the solution was also given (cf. Theorem 4.2 in [11]). The existence and uniqueness of solutions for a generalization of the fractional Zener wave equation proposed by Enelund and Josefson

[8], in the case of mixed homogenous Dirichlet/nonhomogeneous Neumann boundary conditions on bounded polytopal domains in two and three space dimensions, were proved by Saedpanah in [15]; and, under suitable restrictions on the domain Ω and the data, weak solutions of the model were shown in [15] to possess additional regularity. In an earlier work, Larsson and Saedpanah [12] showed the well-posedness of the homogeneous Dirichlet problem for this model using techniques from linear semigroup theory. The weak formulation of the evolution equation (2.5) that we study here differs from the one considered in [15]; indeed, equation $(2.7)_1$ in [15] was arrived at by using Laplace transform techniques on the constitutive law to obtain an explicit expression for the stress tensor in terms of the strain tensor, which was then substituted into the equation of motion to eliminate the stress tensor; whereas, as we shall explain below, we Laplace transform the equation of motion as well as the constitutive law and we then eliminate the Laplace transform of the stress tensor from the transformed equation of motion. Furthermore, in both [12] and [15] the fractional derivative featuring in the constitutive law was the left Riemann–Liouville derivative rather than the Caputo derivative considered here, and the initial response for the stress tensor was assumed to follow Hooke's law.

The aim of the present work is to explore the question of existence and uniqueness of weak solutions to the initial-boundary-value problem (1.1)–(1.4) without the additional assumption that the initial response for the stress follows Hooke's law. In the absence of this extra assumption on the initial stress the analysis of the model is considerably more complicated; nevertheless, we are able to show (cf. Theorem 4.1 below) that the model (1.1)–(1.4) admits a unique weak solution for any $\mathbf{f} \in \mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^3)$, and arbitrary initial data $\mathbf{g} \in [\mathrm{H}^1_0(\Omega)]^3$, $\mathbf{h} \in [\mathrm{L}^2(\Omega)]^3$, and $\mathbf{S} = \mathbf{S}^{\mathrm{T}} \in [\mathrm{L}^2(\Omega)]^{3\times 3}$, without any additional restrictions on the choice of $\mathbf{S}$.

To this end, our first objective is to transform the fractional Zener model (1.1)–(1.4) to a form in which it is amenable to mathematical analysis. We shall therefore Laplace-transform the equation of motion (1.1) (where it will be understood that the source term $\mathbf{f}$ is extended by $\mathbf{0}$ from $(0,T] \times \Omega$ to $(0,\infty) \times \Omega$), as well as the constitutive law (1.4) with respect to the temporal variable t (again with the understanding that, for the moment, $t \in (0,\infty)$ rather than $t \in (0,T]$ with $T < \infty$). This will enable us to eliminate the stress tensor $\boldsymbol{\sigma}$ from the equation of motion in terms of the strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$, resulting in a second-order nonlocal evolution equation (cf. (2.5) below), which will then be the focus of our subsequent analysis. We shall concentrate on the proof of existence and uniqueness of weak solutions, and the continuous dependence of weak solutions on the data. Specifically, we shall show that the constitutive law (1.4), when coupled with (1.1)–(1.3), gives rise to a well-posed mathematical model: by using a compactness argument we shall prove the existence of a weak solution to the model and will prove that weak solutions thus constructed satisfy an energy inequality, which bounds appropriate norms of the solution in terms of norms of the initial data and the source term; we shall also show that weak solutions are unique.

2. **Zener's model as a fractional evolution equation**

The aim of this section is to merge the equation of motion (1.1) and the constitutive law (1.4) into a single evolution equation, which we shall then subject to mathematical analysis. We proceed by eliminating the stress tensor $\boldsymbol{\sigma}$ from (1.1) by Laplace transforming both (1.1) and the constitutive law (1.4).

The Laplace transform with respect to the variable t of a function f defined on $(0,\infty)$ such that $\int_0^\infty |f(t)|\,\mathrm{e}^{-at}\,\mathrm{d}t < \infty$ for some $a \in \mathbb{R}$, is defined by

$$\mathcal{L}(f)(p) = \tilde{f}(p) := \int_0^\infty f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t, \qquad \text{for} \quad p \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}\, p \geq a.$$

Then, for any function $f \in \mathrm{C}([0,\infty)) \cap \mathrm{C}^1((0,\infty))$ such that one has $\int_0^\infty (|\dot{f}(t)|+|f(t)|)\,\mathrm{e}^{-at}\,\mathrm{d}t < \infty$ for some $a \in \mathbb{R}$, straightforward calculations yield that

$$\mathcal{L}(\dot{f})(p) = p\tilde{f}(p) - f(0), \qquad \mathrm{Re}\, p \geq a,$$

where the symbol $\cdot$ over a t-dependent function denotes its derivative with respect to t, and, similarly, $\cdot\cdot$ over a t-dependent function denotes its second derivative with respect to t. As

$$\mathcal{L}((\cdot)^{-\alpha})(p) = \Gamma(1-\alpha)p^{\alpha-1}, \qquad \mathrm{Re}\, p > 0, \quad \alpha \in (0,1),$$

by noting that

$$D_t^\alpha f = \frac{1}{\Gamma(1-\alpha)}\left[\dot{f} *_t (\cdot)^{-\alpha}\right],$$

where the convolution $*_t$ is defined by $(f *_t g)(t) := \int_0^t f(s)g(t-s)\,\mathrm{d}s$, we have that

$$\begin{aligned}\mathcal{L}(D_t^\alpha f)(p) &= \frac{1}{\Gamma(1-\alpha)}\mathcal{L}\left[\dot{f} *_t (\cdot)^{-\alpha}\right](p) = \frac{1}{\Gamma(1-\alpha)}\mathcal{L}(\dot{f})(p)\,\mathcal{L}((\cdot)^{-\alpha})(p)\\ &= p^\alpha \tilde{f}(p) - p^{\alpha-1}f(0), \qquad \mathrm{Re}\, p \geq a, \quad \alpha \in (0,1).\end{aligned}$$

Consider the Mittag–Leffler function

$$E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C}, \quad \alpha > 0, \quad \beta > 0.$$

Letting

$$e_\alpha(t,\gamma) := E_{\alpha,1}(-\gamma t^\alpha), \qquad t\in[0,\infty), \quad \gamma>0,$$

one has that

$$\mathcal{L}(e_\alpha(\cdot,\gamma))(p) = \frac{p^{\alpha-1}}{p^\alpha+\gamma} \qquad \text{for } \operatorname{Re} p > \gamma^{\frac{1}{\alpha}}. \tag{2.1}$$

Henceforth, for the sake of simplicity, we shall write $e_{\alpha,\gamma}(t)$ instead of $e_\alpha(t,\gamma)$, and restrict ourself to the range $\alpha\in(0,1)$ of relevance to us in the present context. As $e_{\alpha,\gamma}(0)=1$, it follows that

$$\mathcal{L}(\dot e_{\alpha,\gamma})(p) = p\,\tilde e_{\alpha,\gamma}(p) - 1 = \frac{p^\alpha}{p^\alpha+\gamma} - \mathcal{L}(\delta), \qquad \operatorname{Re} p > \gamma^{\frac{1}{\alpha}},$$

where δ is the Dirac distribution concentrated at $t=0$. Thus, now with the Laplace transform acting in the sense of tempered distributions

$$\mathcal{L}(\dot e_{\alpha,\gamma}+\delta)(p) = \frac{p^\alpha}{p^\alpha+\gamma}, \qquad \operatorname{Re} p > \gamma^{\frac{1}{\alpha}}.$$

We note in passing that, for a tempered distribution $f\in\mathcal{S}'$, with $\operatorname{supp}(f)\subset[0,\infty)$, one defines $\mathcal{L}(f)(p) = \tilde f(p) := \langle f, \eta\,\mathrm{e}^{-p\cdot}\rangle$, for Re $p>0$, where $\eta\in C^\infty(\mathbb{R})$ is such that $\eta(t)\equiv 0$ for $t\le -2$ and $\eta(t)\equiv 1$ for $t\ge -1$.

As a consequence of this identity we have that

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1+\tau^\alpha p^\alpha}{1+p^\alpha}\right) &= \mathcal{L}^{-1}\left(1+(\tau^\alpha-1)\frac{p^\alpha}{1+p^\alpha}\right)\\ &= \delta + (\tau^\alpha-1)\mathcal{L}^{-1}\left(\frac{p^\alpha}{p^\alpha+1}\right) = \delta + (\tau^\alpha-1)(\dot e_{\alpha,1}+\delta).\end{aligned} \tag{2.2}$$

Following these preparatory considerations, we Laplace-transform the constitutive law (1.4), which yields

$$\begin{aligned}\tilde{\boldsymbol\sigma} + \tau^\alpha\mathcal{L}(D_t^\alpha\boldsymbol\sigma) &= 2\mu\boldsymbol\varepsilon(\tilde{\mathbf{u}}) + \lambda\operatorname{tr}(\boldsymbol\varepsilon(\tilde{\mathbf{u}}))\mathbf{I} + 2\mu\mathcal{L}(D_t^\alpha\boldsymbol\varepsilon(\mathbf{u}))\\ &\quad + \lambda\mathcal{L}(D_t^\alpha(\operatorname{tr}(\boldsymbol\varepsilon(\mathbf{u}))\mathbf{I}), \qquad \tau\in(0,1].\end{aligned}$$

Hence,

$$\begin{aligned}&\tilde{\boldsymbol\sigma}(p) + \tau^\alpha(p^\alpha\tilde{\boldsymbol\sigma}(p) - p^{\alpha-1}\mathbf{S})\\ &\quad = 2\mu\boldsymbol\varepsilon(\tilde{\mathbf{u}}) + \lambda\operatorname{tr}(\boldsymbol\varepsilon(\tilde{\mathbf{u}}))\mathbf{I} + 2\mu(p^\alpha\boldsymbol\varepsilon(\tilde{\mathbf{u}}) - p^{\alpha-1}\boldsymbol\varepsilon(\mathbf{g}))\\ &\qquad + \lambda(p^\alpha\operatorname{tr}(\boldsymbol\varepsilon(\tilde{\mathbf{u}}))\mathbf{I} - p^{\alpha-1}\operatorname{tr}(\boldsymbol\varepsilon(\mathbf{g}))\mathbf{I}).\end{aligned}$$

Equivalently,

$$\begin{aligned}(1+\tau^\alpha p^\alpha)\tilde{\boldsymbol\sigma}(p) &= (1+p^\alpha)(2\mu\boldsymbol\varepsilon(\tilde{\mathbf{u}}) + \lambda\operatorname{tr}(\boldsymbol\varepsilon(\tilde{\mathbf{u}}))\mathbf{I})\\ &\quad + p^{\alpha-1}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol\varepsilon(\mathbf{g}) - \lambda\operatorname{tr}(\boldsymbol\varepsilon(\mathbf{g}))\mathbf{I}),\end{aligned}$$

and therefore

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}(p) &= \frac{1+p^\alpha}{1+\tau^\alpha p^\alpha}(2\mu\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}))\mathbf{I}) \\ &+ \frac{p^{\alpha-1}}{1+\tau^\alpha p^\alpha}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}).\end{aligned} \tag{2.3}$$

Consequently, and by Laplace-transforming the equation of motion (1.1), we deduce that

$$\begin{aligned}\varrho\mathcal{L}(\ddot{\mathbf{u}}) &= \frac{1+p^\alpha}{1+\tau^\alpha p^\alpha}\,\mathrm{Div}(2\mu\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}))\mathbf{I}) \\ &+ \frac{p^{\alpha-1}}{1+\tau^\alpha p^\alpha}\,\mathrm{Div}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) + \tilde{\mathbf{f}},\end{aligned}$$

and, upon multiplying this equality by $\frac{1+\tau^\alpha p^\alpha}{1+p^\alpha}$, we have that

$$\begin{aligned}\varrho\frac{1+\tau^\alpha p^\alpha}{1+p^\alpha}\mathcal{L}(\ddot{\mathbf{u}}) &= \mathrm{Div}(2\mu\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}))\mathbf{I}) \\ &+ \frac{p^{\alpha-1}}{1+p^\alpha}\,\mathrm{Div}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) + \frac{1+\tau^\alpha p^\alpha}{1+p^\alpha}\tilde{\mathbf{f}}.\end{aligned}$$

Hence, by inverse-Laplace-transforming this equality and applying the convolution theorem for the Laplace transform, we obtain

$$\begin{aligned}&\varrho\mathcal{L}^{-1}\left(\frac{1+\tau^\alpha p^\alpha}{1+p^\alpha}\right) *_t \ddot{\mathbf{u}} = \mathrm{Div}(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}) \\ &+ \mathcal{L}^{-1}\left(\frac{p^{\alpha-1}}{1+p^\alpha}\right)\mathrm{Div}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) + \mathcal{L}^{-1}\left(\frac{1+\tau^\alpha p^\alpha}{1+p^\alpha}\right) *_t \mathbf{f}.\end{aligned}$$

Using (2.2) and (2.1) we then deduce that

$$\begin{aligned}\varrho(\delta + (\tau^\alpha - 1)\,(\dot{e}_{\alpha,1} + \delta)) *_t \ddot{\mathbf{u}} &= \mathrm{Div}(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}) \\ &+ e_{\alpha,1}\,\mathrm{Div}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) \\ &+ (\delta + (\tau^\alpha - 1)\,(\dot{e}_{\alpha,1} + \delta)) *_t \mathbf{f},\end{aligned}$$

and therefore

$$\begin{aligned}\varrho\tau^\alpha\ddot{\mathbf{u}} + \varrho\,(\tau^\alpha - 1)\dot{e}_{\alpha,1} *_t \ddot{\mathbf{u}} &= \mathrm{Div}(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}) \\ &+ e_{\alpha,1}\,\mathrm{Div}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) \\ &+ \tau^\alpha\mathbf{f} + (\tau^\alpha - 1)\dot{e}_{\alpha,1} *_t \mathbf{f}.\end{aligned}$$

We now focus on the second term on the left-hand side of this equality. By noting that

$$(f *_t \dot{g})(t) = \frac{\mathrm{d}}{\mathrm{d}t}(f *_t g)(t) - f(t)g(0)$$

we deduce (by suppressing the $\mathbf{x}$-dependence of $\mathbf{u}$ for the sake of notational simplicity) that

$$(\dot{e}_{\alpha,1} *_t \ddot{\mathbf{u}})(t) = \frac{\partial}{\partial t}(\dot{e}_{\alpha,1} *_t \dot{\mathbf{u}})(t) - \dot{e}_{\alpha,1}(t)\dot{\mathbf{u}}(0) = \frac{\partial}{\partial t}(\dot{e}_{\alpha,1} *_t \dot{\mathbf{u}})(t) - \dot{e}_{\alpha,1}(t)\mathbf{h}.$$

Consequently,

$$\begin{aligned}
&\varrho\tau^{\alpha}\ddot{\mathbf{u}} + \varrho(\tau^{\alpha} - 1)\left[\frac{\partial}{\partial t}(\dot{e}_{\alpha,1} *_t \dot{\mathbf{u}}) - \dot{e}_{\alpha,1}\mathbf{h}\right] \\
&\quad = \operatorname{Div}(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}) + e_{\alpha,1}\operatorname{Div}(\tau^{\alpha}\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) \\
&\qquad + \tau^{\alpha}\mathbf{f} + (\tau^{\alpha} - 1)\dot{e}_{\alpha,1} *_t \mathbf{f},
\end{aligned}$$

which upon rearrangement yields

$$\begin{aligned}
\tau^{\alpha}\varrho\ddot{\mathbf{u}} + (1 - \tau^{\alpha})\frac{\partial}{\partial t}(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}) &= \operatorname{Div}(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}) \\
&+ (\tau^{\alpha} - 1)\,\dot{e}_{\alpha,1}\,\varrho\mathbf{h} + e_{\alpha,1}\operatorname{Div}(\tau^{\alpha}\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) \\
&+ \tau^{\alpha}\mathbf{f} + (\tau^{\alpha} - 1)\dot{e}_{\alpha,1} *_t \mathbf{f}.
\end{aligned} \tag{2.4}$$

By introducing the function

$$\begin{aligned}
\mathbf{b} &:= (\tau^{\alpha} - 1)\,\dot{e}_{\alpha,1}\,\varrho\mathbf{h} + e_{\alpha,1}\operatorname{Div}(\tau^{\alpha}\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) \\
&\quad + \tau^{\alpha}\mathbf{f} + (\tau^{\alpha} - 1)\dot{e}_{\alpha,1} *_t \mathbf{f}
\end{aligned}$$

that collects the terms involving the initial data $\mathbf{g}$, $\mathbf{h}$, $\mathbf{S}$ and the load vector $\mathbf{f}$ on the right-hand side of (2.4), the equation (2.4) takes the following more compact form:

$$\begin{aligned}
&\tau^{\alpha}\varrho\ddot{\mathbf{u}} + (1 - \tau^{\alpha})\,\frac{\partial}{\partial t}(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}})(t) \\
&\qquad = \operatorname{Div}(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}) + \mathbf{b}.
\end{aligned} \tag{2.5}$$

We shall refer to equation (2.5) as the *fractional Zener wave equation*, our terminology being motivated by the fact that for $\tau \in (0,1)$ (2.5) is a second-order wave equation with nonlocal/fractional-order damping.

Next we shall derive a formal energy identity for the initial-boundary-value problem (1.2), (1.3), (2.5).

3. **Formal energy estimate for the model**

We begin the analysis of the problem by establishing a formal energy inequality, which we shall later rigorously prove by means of an abstract Galerkin approximation. We shall then use the energy inequality satisfied by the sequence of Galerkin approximations in conjunction with a compactness argument to show the existence of weak solutions to the initial-boundary-value problem (1.2), (1.3), (2.5) under consideration, and we shall also prove the uniqueness of weak solutions. For the moment, though, we shall postulate the existence of sufficiently smooth solutions in order to proceed with the formal derivation of an energy identity for the model.

To this end we shall take the scalar product of (2.5) with $\dot{\mathbf{u}}$, integrate the resulting equality over Ω, and perform partial integration with respect to the spatial variable $\mathbf{x}$, noting that $\mathbf{u}$, and therefore also $\dot{\mathbf{u}}$, satisfies a homogeneous Dirichlet boundary condition on $(0,T]\times\partial\Omega$. In order to avoid notational clutter, whenever the function $\mathbf{f}$ is extended by $\mathbf{0}$ from $(0,T]\times\Omega$ to $(0,\infty)\times\Omega$ the extended function will be denoted by the same symbol as the original function.

As will be seen below, it is significant for the derivation of the energy identity, which guarantees continuous dependence of the solution on the data, that:

- $\tau\in(0,1]$, by hypothesis; and
- $e_{\alpha,1}\geq 0$, $-\dot{e}_{\alpha,1}\geq 0$ and $\ddot{e}_{\alpha,1}\geq 0$ on $(0,T]$, with $\dot{e}_{\alpha,1}\in \mathrm{L}^1((0,T))$ and $\ddot{e}_{\alpha,1}\in \mathrm{L}^1_{\mathrm{loc}}((0,T))$ for all $T>0$.

We note in passing that by a similar reasoning the discussion below can be replicated in the case of the standard (integer-order) Zener model, corresponding to $\alpha=1$, but since the analysis of that model is much simpler we shall not include it here and will confine ourselves to the fractional-order Zener model, with $\alpha\in(0,1)$. An identical comment applies to the case of a Hookean solid, corresponding to taking $\alpha=0$ in (1.4).

By formally testing the equation (2.5) with $\dot{\mathbf{u}}$ and noting that $\dot{\mathbf{u}}$ satisfies a homogeneous Dirichlet boundary condition on $(0,T]\times\partial\Omega$ we deduce, by partial integration with respect to the spatial variable $\mathbf{x}$, that, for any $t\in(0,T]$,

$$
\begin{aligned}
&\frac{\tau^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|\dot{\mathbf{u}}(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x} + (1-\tau^\alpha)\int_\Omega \varrho\frac{\partial}{\partial t}(-\dot{e}_{\alpha,1} *_t \dot{\mathbf{u}})(t,\mathbf{x})\cdot\dot{\mathbf{u}}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\\
&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x})))|^2\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{b}(t,\mathbf{x})\cdot\dot{\mathbf{u}}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Hence, by integration over $t\in(0,T]$ and noting the initial conditions (1.2), we deduce that

$$
\begin{aligned}
&\frac{\tau^\alpha}{2}\int_\Omega \varrho|\dot{\mathbf{u}}(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\\
&\qquad+(1-\tau^\alpha)\int_0^t\int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}})(s,\mathbf{x})\cdot\sqrt{\varrho}\dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\\
&\qquad+\frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x})))|^2\,\mathrm{d}\mathbf{x}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^t \int_\Omega \mathbf{b}\cdot\dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \frac{\tau^\alpha}{2}\int_\Omega \varrho|\mathbf{h}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x})))|^2\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.1}
$$

To proceed, we need to show that the second term on the left-hand side of (3.1) is nonnegative, and that $\dot{\mathbf{u}}$ can be eliminated from the right-hand side by absorbing it into the terms appearing on the left-hand side. Once the nonnegativity of the second term on the left-hand side of (3.1) has been verified, the identity (3.1) can be viewed as expressing balance of the total energy. In particular, when the load vector $\mathbf{f} = \mathbf{0}$ and the initial data are such that $\mathbf{b} = \mathbf{0}$, we have that, for $t \in (0,T]$,

$$
\begin{aligned}
&\frac{\tau^\alpha}{2}\int_\Omega \varrho|\dot{\mathbf{u}}(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \\
&\quad + (1-\tau^\alpha)\int_0^t\int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}})(s,\mathbf{x})\cdot\sqrt{\varrho}\dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x})))|^2\,\mathrm{d}\mathbf{x} \\
&= \frac{\tau^\alpha}{2}\int_\Omega \varrho\,|\mathbf{h}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x})))|^2\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.2}
$$

Even more specifically, if $\mathbf{f} = \mathbf{0}$ and $\tau = 1$, and $\mathbf{S}$ is related to $\boldsymbol{\varepsilon}(\mathbf{g})$ through Hooke's law (i.e., $\mathbf{S} = 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))$), whereby also $\mathbf{b} = \mathbf{0}$, then the second term on the left-hand side of (3.1) (which, thanks to Lemma 3.1 below, can be viewed as an energy dissipation term,) is absent, as is the first term on the right-hand side of (3.1), and we have conservation of the total energy:

$$
\begin{aligned}
\mathcal{E}(t) &:= \frac{1}{2}\int_\Omega \varrho|\dot{\mathbf{u}}(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x})))|^2\,\mathrm{d}\mathbf{x} \\
&= \mathcal{E}(0) \qquad \forall\, t \in [0,T].
\end{aligned}
$$

Returning to the general case, to show the nonnegativity of the second term on the left-hand side of (3.1) we invoke the following result (cf. Lemma 1.7.2 in [16], whose proof is based on the identity stated in Lemma 2.3.1 in the work of Zacher [18]; see also identity (9) in [17]).

Lemma 3.1. *Let $\mathcal{H}$ be a separable Hilbert space over the field of real numbers, with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$, and let $T > 0$. Then, for any $k \in \mathrm{L}^1(0,T)$ such that $k \geq 0$, $\dot{k} \in \mathrm{L}^1_{\mathrm{loc}}(0,T)$, and $\dot{k} \leq 0$, and any $v \in \mathrm{L}^2((0,T);\mathcal{H})$, the following inequality holds:*

$$\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}(k *_t v)(s), v(s)\right)_{\mathcal{H}} \mathrm{d}s \geq \frac{1}{2}(k *_t \|v(\cdot)\|^2_{\mathcal{H}})(t) + \frac{1}{2}\int_0^t k(s)\|v(s)\|^2_{\mathcal{H}}\,\mathrm{d}s$$

for all $t \in (0,T]$, each of the two terms on the right-hand side of the inequality being nonnegative.

Taking $k(t) = -\dot{e}_{\alpha,1}(t)(>0)$, $t \in (0,T]$, $\mathcal{H} = \mathrm{L}^2_\varrho(\Omega)$, equipped with the inner product and norm (and analogous notations for norms of weighted Lebesgue spaces, used in what follows, with weight functions $1/\varrho$, μ, $1/\mu$, and λ instead of ϱ) defined by

$$(\mathbf{v},\mathbf{w})_{\mathrm{L}^2_\varrho(\Omega)} := \int_\Omega \varrho(\mathbf{x})\,\mathbf{v}(\mathbf{x})\cdot\mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad \|\mathbf{v}\|_{L^2_\varrho(\Omega)} := (\mathbf{v},\mathbf{v})^{\frac{1}{2}}_{\mathrm{L}^2_\varrho(\Omega)},$$

and $v = \dot{\mathbf{u}}$ in Lemma 3.1, we deduce that the second term on the left-hand side of (3.1) is nonnegative.

It remains to show that the function $\dot{\mathbf{u}}$, appearing in the integrand of the first integral on the right-hand side, can be absorbed into the left-hand side. To this end, we recall that

$$\begin{aligned}\mathbf{b} := {} & (\tau^\alpha - 1)\,\dot{e}_{\alpha,1}\,\varrho\mathbf{h} + e_{\alpha,1}\,\mathrm{Div}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) \\ & + \tau^\alpha\mathbf{f} + (\tau^\alpha - 1)\dot{e}_{\alpha,1} *_t \mathbf{f},\end{aligned}$$

and we denote by $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$, respectively, the four terms whose sum is $\mathbf{b}$.

Clearly, because the function $t \in [0,\infty) \mapsto e_{\alpha,1}(t)$ is positive, strictly monotonic decreasing, and $e_{\alpha,1}(0) = 1$, we have by the Cauchy–Schwarz inequality that

$$\begin{aligned}&\int_0^t \int_\Omega \mathbf{T}_1(s,\mathbf{x})\cdot\dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\ &\leq (1-\tau^\alpha)\int_0^t (-\dot{e}_{\alpha,1}(s))\int_\Omega \varrho|\mathbf{h}(\mathbf{x})||\dot{\mathbf{u}}(s,\mathbf{x})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\ &\leq (1-\tau^\alpha)\int_0^t (-\dot{e}_{\alpha,1}(s))\,\|\mathbf{h}\|_{\mathrm{L}^2_\varrho(\Omega)}\,\|\dot{\mathbf{u}}(s,\cdot)\|_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s \\ &= (1-\tau^\alpha)\,\|\mathbf{h}\|_{\mathrm{L}^2_\varrho(\Omega)}\int_0^t (-\dot{e}_{\alpha,1}(s))\,\|\dot{\mathbf{u}}(s,\cdot)\|_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s \\ &\leq (1-\tau^\alpha)\,\|\mathbf{h}\|_{\mathrm{L}^2_\varrho(\Omega)}\left(\int_0^t (-\dot{e}_{\alpha,1}(s))\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t (-\dot{e}_{\alpha,1}(s))\|\dot{\mathbf{u}}(s,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq (1-\tau^\alpha)\,\|\mathbf{h}\|_{\mathrm{L}^2_\varrho(\Omega)}\,(e_{\alpha,1}(0) - e_{\alpha,1}(t))^{\frac{1}{2}}\left(\int_0^t (-\dot{e}_{\alpha,1}(s))\|\dot{\mathbf{u}}(s,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s\right)^{\frac{1}{2}}.\end{aligned}$$

By bounding the nonnegative factor $(e_{\alpha,1}(0) - e_{\alpha,1}(t))^{\frac{1}{2}}$ above by 1, for any $\delta_1 > 0$, to be fixed,

$$\begin{aligned}\int_0^t \int_\Omega \mathbf{T}_1(s,\mathbf{x}) \cdot \dot{\mathbf{u}}(s,\mathbf{x}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\ \leq \frac{(1-\tau^\alpha)^2}{4\delta_1\tau^\alpha}\|\mathbf{h}\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \tau^\alpha\delta_1 \int_0^t (-\dot{e}_{\alpha,1}(s))\|\dot{\mathbf{u}}(s,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s.\end{aligned} \tag{3.3}$$

Next, by partial integration with respect to the temporal variable followed by partial integration with respect to the spatial variable, we have, upon defining $\boldsymbol{\kappa}_0 := \tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}$, that

$$\int_0^t \int_\Omega \mathbf{T}_2(s,\mathbf{x}) \cdot \dot{\mathbf{u}}(s,\mathbf{x}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}s = \int_0^t e_{\alpha,1}(s)\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega \mathrm{Div}\,\boldsymbol{\kappa}_0(\mathbf{x})\cdot\mathbf{u}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\right]\mathrm{d}s$$

$$=\left[e_{\alpha,1}(s)\int_\Omega \mathrm{Div}\,\boldsymbol{\kappa}_0(\mathbf{x})\cdot\mathbf{u}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\right]_{s=0}^{s=t} - \int_0^t \dot{e}_{\alpha,1}(s)\left[\int_\Omega \mathrm{Div}\,\boldsymbol{\kappa}_0(\mathbf{x})\cdot\mathbf{u}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\right]\mathrm{d}s$$

$$=\left[-e_{\alpha,1}(s)\int_\Omega \boldsymbol{\kappa}_0(\mathbf{x}) : \nabla\boldsymbol{\kappa}_0(\mathbf{x}) : \nabla\mathbf{u}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\right]\mathrm{d}s.$$

Now, letting $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ denote the set of all symmetric 3×3 matrices with real entries, and noting that for any $A \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and any $B \in \mathbb{R}^{3\times3}$ one has that $A : B = A : \frac{1}{2}(B + B^{\mathrm{T}})$, we deduce that

$$\begin{aligned}\int_0^t \int_\Omega \mathbf{T}_2(s,\mathbf{x}) \cdot \dot{\mathbf{u}}(s,\mathbf{x}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}s &= \int_0^t \dot{e}_{\alpha,1}(s)\left[\int_\Omega \boldsymbol{\kappa}_0(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{u}(s,\mathbf{x}))\,\mathrm{d}\mathbf{x}\right]\mathrm{d}s \\ &+ \left[e_{\alpha,1}(0)\int_\Omega \boldsymbol{\kappa}_0(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x}))\,\mathrm{d}\mathbf{x}\right] - \left[e_{\alpha,1}(t)\int_\Omega \boldsymbol{\kappa}_0(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{u}(t,\mathbf{x}))\,\mathrm{d}\mathbf{x}\right] \\ \leq \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)} &\int_0^t (-\dot{e}_{\alpha,1}(s))\|\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot))\|_{\mathrm{L}^2_\mu(\Omega)}\,\mathrm{d}s + \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)}\|\boldsymbol{\varepsilon}(\mathbf{g})\|_{\mathrm{L}^2_\mu(\Omega)} \\ &+ \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)}\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|_{\mathrm{L}^2_\mu(\Omega)},\end{aligned}$$

where in the transition to the right-hand side of the last inequality we have used that $e_{\alpha,1}(0) = 1$ and that $t \in [0,\infty) \mapsto e_{\alpha,1}(t)$ is positive and monotonic decreasing. Hence, by the Cauchy–Schwarz inequality, and with a suitable real number $\delta_2 > 0$, to be fixed below,

$$
\begin{aligned}
&\int_0^t \int_\Omega \mathbf{T}_2(s,\mathbf{x}) \cdot \dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&\quad \le \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)} \left(\int_0^t (-\dot{e}_{\alpha,1}(s))\,\mathrm{d}s\right)^{\frac12} \left(\int_0^t (-\dot{e}_{\alpha,1}(s))\|\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot))\|^2_{\mathrm{L}^2_\mu(\Omega)}\,\mathrm{d}s\right)^{\frac12} \\
&\qquad + \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)}\|\boldsymbol{\varepsilon}(\mathbf{g})\|_{\mathrm{L}^2_\mu(\Omega)} + \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)}\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|_{\mathrm{L}^2_\mu(\Omega)} \\
&\quad \le \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)} \left(\int_0^t (-\dot{e}_{\alpha,1}(s))\|\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot))\|^2_{\mathrm{L}^2_\mu(\Omega)}\,\mathrm{d}s\right)^{\frac12} \\
&\qquad + \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)}\|\boldsymbol{\varepsilon}(\mathbf{g})\|_{\mathrm{L}^2_\mu(\Omega)} + \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)}\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|_{\mathrm{L}^2_\mu(\Omega)} \\
&\quad \le \delta_2 \int_0^t (-\dot{e}_{\alpha,1}(s))\|\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot))\|^2_{\mathrm{L}^2_\mu(\Omega)}\,\mathrm{d}s + \frac{1}{4\delta_2}\|\boldsymbol{\kappa}_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)} \\
&\qquad + \delta_2\|\boldsymbol{\varepsilon}(\mathbf{g})\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{4\delta_2}\|\boldsymbol{\kappa}_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)} \\
&\qquad + \delta_2\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{4\delta_2}\|\boldsymbol{\kappa}_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)}.
\end{aligned} \tag{3.4}
$$

Next, for a positive real number δ_3, to be fixed below,

$$
\begin{aligned}
\int_0^t \int_\Omega \mathbf{T}_3(s,\mathbf{x}) \cdot \dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x} &= \tau^\alpha \int_0^t \int_\Omega \mathbf{f}(s,\mathbf{x}) \cdot \dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x} \\
&\le \tau^\alpha \int_0^t \|\mathbf{f}(s)\|_{\mathrm{L}^2_{1/\varrho}(\Omega)}\|\dot{\mathbf{u}}(s)\|_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s \\
&\le \delta_3 \int_0^t \|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{\tau^{2\alpha}}{4\delta_3}\int_0^t \|\mathbf{f}(s)\|^2_{\mathrm{L}^2_{1/\varrho}(\Omega)}\,\mathrm{d}s.
\end{aligned} \tag{3.5}
$$

Finally, by the Cauchy–Schwarz inequality with respect to $\mathbf{x}$, Minkowski's integral inequality, the negativity of $\dot{e}_{\alpha,1}$, the bound $\|-\dot{e}_{\alpha,1}\|_{\mathrm{L}^1(0,t)} = 1 - e_{\alpha,1}(t) \le 1$, Young's inequality for the (Laplace) convolution $*_t$ (whose proof we have included at the end of this section for the sake of completeness; cf. Lemma 3.2), and with $\delta_4 > 0$ to be fixed below, we have that

$$
\begin{aligned}
&\int_0^t \int_\Omega \mathbf{T}_4(s,\mathbf{x}) \cdot \dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x} \\
&\quad = (1-\tau^\alpha)\int_0^t \int_\Omega (-\dot{e}_{\alpha,1} *_s \mathbf{f})(s,\mathbf{x}) \cdot \dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x} \\
&\quad \le (1-\tau^\alpha)\int_0^t \|-\dot{e}_{\alpha,1} *_s \mathbf{f}(s)\|_{\mathrm{L}^2_{1/\varrho}(\Omega)}\|\dot{\mathbf{u}}(s)\|_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le (1-\tau^\alpha)\int_0^t (-\dot e_{\alpha,1} *_s \|\mathbf{f}\|_{\mathrm{L}^2_{1/\varrho}(\Omega)})(s)\|\dot{\mathbf{u}}(s)\|_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s\\
&\le \delta_4\int_0^t \|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{(1-\tau^\alpha)^2}{4\delta_4}\int_0^t |(-\dot e_{\alpha,1} *_s \|\mathbf{f}\|_{\mathrm{L}^2_{1/\varrho}(\Omega)})(s)|^2\,\mathrm{d}s\\
&= \delta_4\int_0^t \|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{(1-\tau^\alpha)^2}{4\delta_4}\Big[\|(-\dot e_{\alpha,1}) *_s \|\mathbf{f}\|_{\mathrm{L}^2_{1/\varrho}(\Omega)}\|_{\mathrm{L}^2(0,t)}\Big]^2\\
&\le \delta_4\int_0^t \|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{(1-\tau^\alpha)^2}{4\delta_4}\Big[\|(-\dot e_{\alpha,1})\|_{\mathrm{L}^1(0,t)}\|\|\mathbf{f}\|_{\mathrm{L}^2_{1/\varrho}(\Omega)}\|_{\mathrm{L}^2(0,t)}\Big]^2\\
&\le \delta_4\int_0^t \|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{(1-\tau^\alpha)^2}{4\delta_4}\int_0^t \|\mathbf{f}(s)\|^2_{\mathrm{L}^2_{1/\varrho}(\Omega)}\,\mathrm{d}s.
\end{aligned}
\tag{3.6}
$$

By substituting (3.3)–(3.6) into (3.2) we deduce that

$$
\begin{aligned}
&\frac{\tau^\alpha}{2}\|\dot{\mathbf{u}}(t)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + (1-\tau^\alpha)\int_0^t\int_\Omega \frac{\partial}{\partial s}(-\dot e_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}})(s,\mathbf{x})\cdot\sqrt{\varrho}\dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\\
&\qquad + \|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\\
&\le \frac{\tau^\alpha}{2}\|\mathbf{h}\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \|\boldsymbol{\varepsilon}(\mathbf{g})\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\\
&\qquad + \tau^\alpha\delta_1\int_0^t (-\dot e_{\alpha,1}(s))\|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{(1-\tau^\alpha)^2}{4\delta_1\tau^\alpha}\|\mathbf{h}\|^2_{\mathrm{L}^2_\varrho(\Omega)}\\
&\qquad + \delta_2\int_0^t (-\dot e_{\alpha,1}(s))\|\boldsymbol{\varepsilon}(\mathbf{u}(s))\|^2_{\mathrm{L}^2_\mu(\Omega)}\,\mathrm{d}s + \frac{1}{4\delta_2}\|\boldsymbol{\kappa}_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)}\\
&\qquad + \delta_2\|\boldsymbol{\varepsilon}(\mathbf{g})\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{4\delta_2}\|\boldsymbol{\kappa}_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)}\\
&\qquad + \delta_2\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{4\delta_2}\|\boldsymbol{\kappa}_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)}\\
&\qquad + \delta_3\int_0^t \|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{\tau^{2\alpha}}{4\delta_3}\int_0^t \|\mathbf{f}(s)\|^2_{\mathrm{L}^2_{1/\varrho}(\Omega)}\,\mathrm{d}s\\
&\qquad + \delta_4\int_0^t \|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{(1-\tau^\alpha)^2}{4\delta_4}\int_0^t \|\mathbf{f}(s)\|^2_{\mathrm{L}^2_{1/\varrho}(\Omega)}\,\mathrm{d}s.
\end{aligned}
\tag{3.7}
$$

We now fix $\delta_1 = \delta_2 = \frac{1}{2}$ and $\delta_3 = \delta_4 = \frac{\tau^\alpha}{4}$. The inequality (3.7) then takes the following form, for $t \in (0,T]$:

$$
\begin{aligned}
&\frac{\tau^\alpha}{2}\|\dot{\mathbf{u}}(t)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + (1-\tau^\alpha)\int_0^t\int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}})(s,\mathbf{x})\cdot\sqrt{\varrho}\dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\\
&\qquad + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\\
&\leq \frac{\tau^{2\alpha}+(1-\tau^\alpha)^2}{2\tau^\alpha}\|\mathbf{h}\|^2_{\mathrm{L}^2_\varrho(\Omega)}\\
&\qquad + \frac{3}{2}\|\boldsymbol{\varepsilon}(\mathbf{g})\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\|^2_{\mathrm{L}^2_\lambda(\Omega)} + \frac{3}{2}\|\boldsymbol{\kappa}_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)}\\
&\qquad + \frac{\tau^{2\alpha}+(1-\tau^\alpha)^2}{\tau^\alpha}\int_0^t \|\mathbf{f}(s)\|^2_{\mathrm{L}^2_{1/\varrho}(\Omega)}\,\mathrm{d}s\\
&+ \frac{\tau^\alpha}{2}\int_0^t(1-\dot{e}_{\alpha,1}(s))\|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s + \frac{1}{2}\int_0^t(-\dot{e}_{\alpha,1}(s))\|\boldsymbol{\varepsilon}(\mathbf{u}(s))\|^2_{\mathrm{L}^2_\mu(\Omega)}\,\mathrm{d}s.
\end{aligned}
\tag{3.8}
$$

Now, consider the following two nonnegative functions defined on $[0,T]$:

$$
\begin{aligned}
y(t) &:= \frac{\tau^\alpha}{2}\|\dot{\mathbf{u}}(t)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t)))\|^2_{\mathrm{L}^2_\lambda(\Omega)},\\
z(t) &:= (1-\tau^\alpha)\int_0^t\int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}})(s,\mathbf{x})\cdot\sqrt{\varrho}\dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x},
\end{aligned}
$$

and let

$$
\begin{aligned}
A(t) := {}& \frac{\tau^{2\alpha}+(1-\tau^\alpha)^2}{2\tau^\alpha}\|\mathbf{h}\|^2_{\mathrm{L}^2_\varrho(\Omega)}\\
&+ \frac{3}{2}\|\boldsymbol{\varepsilon}(\mathbf{g})\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\\
&+ \frac{3}{2}\|\boldsymbol{\kappa}_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)} + \frac{\tau^{2\alpha}+(1-\tau^\alpha)^2}{\tau^\alpha}\int_0^t \|\mathbf{f}(s)\|^2_{\mathrm{L}^2_{1/\varrho}(\Omega)}\,\mathrm{d}s.
\end{aligned}
\tag{3.9}
$$

Clearly, $0 \leq A(t) \leq A(T) =: A$. The inequality (3.8) then implies that

$$
y(t) + z(t) \leq A(t) + \int_0^t (1-\dot{e}_{\alpha,1}(s))y(s)\,\mathrm{d}s.
$$

Since $t \in [0,T] \mapsto A(t)$ is a nonnegative and nondecreasing function, by Gronwall's lemma we have that, for $t \in (0,T]$,

$$
y(t) + z(t) \leq A(t)\exp\left(\int_0^t (1-\dot{e}_{\alpha,1}(s))\,\mathrm{d}s\right) = A(t)\exp(t+1-e_{\alpha,1}(t)).
$$

In other words, with

$$
A(t) = A(\tau^\alpha, \|\mathbf{h}\|_{\mathrm{L}^2_\varrho(\Omega)}, \|\boldsymbol{\varepsilon}(\mathbf{g})\|_{\mathrm{L}^2_\mu(\Omega)}, \|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\|_{\mathrm{L}^2_\lambda(\Omega)}, \|\boldsymbol{\kappa}_0\|_{\mathrm{L}^2_{1/\mu}(\Omega)}, \|\mathbf{f}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2_{1/\varrho}(\Omega))})
$$

defined by the expression (3.9) for $t \in [0,T]$, the following energy inequality holds for all $t \in [0,T]$:

$$
\begin{aligned}
&\frac{\tau^\alpha}{2}\|\dot{\mathbf{u}}(t)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}(t))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}, \\
&\quad + (1-\tau^\alpha)\int_0^t \int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}})(s,\mathbf{x}) \cdot \sqrt{\varrho}\dot{\mathbf{u}}(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x} \\
&\le A(t)\exp(t+1-e_{\alpha,1}(t)).
\end{aligned}
\tag{3.10}
$$

Thus, assuming the existence of a (sufficiently smooth) solution $\mathbf{u}$ to (1.2), (1.3), (2.5), with

$$
\begin{aligned}
\mathbf{g} \in [\mathrm{H}^1_0(\Omega)]^3, \quad \mathbf{h} \in [\mathrm{L}^2(\Omega)]^3, \\
\mathbf{S} = \mathbf{S}^{\mathrm{T}} \in [\mathrm{L}^2(\Omega)]^{3\times 3}, \quad \mathbf{f} \in \mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^3),
\end{aligned}
\tag{3.11}
$$

recalling that, by hypothesis (1.5), $\varrho, \mu, \lambda \in \mathrm{L}^\infty(\Omega)$, ϱ and μ are bounded below by positive constants ϱ_0 and μ_0, respectively, and $\lambda \ge 0$ a.e. on Ω, the energy inequality (3.10) holds, with $A(t) < \infty$ for all $t \in [0,T]$.

We emphasize here the significance of our assumption that $\tau \in (0,1]$: the positivity of τ is necessary in order to ensure that the factor $A(t)$ (cf. (3.9)) appearing on the right-hand side of the energy inequality (3.10) is finite, while $\tau \le 1$ $(= \rho)$ ensures that the prefactor of the last term on the left-hand side of (3.10), which can be viewed as a nonnegative energy dissipation term thanks to Lemma 3.1, is nonnegative, whereby the entire left-hand side of (3.10) is nonnegative.

Remark 3.1. We remark that if $\mathbf{S}$ is chosen so that $\tau^\alpha \mathbf{S} = 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))$, then $\boldsymbol{\kappa}_0 = \mathbf{0}$, and therefore also $\mathbf{T}_2 = \mathbf{0}$. The energy inequality (3.10) is then simpler and sharper, which can be seen by erasing all terms containing δ_2 from the right-hand side of (3.7), and making the same choices of δ_1, δ_3 and δ_4 as above. In the special case of $\lambda = 0$ this particular choice of the initial stress $\mathbf{S}$, namely $\mathbf{S} = 2\mu(1/\tau)^\alpha\boldsymbol{\varepsilon}(\mathbf{g})$, in our initial condition $(1.2)_3$ results in the same initial condition as the one stated in equation (13) in the work of Freed and Diethelm [9] (recall that we scaled ρ to 1, so $(\rho/\tau)^\alpha = (1/\tau)^\alpha$). We shall proceed without making this restrictive assumption on $\mathbf{S}$, and continue to study the general case when $\tau^\alpha\mathbf{S}$ is not required to be equal to $2\mu\boldsymbol{\varepsilon}(\mathbf{g}) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))$.

In the next section we shall use a compactness argument, based on a sequence of spatial Galerkin approximations to the problem, to show the existence of a (unique) weak solution.

We close this section with the proof of Young's inequality for Laplace-type convolution, which we used in the derivation of the energy inequality.

The proof of this result in the case of Fourier-type convolution is standard; in the case of Laplace-type convolution the argument proceeds along similar lines, with minor modifications; we have included its statement and proof for the convenience of the reader.

LEMMA 3.2. *Let $p, q, r \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, and let $f \in \mathrm{L}^p(0,t)$ and $g \in \mathrm{L}^q(0,t)$ for some $t > 0$; then $s \in [0,t] \mapsto (f *_s g)(s) := \int_0^s f(s-u)g(u)\,\mathrm{d}u \in \mathrm{L}^r(0,t)$, and*

$$\|f *_s g\|_{\mathrm{L}^r(0,t)} \leq \|f\|_{\mathrm{L}^p(0,t)} \|g\|_{\mathrm{L}^q(0,t)}.$$

P r o o f. If $p = \infty$, then necessarily $q = 1$ and $r = \infty$, and if $q = \infty$, then necessarily $p = 1$ and $r = \infty$. Since for $r = \infty$ the result is a direct consequence of Hölder's inequality, we shall concentrate here on the nontrivial case when $p, q, r \in [1, \infty)$. Let $\overline{f} := \chi_{[0,t]} f$ and $\overline{g} := \chi_{[0,t]} g$, where $\chi_{[0,t]}$ is the characteristic function of the interval $[0, t]$. By Young's convolution inequality for functions defined on the real line and $*$ signifying convolution over the real line, we have that

$$\|\overline{f} * \overline{g}\|_{\mathrm{L}^r(\mathbb{R})} \leq \|\overline{f}\|_{\mathrm{L}^p(\mathbb{R})} \|\overline{g}\|_{\mathrm{L}^q(\mathbb{R})} = \|f\|_{\mathrm{L}^p(0,t)} \|g\|_{\mathrm{L}^q(0,t)}.$$

One the other hand, the left-hand side of the inequality can be bounded from below as follows:

$$\|\overline{f} * \overline{g}\|_{\mathrm{L}^r(\mathbb{R})} \geq \|\overline{f} * \overline{g}\|_{\mathrm{L}^r(0,t)} = \left[\int_0^t \left|\int_{-\infty}^{+\infty} \overline{f}(s-v)\,\overline{g}(v)\,\mathrm{d}v\right|^r \mathrm{d}s\right]^{\frac{1}{r}}$$
$$= \left[\int_0^t \left|\int_{\max(0,s-t)}^{\min(s,t)} \overline{f}(s-v)\,\overline{g}(v)\,\mathrm{d}v\right|^r \mathrm{d}s\right]^{\frac{1}{r}}$$
$$= \left[\int_0^t \left|\int_0^s f(s-v)\,g(v)\,\mathrm{d}v\right|^r \mathrm{d}s\right]^{\frac{1}{r}} = \|f *_s g\|_{\mathrm{L}^r(0,t)},$$

and that completes the proof. □

4. **Existence of weak solutions**

Hereafter $\mathrm{W}^{s,p}(D)$ will denote the Sobolev space of real-valued functions defined on a bounded open set $D \subset \mathbb{R}^d$, $d \geq 1$, with differentiability index $s > 0$ and integrability index $p \in [1, \infty]$ (cf. [1]). When $p = 2$, we shall write $\mathrm{H}^s(\Omega)$ instead of $\mathrm{W}^{s,2}(D)$ and $\mathrm{H}^s_0(D)$ will denote the closure of $\mathrm{C}^\infty_0(D)$ in $\mathrm{H}^s(D)$. When D is a bounded open Lipschitz domain and $s \in (\frac{1}{2}, \frac{3}{2})$, elements of $\mathrm{H}^s_0(D)$ have zero trace on ∂D; for such s, $\mathrm{H}^{-s}(D)$ will denote the dual space of $\mathrm{H}^s_0(D)$.

For a Banach space $\mathcal{B}$, we shall denote by $\mathrm{L}^p(0,T;\mathcal{B})$ and $\mathrm{W}^{s,p}(0,T;\mathcal{B})$, respectively, the associated Lebesgue and Sobolev space of $\mathcal{B}$-valued mappings defined on the open interval $(0,T)$, and $\mathrm{C}([0,T];\mathcal{B})$ will signify the set of all uniformly continuous $\mathcal{B}$-valued functions defined on $[0,T]$. Furthermore, $\mathrm{C}^{0,1}([0,T];\mathcal{B})$ will denote the space of Lipschitz-continuous $\mathcal{B}$-valued functions defined on $[0,T]$. Suppose that $\mathcal{H}$ is a Hilbert space over the field of real numbers with inner product $(\cdot,\cdot)_{\mathcal{H}}$. We shall denote by $\mathrm{C}_w([0,T];\mathcal{H})$ the linear space of all weakly continuous functions from $[0,T]$ into $\mathcal{H}$, i.e., the set of all functions $v \in \mathrm{L}^\infty(0,T;\mathcal{H})$ such that $t \in [0,T] \mapsto (v(t),w) \in \mathbb{R}$ is a continuous function on $[0,T]$ for each $w \in \mathcal{H}$.

Our objective in this section is to show the existence and the uniqueness of a *weak solution* to the problem (1.2), (1.3), (2.5), defined as follows.

Definition 4.1 (Weak solution). Suppose that the initial data $\mathbf{g}$, $\mathbf{h}$, $\mathbf{S}$ and the source term $\mathbf{f}$ satisfy (3.11), and assume that $\tau \in (0,1]$, $\alpha \in (0,1)$, and ϱ, μ, and λ are as in (1.5). A function

$$\begin{gathered}\mathbf{u} \in \mathrm{C}_w([0,T];[\mathrm{H}^1_0(\Omega)]^3), \qquad \text{with} \\ \dot{\mathbf{u}} \in \mathrm{C}_w([0,T];\,[\mathrm{L}^2(\Omega)]^3), \quad \text{and} \quad (-\dot{e}_{\alpha,1})^{\frac{1}{2}}\,\dot{\mathbf{u}} \in \mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^3),\end{gathered} \tag{4.1}$$

satisfying the equality

$$\begin{aligned}\tau^\alpha \int_0^T (\varrho\mathbf{u}(s,\cdot),\ddot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s - (1-\tau^\alpha)\int_0^T ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot),\dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s \\ + \int_0^T \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)))\mathbf{I}\,,\boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s \\ = -\tau^\alpha(\varrho\mathbf{g},\dot{\mathbf{v}}(0,\cdot)) + \tau^\alpha(\varrho\mathbf{h},\mathbf{v}(0,\cdot)) + \int_0^T \langle \mathbf{b}(s,\cdot),\mathbf{v}(s,\cdot)\rangle\,\mathrm{d}s\end{aligned} \tag{4.2}$$

for all $\mathbf{v} \in \mathrm{W}^{2,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}^1_0(\Omega)]^3)$ with $\mathbf{v}(T,\cdot) = 0$ and $\dot{\mathbf{v}}(T,\cdot) = 0$, and

$$\begin{aligned}\mathbf{b} := (\tau^\alpha - 1)\,\dot{e}_{\alpha,1}\,\varrho\mathbf{h} \,+\, e_{\alpha,1}\,\mathrm{Div}(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}) \\ + \tau^\alpha\mathbf{f} + (\tau^\alpha - 1)\dot{e}_{\alpha,1} *_t \mathbf{f},\end{aligned} \tag{4.3}$$

is called a weak solution to the problem (1.2), (1.3), (2.5).

In (4.2) and throughout the rest of the paper $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $[\mathrm{H}^{-1}(\Omega)]^3$ and $[\mathrm{H}^1_0(\Omega)]^3$, and $(\cdot,\cdot)$ is the inner product of $[\mathrm{L}^2(\Omega)]^3$. We note that, for $\alpha \in (0,1)$,

$$-\dot{e}_{\alpha,1}(t) \sim \frac{\alpha\,t^{\alpha-1}}{\Gamma(\alpha+1)} \qquad \text{as } t \to 0_+, \tag{4.4}$$

and hence, by noting from (4.3) the additive structure of $\mathbf{b}$, we have that

$$\mathbf{b} \in \mathrm{L}^p(0,T;[\mathrm{L}^2(\Omega)]^3) + \mathrm{W}^{1,p}(0,T;[\mathrm{H}^{-1}(\Omega)]^3) + \mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^3) \qquad \forall\, p \in \left[1, \tfrac{1}{1-\alpha}\right).$$

The function $\boldsymbol{\sigma}$ has been eliminated in the transition from (1.2), (1.3), (2.5) to the weak formulation (4.2), (4.3), and the initial condition $\boldsymbol{\sigma}(0,\cdot) = \mathbf{S}(\cdot)$ has been encoded into (4.2), (4.3). Motivated by (2.3), for a weak solution $\mathbf{u}$, whose existence and uniqueness we will show in Theorem 4.1 below, we therefore *define* the associated stress tensor $\boldsymbol{\sigma}$ by

$$\begin{aligned}\boldsymbol{\sigma}(t,\cdot) := \mathcal{L}^{-1}\left(\frac{1+p^\alpha}{1+\tau^\alpha p^\alpha}\right) *_t \left(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)))\mathbf{I}\right)\\ + \mathcal{L}^{-1}\left(\frac{p^{\alpha-1}}{1+\tau^\alpha p^\alpha}\right)\left(\tau^\alpha \mathbf{S}(\cdot) - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}(\cdot)) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}(\cdot)))\mathbf{I}\right).\end{aligned} \tag{4.5}$$

Consider the bilinear form $a(\cdot,\cdot)$ on $[\mathrm{H}^1_0(\Omega)]^3 \times [\mathrm{H}^1_0(\Omega)]^3$, defined by

$$a(\mathbf{w},\mathbf{v}) := (2\mu\boldsymbol{\varepsilon}(\mathbf{w}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{w}))\mathbf{I}, \boldsymbol{\varepsilon}(\mathbf{v})) \qquad \forall\, \mathbf{w},\mathbf{v} \in [\mathrm{H}^1_0(\Omega)]^3,$$

and observe that

$$a(\mathbf{w},\mathbf{v}) = (2\mu\boldsymbol{\varepsilon}(\mathbf{w}), \boldsymbol{\varepsilon}(\mathbf{v})) + (\lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{w})), \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{v}))) \qquad \forall\, \mathbf{w},\mathbf{v} \in [\mathrm{H}^1_0(\Omega)]^3.$$

Clearly, $a(\mathbf{w},\mathbf{v}) = a(\mathbf{v},\mathbf{w})$, and there exist positive real numbers c_1 and c_0 such that $a(\mathbf{w},\mathbf{v}) \le c_1 \|\mathbf{w}\|_{\mathrm{H}^1(\Omega)} \|\mathbf{v}\|_{\mathrm{H}^1(\Omega)}$ for all $\mathbf{w},\mathbf{v} \in [\mathrm{H}^1_0(\Omega)]^3$ (by the Cauchy–Schwarz inequality), and $a(\mathbf{v},\mathbf{v}) \ge c_0 \|\mathbf{v}\|^2_{\mathrm{H}^1(\Omega)}$ for all $\mathbf{v} \in [\mathrm{H}^1_0(\Omega)]^3$ (by Korn's inequality). Hence, $a(\cdot,\cdot)$ is a symmetric, bounded, and coercive bilinear form on $[\mathrm{H}^1_0(\Omega)]^3 \times [\mathrm{H}^1_0(\Omega)]^3$. Furthermore, by Rellich's theorem, the infinite-dimensional separable Hilbert space $[\mathrm{H}^1_0(\Omega)]^3$ is compactly and densely embedded into the infinite-dimensional separable Hilbert space $[\mathrm{L}^2(\Omega)]^3$.

To proceed, we require the following version of the Hilbert–Schmidt theorem [9].

Lemma 4.1. *Let $\mathcal{H}$ and $\mathcal{V}$ be separable Hilbert spaces, with $\mathcal{V}$ compactly embedded into $\mathcal{H}$ and $\overline{\mathcal{V}} = \mathcal{H}$ in the norm of $\mathcal{H}$. Let $a\colon \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ be a nonzero, symmetric, bounded and coercive bilinear form. Then, there exist sequences of real numbers $(\lambda_n)_{n\in\mathbb{N}}$ and unit $\mathcal{H}$-norm members $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{V}$, which solve the following problem:* Find $\lambda \in \mathbb{R}$ and $e \in \mathcal{H}\setminus\{0\}$ such that

$$a(e,v) = \lambda(e,v)_{\mathcal{H}} \quad \forall\, v \in \mathcal{V}. \tag{4.6}$$

The λ_n, which can be assumed to be in increasing order with respect to n, are positive, bounded from below away from 0, and $\lim_{n\to\infty} \lambda_n = \infty$.

Additionally, the e_n form an $\mathcal{H}$-orthonormal system whose $\mathcal{H}$-closed span is $\mathcal{H}$ and the rescaling $e_n/\sqrt{\lambda_n}$ gives rise to an a-orthonormal system whose a-closed span is $\mathcal{V}$.

We are now ready to formulate the main result of this section.

THEOREM 4.1. *Suppose that the initial data* $\mathbf{g}$, $\mathbf{h}$, $\mathbf{S}$ *and the source term* $\mathbf{f}$ *satisfy* (3.11), *and assume that* $\tau \in (0,1]$, $\alpha \in (0,1)$, *and* ϱ, μ, *and* λ *are as in* (1.5). *Then, the weak formulation* (4.2), (4.3) *of the problem* (1.2), (1.3), (2.5) *has a (weak) solution in the sense of Definition 4.1 such that*

$$\mathbf{u} \in \mathrm{C}([0,T];[\mathrm{H}^s_0(\Omega)]^3) \qquad \text{for all } s \in (\tfrac{1}{2},1),$$

and

$$\tau^\alpha \varrho\dot{\mathbf{u}} + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}) \in \mathrm{W}^{1,p}(0,T;[\mathrm{H}^{-1}(\Omega)]^3), \qquad \alpha \in (0,1),$$

for all $p \in [1,2]$ *satisfying* $p < \frac{1}{1-\alpha}$. *Furthermore,* $\mathbf{u}$ *satisfies the energy inequality*

$$\begin{aligned}\frac{\tau^\alpha}{2}\|\dot{\mathbf{u}}(t')\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}(t'))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t')))\|^2_{\mathrm{L}^2_\lambda(\Omega)} \\ + \frac{1-\tau^\alpha}{2}\int_0^{t'} -\dot{e}_{\alpha,1}(s)\|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s \le 3A(t)\exp(t+1-e_{\alpha,1}(t)),\end{aligned} \tag{4.7}$$

for all $t \in (0,T]$ *and a.e.* $t' \in (0,t]$, *where* $A(t)$ *is defined by* (3.9) *for* $t \in [0,T]$.

The initial condition $\mathbf{u}(0,\cdot) = \mathbf{g}(\cdot)$ *is satisfied in the sense of continuous functions from* $[0,T]$ *into* $[\mathrm{L}^2(\Omega)]^3$ *and the initial condition* $\dot{\mathbf{u}}(0,\cdot) = \mathbf{h}(\cdot)$ *is satisfied as an equality in* $\mathrm{C}_w([0,T],[\mathrm{L}^2(\Omega)]^3)$. *Furthermore, the weak solution* $\mathbf{u}$ *is unique and depends continuously on the data* $\mathbf{g}$, $\mathbf{h}$, $\mathbf{S}$, *and* $\mathbf{f}$.

The stress tensor $\boldsymbol{\sigma}$, *defined by* (4.5) *in terms of the unique weak solution* $\mathbf{u}$ *of* (4.2), (4.3), *satisfies the initial condition* $\boldsymbol{\sigma}(0,\cdot) = \mathbf{S}(\cdot)$ *as an equality in* $\mathrm{C}_w([0,T],[\mathrm{L}^2(\Omega)]^{3\times 3})$.

Proof. *Step 1*: *Existence of solutions.* We begin by showing the existence of a weak solution. We shall use Lemma 4.1 with $\mathcal{H} = [\mathrm{L}^2_\varrho(\Omega)]^3 \simeq [\mathrm{L}^2(\Omega)]^3$ equipped with the inner product defined by $(\mathbf{w},\mathbf{v})_{\mathcal{H}} := (\varrho\mathbf{w},\mathbf{v})$, $\mathcal{V} = [\mathrm{H}^1_0(\Omega)]^3$, to generate an $\mathcal{H}$-orthonormal Galerkin basis $(\boldsymbol{\varphi}_n)_{n\in\mathbb{N}} \subset [\mathrm{H}^1_0(\Omega)]^3$, whose $[\mathrm{L}^2(\Omega)]^3$-closed span is $[\mathrm{L}^2(\Omega)]^3$ and the rescaling $\boldsymbol{\varphi}_n/\sqrt{\lambda_n}$ gives rise to an a-orthonormal system whose a-closed span is $[\mathrm{H}^1_0(\Omega)]^3$; $(\lambda_n)_{n\in\mathbb{N}}$ is a countably infinite sequence of positive eigenvalues, bounded away from 0, and $\lim_{n\to\infty}\lambda_n = \infty$, defined by $a(\boldsymbol{\varphi}_n,\mathbf{v}) = \lambda_n(\varrho\boldsymbol{\varphi}_n,\mathbf{v})$ for all $\mathbf{v} \in [\mathrm{H}^1_0(\Omega)]^3$.

Let $\mathcal{V}_n := \mathrm{span}\{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n\}$, and let $P_n\mathbf{v} \in \mathcal{V}_n$ denote the orthogonal projection of $\mathbf{v} \in [\mathrm{L}^2_\varrho(\Omega)]^3$, in the inner product of $[\mathrm{L}^2_\varrho(\Omega)]^3$, onto $\mathcal{V}_n$. We seek a Galerkin approximation $\mathbf{u}_n : [0,T] \mapsto \mathbf{u}_n(t) \in \mathcal{V}_n$ of the form

$$\mathbf{u}_n(t,\mathbf{x}) := \sum_{k=1}^{n} \beta_k(t)\boldsymbol{\varphi}_k(\mathbf{x}) \tag{4.8}$$

satisfying

$$\begin{aligned} \tau^\alpha(\varrho\ddot{\mathbf{u}}_n, \mathbf{v}) + (1-\tau^\alpha)\left(\frac{\partial}{\partial t}(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}_n), \mathbf{v}\right) \\ + \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}_n) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n))\mathbf{I}\,, \boldsymbol{\varepsilon}(\mathbf{v})\big) = \langle \mathbf{b}, \mathbf{v}\rangle \end{aligned} \tag{4.9}$$

for all $\mathbf{v} \in \mathcal{V}_n$, together with the initial conditions

$$\mathbf{u}_n(0,\cdot) = P_n\mathbf{g} \qquad \text{and} \qquad \dot{\mathbf{u}}_n(0,\cdot) = P_n\mathbf{h}.$$

Equivalently,

$$\beta_k(0) = (\varrho\mathbf{g}, \boldsymbol{\varphi}_k) \qquad \text{and} \qquad \dot{\beta}_k(0) = (\varrho\mathbf{h}, \boldsymbol{\varphi}_k), \qquad \text{for } k = 1, \dots, n.$$

Hence,

$$\|\mathbf{u}_n(0,\cdot)\|_{\mathrm{L}^2_\varrho(\Omega)} \le \|\mathbf{g}\|_{\mathrm{L}^2_\varrho(\Omega)} \qquad \text{and} \qquad \|\dot{\mathbf{u}}_n(0,\cdot)\|_{\mathrm{L}^2_\varrho(\Omega)} \le \|\mathbf{h}\|_{\mathrm{L}^2_\varrho(\Omega)},$$

and

$$\begin{aligned} a(\mathbf{u}_n(0,\cdot), \mathbf{u}_n(0,\cdot)) &= \sum_{k,\ell=1}^{n} \beta_k(0)\beta_\ell(0) a(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_\ell) \\ &= \sum_{k,\ell=1}^{n} \beta_k(0)\beta_\ell(0)\lambda_k(\varrho\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_\ell) = \sum_{k=1}^{n}[\beta_k(0)]^2\lambda_k\|\boldsymbol{\varphi}_k\|^2_{\mathrm{L}^2_\varrho(\Omega)} \\ &= \sum_{k=1}^{n}[(\varrho\mathbf{g}, \boldsymbol{\varphi}_k)]^2\lambda_k\|\boldsymbol{\varphi}_k\|^2_{\mathrm{L}^2_\varrho(\Omega)} \le \sum_{k=1}^{\infty}[(\varrho\mathbf{g}, \boldsymbol{\varphi}_k)]^2\lambda_k\|\boldsymbol{\varphi}_k\|^2_{\mathrm{L}^2_\varrho(\Omega)} \\ &= \sum_{k,\ell=1}^{\infty}(\varrho\mathbf{g}, \boldsymbol{\varphi}_k)(\varrho\mathbf{g}, \boldsymbol{\varphi}_\ell)\lambda_k(\varrho\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_\ell) = a\left(\sum_{k=1}^{\infty}(\varrho\mathbf{g}, \boldsymbol{\varphi}_k)\boldsymbol{\varphi}_k, \sum_{\ell=1}^{\infty}(\varrho\mathbf{g}, \boldsymbol{\varphi}_\ell)\boldsymbol{\varphi}_\ell\right) \\ &= a(\mathbf{g}, \mathbf{g}). \end{aligned}$$

Thus, by the coercivity and the boundedness of the bilinear form $a(\cdot,\cdot)$ on $[\mathrm{H}^1_0(\Omega)]^3 \times [\mathrm{H}^1_0(\Omega)]^3$, also

$$c_0\|\mathbf{u}_n(0,\cdot)\|^2_{\mathrm{H}^1(\Omega)} = c_0\|P_n\mathbf{g}\|^2_{\mathrm{H}^1(\Omega)} \le c_1\|\mathbf{g}\|^2_{\mathrm{H}^1(\Omega)}.$$

Hence, the orthogonal projector P_n has operator norm $\|P_n\|_{\mathcal{L}([\mathrm{L}^2_\varrho(\Omega)]^3, [\mathrm{L}^2_\varrho(\Omega)]^3)}$ bounded by 1, uniformly in n, and it is, simultaneously, a bounded linear operator from the Sobolev space $[\mathrm{H}^1_0(\Omega)]^3$ into $\mathcal{V}_n \subset [\mathrm{H}^1_0(\Omega)]^3$, with operator norm $\|P_n\|_{\mathcal{L}([\mathrm{H}^1(\Omega)]^3, [\mathrm{H}^1(\Omega)]^3)}$ bounded by $(c_1/c_0)^{1/2}$, uniformly in n.

We begin by showing the existence of a unique Galerkin approximation $t \in [0,T] \mapsto \mathbf{u}_n(t) \in \mathcal{V}_n$. By substituting (4.8) into (4.9) and taking $\mathbf{v} =$

$\boldsymbol{\varphi}_m \in \mathcal{V}_n$ for $m = 1, \dots, n$ and noting the orthonormality $(\varrho\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_m) = \delta_{k,m}$ for $k, m = 1, \dots, n$, we have that

$$\tau^\alpha \ddot{\beta}_m + (1-\tau^\alpha)\frac{\mathrm{d}}{\mathrm{d}t}(-\dot{e}_{\alpha,1} *_t \dot{\beta}_m) + \lambda_m \beta_m = \langle \mathbf{b}, \boldsymbol{\varphi}_m \rangle, \qquad m = 1, \dots, n, \tag{4.10}$$

with $\langle \mathbf{b}, \boldsymbol{\varphi}_m \rangle \in \mathrm{L}^p(0,T)$ for all $p \in \big[1, \frac{1}{1-\alpha}\big)$, in conjunction with the initial conditions $\beta_m(0) = (\varrho\mathbf{g}, \boldsymbol{\varphi}_m)$, $\dot{\beta}_m(0) = (\varrho\mathbf{h}, \boldsymbol{\varphi}_m)$, $m = 1, \dots, n$.

The existence of a unique solution β_m to this problem, with $\dot{\beta}_m \in \mathrm{AC}([0,T])$ for each $m \in \{1, \dots, n\}$ is easily shown: by letting $\gamma_m := \dot{\beta}_m$, (4.10) can be rewritten as a first-order system for the two-component function $t \in [0,T] \mapsto (\beta_m(t), \gamma_m(t))^{\mathrm{T}} \in \mathbb{R}^2$, and then, because $\langle \mathbf{b}, \boldsymbol{\varphi}_m \rangle \in \mathrm{L}^1(0,T)$, integration of this system over $[0,t]$, with $t \in (0,T]$ yields an integral equation to which one can apply Banach's fixed point theorem in the complete metric space $\mathrm{C}([0,T]) \times \mathrm{C}([0,T])$ to deduce the existence of a unique absolutely continuous solution $(\beta_m, \gamma_m)^{\mathrm{T}}$, defined on a "maximal" interval $[0, t_*] \subset [0,T]$. If t_* were strictly less than T, then it would follow that $|\beta_m(t)| + |\gamma_m(t)| \to +\infty$ as $t \to t_*$; the *a priori* bound (4.12), which we shall prove below, however rules out this possibility; therefore $t_* = T$. Thus we deduce the existence of a unique Galerkin approximation $t \in [0,T] \mapsto \mathbf{u}_n(t) \in \mathcal{V}_n$, with $\dot{\mathbf{u}}_n \in \mathrm{AC}([0,T]; \mathcal{V}_n)$.

By taking $\mathbf{v} = \boldsymbol{\varphi}_m$ in (4.9), multiplying the resulting equality with $\beta_m(t)$ and summing over $m = 1, \dots, n$, we deduce that

$$\begin{aligned}
&\frac{\tau^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|\dot{\mathbf{u}}_n(t,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \\
&\quad + (1-\tau^\alpha)\int_\Omega \frac{\partial}{\partial t}(-\dot{e}_{\alpha,1} *_t \sqrt{\varrho}\dot{\mathbf{u}}_n)(t,\mathbf{x}) \cdot \sqrt{\varrho}\dot{\mathbf{u}}_n(t,\mathbf{x}) \,\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{u}_n(t,\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n(t,\mathbf{x})))|^2 \,\mathrm{d}\mathbf{x} \;=\; \langle \mathbf{b}(t,\cdot), \dot{\mathbf{u}}_n(t,\cdot) \rangle.
\end{aligned}$$

Hence, by integration over $t \in (0,T]$ and noting the initial conditions satisfied by $\mathbf{u}_n$ we deduce that

$$\begin{aligned}
&\frac{\tau^\alpha}{2}\int_\Omega \varrho|\dot{\mathbf{u}}_n(t,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \\
&\quad + (1-\tau^\alpha)\int_0^t\int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}}_n)(s,\mathbf{x}) \cdot \sqrt{\varrho}\dot{\mathbf{u}}_n(s,\mathbf{x}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{u}_n(t,\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n(t,\mathbf{x})))|^2 \,\mathrm{d}\mathbf{x}
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \langle \mathbf{b}(s,\cdot), \dot{\mathbf{u}}_n(s,\cdot)\rangle \,\mathrm{d}s \\
&\quad + \frac{\tau^\alpha}{2}\int_\Omega \varrho|\dot{\mathbf{u}}_n(0,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{u}_n(0,\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n(0,\mathbf{x})))|^2 \,\mathrm{d}\mathbf{x} \\
&= \int_0^t \langle \mathbf{b}(s,\cdot), \dot{\mathbf{u}}_n(s,\cdot)\rangle \,\mathrm{d}s + \frac{\tau^\alpha}{2}\|\dot{\mathbf{u}}_n(0,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}a(\mathbf{u}_n(0,\cdot),\mathbf{u}_n(0,\cdot)) \\
&\le \int_0^t \langle \mathbf{b}(s,\cdot), \dot{\mathbf{u}}_n(s,\cdot)\rangle \,\mathrm{d}s + \frac{\tau^\alpha}{2}\|\mathbf{h}\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}a(\mathbf{g},\mathbf{g}) \\
&= \int_0^t \langle \mathbf{b}(s,\cdot), \dot{\mathbf{u}}_n(s,\cdot)\rangle \,\mathrm{d}s + \frac{\tau^\alpha}{2}\int_\Omega \varrho|\mathbf{h}|^2 \,\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x})))|^2 \,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\tau^\alpha}{2}\int_\Omega \varrho|\dot{\mathbf{u}}_n(t,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \\
&\quad + (1-\tau^\alpha)\int_0^t\int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}}_n)(s,\mathbf{x})\cdot\sqrt{\varrho}\dot{\mathbf{u}}_n(s,\mathbf{x}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{u}_n(t,\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n(t,\mathbf{x})))|^2 \,\mathrm{d}\mathbf{x} \qquad (4.11)\\
&\le \int_0^t \langle \mathbf{b}(s,\cdot), \dot{\mathbf{u}}_n(s,\cdot)\rangle \,\mathrm{d}s \\
&\quad + \frac{\tau^\alpha}{2}\int_\Omega \varrho|\mathbf{h}|^2 \,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega 2\mu|\boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x}))|^2 + \lambda|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}(\mathbf{x})))|^2 \,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

We can now repeat the procedure (this time rigorously, as $\mathbf{u}_n$ possesses the necessary regularity properties) leading from (3.1) to the energy inequality (3.10), with $\mathbf{u}$ replaced by $\mathbf{u}_n$ throughout, resulting in the uniform bound

$$
\begin{aligned}
&\frac{\tau^\alpha}{2}\|\dot{\mathbf{u}}_n(t)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}_n(t))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n(t)))\|^2_{\mathrm{L}^2_\lambda(\Omega)} \\
&\quad + (1-\tau^\alpha)\int_0^t\int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}}_n)(s)\cdot\sqrt{\varrho}\dot{\mathbf{u}}_n(s) \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \qquad (4.12)\\
&\le A(t)\exp(t+1-e_{\alpha,1}(t)),
\end{aligned}
$$

for all $t \in (0,T]$, with $A(t)$ again defined by the expression (3.9).

We are now ready to pass to the limit $n \to \infty$. To this end, we fix an integer N and choose a function $\mathbf{v} \in \mathrm{C}^2_0([0,T);[\mathrm{H}^1_0(\Omega)]^3)$ of the form

$$
\mathbf{v}(t,\mathbf{x}) := \sum_{k=1}^N \alpha_k(t)\boldsymbol{\varphi}_k(\mathbf{x}), \qquad (4.13)
$$

where $\alpha_k \in \mathrm{C}^2_0([0,T))$ for $k=1,\dots,N$, i.e., $\alpha_k \in \mathrm{C}^2([0,T])$ and has compact support in the half-open interval $[0,T)$. We then choose $n \geq N$ in (4.9), take $\mathbf{v} = \boldsymbol{\varphi}_k$ as test function in (4.9) for $k \in \{1,\dots,N\}$, multiply the resulting equality with α_k, sum through $k=1,\dots,N$, and perform partial integrations in the first and the second term on the left-hand side to deduce that

$$\begin{aligned}
&\tau^\alpha(\varrho\mathbf{u}_n(0,\cdot),\dot{\mathbf{v}}(0,\cdot)) - \tau^\alpha(\varrho\dot{\mathbf{u}}_n(0,\cdot),\mathbf{v}(0,\cdot)) + \tau^\alpha\int_0^T(\varrho\mathbf{u}_n(s,\cdot),\ddot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s\\
&\quad - (1-\tau^\alpha)((-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}_n)(0,\cdot),\mathbf{v}(0,\cdot))\\
&\quad - (1-\tau^\alpha)\int_0^T((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}}_n)(s,\cdot),\dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s\\
&\quad + \int_0^T\big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}_n(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n(s,\cdot)))\mathbf{I}\,,\boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s\\
&= \int_0^T\langle\mathbf{b}(s,\cdot),\mathbf{v}(s,\cdot)\rangle\,\mathrm{d}s
\end{aligned}$$

vskip -3pt for all $\mathbf{v}$ as in (4.13) with N fixed, and with any $n \geq N$.

Thus, because $(\varrho\mathbf{u}_n(0,\cdot),\dot{\mathbf{v}}(0,\cdot)) = (\varrho\mathbf{g},\dot{\mathbf{v}}(0,\cdot))$ and $(\varrho\dot{\mathbf{u}}_n(0,\cdot),\mathbf{v}(0,\cdot)) = (\varrho\mathbf{h},\mathbf{v}(0,\cdot))$ for all $\mathbf{v} \in \mathcal{V}_n$, and therefore (since $n \geq N$) also for all $\mathbf{v}$ of the form (4.13), and as $((-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}_n)(0,\cdot) = 0$, we have that

$$\begin{aligned}
&\tau^\alpha\int_0^T(\varrho\mathbf{u}_n(s,\cdot),\ddot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s - (1-\tau^\alpha)\int_0^T((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}}_n)(s,\cdot),\dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s\\
&\quad + \int_0^T\big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}_n(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n(s,\cdot)))\mathbf{I}\,,\boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s \qquad (4.14)\\
&= -\tau^\alpha(\varrho\mathbf{g},\dot{\mathbf{v}}(0,\cdot)) + \tau^\alpha(\varrho\mathbf{h},\mathbf{v}(0,\cdot)) + \int_0^T\langle\mathbf{b}(s,\cdot),\mathbf{v}(s,\cdot)\rangle\,\mathrm{d}s.
\end{aligned}$$

As $0 \leq A(t) \leq A(T) =: A$ and $\exp(t+1-e_{\alpha,1}(t)) \leq \exp(T+1)$, it follows from the energy estimate (4.12) and Lemma 3.1 that

- $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is a bounded sequence in the space $\mathrm{L}^\infty(0,T;[\mathrm{H}^1_0(\Omega)]^3)$;
- $(\dot{\mathbf{u}}_n)_{n\in\mathbb{N}}$ is a bounded sequence in the space $\mathrm{L}^\infty(0,T;[\mathrm{L}^2_\varrho(\Omega)]^3) \simeq \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^3)$;
- $((-\dot{e}_{\alpha,1})^{\frac{1}{2}}\,\dot{\mathbf{u}}_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{L}^2(0,T;[\mathrm{L}^2_\varrho(\Omega)]^3) \simeq \mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^3)$.

Thus, by the Banach–Alaoglu theorem there exists a subsequence $(\mathbf{u}_{n_\ell})_{\ell=1}^\infty$ such that

$$\begin{cases} \mathbf{u}_{n_\ell} \rightharpoonup \mathbf{u} & \text{weakly* in } \mathrm{L}^\infty(0,T;[\mathrm{H}^1_0(\Omega)]^3), \\ \dot{\mathbf{u}}_{n_\ell} \rightharpoonup \dot{\mathbf{u}} & \text{weakly* in } \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^3), \\ (-\dot{e}_{\alpha,1})^{\frac{1}{2}}\,\dot{\mathbf{u}}_{n_\ell} \rightharpoonup (-\dot{e}_{\alpha,1})^{\frac{1}{2}}\,\dot{\mathbf{u}} & \text{weakly in } \mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^3). \end{cases} \tag{4.15}$$

Furthermore, because for any $s \in (\frac{1}{2}, 1)$ the Sobolev space $[\mathrm{H}^1_0(\Omega)]^3$ is compactly embedded into the fractional-order Sobolev space $[\mathrm{H}^s_0(\Omega)]^3$, which is, in turn, continuously embedded into $[\mathrm{L}^2(\Omega)]^3$, it follows from the Aubin–Lions–Simon lemma (cf. [5]) and the first two bullet points above that

$$\mathbf{u}_{n_\ell} \to \mathbf{u} \qquad \text{strongly in } \mathrm{C}([0,T];[\mathrm{H}^s_0(\Omega)]^3), \qquad s \in (\tfrac{1}{2}, 1), \tag{4.16}$$

and therefore also

$$\mathbf{u}_{n_\ell} \to \mathbf{u} \qquad \text{strongly in } \mathrm{C}([0,T];[\mathrm{L}^2(\Omega)]^3). \tag{4.17}$$

We take $n = n_\ell$ in (4.14) and pass to the limit $\ell \to \infty$ with $\mathbf{v}$ fixed. It then follows that

$$\begin{aligned} \tau^\alpha \int_0^T (\varrho\mathbf{u}(s,\cdot), \ddot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s - (1-\tau^\alpha) \int_0^T ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s \\ + \int_0^T \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)))\mathbf{I}\,, \boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s \\ = -\tau^\alpha(\varrho\mathbf{g}, \dot{\mathbf{v}}(0,\cdot)) + \tau^\alpha(\varrho\mathbf{h}, \mathbf{v}(0,\cdot)) + \int_0^T \langle \mathbf{b}(s,\cdot), \mathbf{v}(s,\cdot)\rangle\,\mathrm{d}s \end{aligned} \tag{4.18}$$

for all $\mathbf{v}$ as in (4.13) above, with N fixed. This equality however holds for all functions $\mathbf{v} \in \mathrm{W}^{2,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}^1_0(\Omega))]^3)$ such that $\mathbf{v}(T,\cdot) = 0$ and $\dot{\mathbf{v}}(T,\cdot) = 0$, as the set of all functions of the form (4.13) is dense in this function space.

We note here that the passage to the limit in the second term on the left-hand side of (4.14), resulting in the second term on the left-hand side of (4.18) proceeds as follows: by Fubini's theorem to interchange the spatial integral with the integral with respect to s, and then by interchanging the order of integration in s and t, we have that

$$\begin{aligned}
&\int_0^T ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}}_{n_\ell})(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s \\
&\quad = -\int_0^T \left(\int_0^t \dot{e}_{\alpha,1}(t-s)\varrho\dot{\mathbf{u}}_{n_\ell}(s,\cdot)\,\mathrm{d}s, \dot{\mathbf{v}}(t,\cdot)\right)\mathrm{d}t \\
&\quad = -\int_0^T \int_0^t \dot{e}_{\alpha,1}(t-s)\,(\varrho\dot{\mathbf{u}}_{n_\ell}(s,\cdot), \dot{\mathbf{v}}(t,\cdot))\,\mathrm{d}s\,\mathrm{d}t \\
&\quad = -\int_0^T \int_s^T \dot{e}_{\alpha,1}(t-s)\,(\varrho\dot{\mathbf{u}}_{n_\ell}(s,\cdot), \dot{\mathbf{v}}(t,\cdot))\,\mathrm{d}t\,\mathrm{d}s \\
&\quad = -\int_0^T \left(\varrho\dot{\mathbf{u}}_{n_\ell}(s,\cdot), \int_s^T \dot{e}_{\alpha,1}(t-s)\dot{\mathbf{v}}(t,\cdot)\,\mathrm{d}t\right)\mathrm{d}s.
\end{aligned}$$

Then, because $s \in [0,T] \mapsto \int_s^T \dot{e}_{\alpha,1}(t-s)\dot{\mathbf{v}}(t,\cdot)\,\mathrm{d}t \in \mathrm{L}^1(0,T;[\mathrm{L}^2(\Omega)]^3)$, noting $(4.15)_2$ yields

$$\begin{aligned}
&\lim_{\ell\to\infty}\int_0^T ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}}_{n_\ell})(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s \\
&\quad = -\lim_{\ell\to\infty}\int_0^T \left(\varrho\dot{\mathbf{u}}_{n_\ell}(s,\cdot), \int_s^T \dot{e}_{\alpha,1}(t-s)\dot{\mathbf{v}}(t,\cdot)\,\mathrm{d}t\right)\mathrm{d}s \\
&\quad = -\int_0^T \left(\varrho\dot{\mathbf{u}}(s,\cdot), \int_s^T \dot{e}_{\alpha,1}(t-s)\dot{\mathbf{v}}(t,\cdot)\,\mathrm{d}t\right)\mathrm{d}s \\
&\quad = \int_0^T ((-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s,
\end{aligned}$$

as has been asserted above. The passages to the limits in the first and third term on the left-hand side of (4.14) are immediate, by using $(4.15)_2$ and $(4.15)_1$, respectively.

We have thus shown the existence of a function $\mathbf{u} \in \mathrm{L}^\infty(0,T;[\mathrm{H}^1_0(\Omega)]^3)$ such that $\dot{\mathbf{u}} \in \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^3)$, satisfying the equation (4.18) for all $\mathbf{v} \in \mathrm{W}^{2,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}^1_0(\Omega))]^3)$ such that $\mathbf{v}(T,\cdot) = 0$ and $\dot{\mathbf{v}}(T,\cdot) = 0$; the proof of the existence of a weak solution is therefore almost complete. It remains to show that $\mathbf{u} \in \mathrm{C}_w([0,T];[\mathrm{H}^1_0(\Omega)]^3)$ and $\dot{\mathbf{u}} \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^3)$.

We begin by recalling that, for any pair of Hilbert spaces $\mathcal{H}$ and $\mathcal{V}$ such that $\mathcal{V}$ is continuously and densely embedded into $\mathcal{H}$, if $v \in \mathrm{L}^\infty(0,T;\mathcal{V})$ and $\dot{v} \in \mathrm{L}^1(0,T;\mathcal{H})$ (whereby $v \in \mathrm{W}^{1,1}(0,T;\mathcal{H}) \subset \mathrm{C}([0,T];\mathcal{H}) \subset \mathrm{C}_w([0,T];\mathcal{H})$), then $v \in \mathrm{C}_w([0,T];\mathcal{V})$ (cf. eq. (8.49) in Lemma 8.1, Ch. 3 of [13]). Therefore, because

$$\mathbf{u} \in \mathrm{L}^\infty(0,T;[\mathrm{H}^1_0(\Omega)]^3) \quad \text{and} \quad \dot{\mathbf{u}} \in \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^3) \subset \mathrm{L}^1(0,T;[\mathrm{L}^2(\Omega)]^3),$$

it follows, with $\mathcal{V} = [\mathrm{H}_0^1(\Omega)]^3$ and $\mathcal{H} = [\mathrm{L}^2(\Omega)]^3$, that $\mathbf{u} \in \mathrm{C}_w([0,T];[\mathrm{H}_0^1(\Omega)]^3)$.

Next, we will show that $\dot{\mathbf{u}} \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^3)$. It follows from (4.9) that $\mathbf{u}_n(t) \in \mathcal{V}_n$, for $t \in [0,T]$, satisfies:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}\big(\tau^\alpha \varrho \dot{\mathbf{u}}_n + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho \dot{\mathbf{u}}_n)\big), \mathbf{v}\right)\\&\qquad = -\big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}_n) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n))\mathbf{I}\,, \boldsymbol{\varepsilon}(\mathbf{v})\big) + \langle \mathbf{b}, \mathbf{v}\rangle\end{aligned} \tag{4.19}$$

for all $\mathbf{v} \in \mathcal{V}_n$. We thus have from (4.19) that, for any $\mathbf{v} \in [\mathrm{H}_0^1(\Omega)]^3$,

$$\begin{aligned}&\left(\frac{\partial}{\partial t}\big(\tau^\alpha \varrho \dot{\mathbf{u}}_n + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho \dot{\mathbf{u}}_n)\big), \mathbf{v}\right)\\&\qquad = \left(\frac{\partial}{\partial t}\big(\varrho\tau^\alpha \dot{\mathbf{u}}_n + \varrho(1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \dot{\mathbf{u}}_n)\big), P_n\mathbf{v}\right)\\&\qquad = -\big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}_n) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n))\mathbf{I}\,, \boldsymbol{\varepsilon}(P_n\mathbf{v})\big) + \langle \mathbf{b}, P_n\mathbf{v}\rangle.\end{aligned}$$

We note that by the energy estimate (4.12) and because $\|P_n\|_{\mathcal{L}([\mathrm{H}^1(\Omega)]^3,[\mathrm{H}^1(\Omega)]^3)}$ is bounded by $(c_1/c_0)^{1/2}$, uniformly in n, there exists a positive constant C, independent of n, such that, for all $\mathbf{v} \in [\mathrm{H}_0^1(\Omega)]^3$,

$$\begin{aligned}&\big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}_n) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n))\mathbf{I}\,, \boldsymbol{\varepsilon}(P_n\mathbf{v})\big)\\&\quad = \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}_n)\,, \boldsymbol{\varepsilon}(P_n\mathbf{v})\big) + \big(\lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n))\mathbf{I}\,, \boldsymbol{\varepsilon}(P_n\mathbf{v})\big)\\&\quad = \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}_n)\,, \boldsymbol{\varepsilon}(P_n\mathbf{v})\big) + \big(\lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n))\,, \mathrm{tr}(\boldsymbol{\varepsilon}(P_n\mathbf{v}))\big)\\&\quad \le \left(2\|\boldsymbol{\varepsilon}(\mathbf{u}_n)\|^2_{\mathrm{L}^2_\mu(\Omega)} + \|\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\right)^{\frac{1}{2}}\\&\qquad \times \left(2\|\boldsymbol{\varepsilon}(P_n\mathbf{v})\|^2_{\mathrm{L}^2_\mu(\Omega)} + \|\mathrm{tr}(\boldsymbol{\varepsilon}(P_n\mathbf{v}))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\right)^{\frac{1}{2}}\\&\quad \le C\left(2\|\boldsymbol{\varepsilon}(P_n\mathbf{v})\|^2_{\mathrm{L}^2_\mu(\Omega)} + \|\mathrm{tr}(\boldsymbol{\varepsilon}(P_n\mathbf{v}))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\right)^{\frac{1}{2}} \le C\|\mathbf{v}\|_{\mathrm{H}^1(\Omega)}.\end{aligned}$$

Also,

$$\begin{aligned}\langle \mathbf{b}, P_n\mathbf{v}\rangle &= (\tau^\alpha - 1)\dot{e}_{\alpha,1}\,(\varrho\mathbf{h}\,, P_n\mathbf{v})\\&\quad - e_{\alpha,1}\,(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}\,, \boldsymbol{\varepsilon}(P_n\mathbf{v}))\\&\quad + \tau^\alpha\,(\mathbf{f}, P_n\mathbf{v}) + (\tau^\alpha - 1)\dot{e}_{\alpha,1} *_t (\mathbf{f}, P_n\mathbf{v}) \qquad \forall\, \mathbf{v} \in [\mathrm{H}_0^1(\Omega)]^3,\end{aligned}$$

and therefore, because $\|P_n\|_{\mathcal{L}([\mathrm{L}^2(\Omega)]^3,[\mathrm{L}^2(\Omega)]^3)} \le 1$ and $\|P_n\|_{\mathcal{L}([\mathrm{H}^1(\Omega)]^3,[\mathrm{H}^1(\Omega)]^3)}$ is bounded by $(c_1/c_0)^{1/2}$, uniformly in n, there exists a positive constant C, independent of n, such that

$$
\begin{aligned}
|\langle \mathbf{b}, P_n\mathbf{v}\rangle| \leq{}& (1-\tau^\alpha)(-\dot{e}_{\alpha,1})\|\mathbf{h}\|_{\mathrm{L}^2_\varrho(\Omega)}\|\mathbf{v}\|_{\mathrm{L}^2_\varrho(\Omega)}\\
&+ Ce_{\alpha,1}\|\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}\|_{\mathrm{L}^2(\Omega)}\|\mathbf{v}\|_{\mathrm{H}^1(\Omega)}\\
&+ \tau^\alpha\|\mathbf{f}\|_{\mathrm{L}^2_{1/\varrho}(\Omega)}\|\mathbf{v}\|_{\mathrm{L}^2_\varrho(\Omega)}\\
&+ (1-\tau^\alpha)\big((-\dot{e}_{\alpha,1}) *_t \|\mathbf{f}\|_{\mathrm{L}^2_{1/\varrho}(\Omega)}\big)\|\mathbf{v}\|_{\mathrm{L}^2_\varrho(\Omega)} \quad \forall\, \mathbf{v}\in[\mathrm{H}^1_0(\Omega)]^3.
\end{aligned}
$$

Thus we deduce, with $\varrho_1 := \|\varrho\|_{\mathrm{L}^\infty(\Omega)}$,

$$
\begin{aligned}
&\left\|\frac{\partial}{\partial t}\big(\tau^\alpha\varrho\dot{\mathbf{u}}_n + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}_n)\big)\right\|_{\mathrm{H}^{-1}(\Omega)}\\
&:= \sup_{\mathbf{v}\in[\mathrm{H}^1_0(\Omega)]^3} \frac{\big\langle \frac{\partial}{\partial t}\big(\varrho\tau^\alpha\dot{\mathbf{u}}_n + \varrho(1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \dot{\mathbf{u}}_n)\big), \mathbf{v}\big\rangle}{\|\mathbf{v}\|_{\mathrm{H}^1_0(\Omega)}}\\
&\leq C + \varrho_1(1-\tau^\alpha)(-\dot{e}_{\alpha,1})\|\mathbf{h}\|_{\mathrm{L}^2(\Omega)}\\
&\quad + Ce_{\alpha,1}\|\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}\|_{\mathrm{L}^2(\Omega)}\\
&\quad + \tau^\alpha\sqrt{\frac{\varrho_1}{\varrho_0}}\|\mathbf{f}\|_{\mathrm{L}^2(\Omega)} + (1-\tau^\alpha)\sqrt{\frac{\varrho_1}{\varrho_0}}\big((-\dot{e}_{\alpha,1}) *_t \|\mathbf{f}\|_{\mathrm{L}^2(\Omega)}\big),
\end{aligned}
$$

which then implies, because of (4.4), for any $p\in[1,2]$ satisfying $p<\frac{1}{1-\alpha}$, that

$$
\begin{aligned}
&\left\|\frac{\partial}{\partial t}\big(\tau^\alpha\varrho\dot{\mathbf{u}}_n + \varrho(1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}_n)\big)\right\|_{\mathrm{L}^p(0,T;\mathrm{H}^{-1}(\Omega))}\\
&\leq CT^{\frac{1}{p}} + \varrho_1(1-\tau^\alpha)\|-\dot{e}_{\alpha,1}\|_{\mathrm{L}^p(0,T)}\|\mathbf{h}\|_{\mathrm{L}^2(\Omega)}\\
&\quad + C\|e_{\alpha,1}\|_{\mathrm{L}^p(0,T)}\|\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}\|_{\mathrm{L}^2(\Omega)}\\
&\quad + \tau^\alpha\sqrt{\frac{\varrho_1}{\varrho_0}}\|\mathbf{f}\|_{\mathrm{L}^p(0,T;\mathrm{L}^2(\Omega))} + (1-\tau^\alpha)\sqrt{\frac{\varrho_1}{\varrho_0}}\|-\dot{e}_{\alpha,1}\|_{\mathrm{L}^1(0,T)}\|\mathbf{f}\|_{\mathrm{L}^p(0,T;\mathrm{L}^2(\Omega))}.
\end{aligned}
$$

Hence, for any $p\in[1,2]$ such that $p<\frac{1}{1-\alpha}$, we have that

$$
\left\|\frac{\partial}{\partial t}\big(\tau^\alpha\varrho\dot{\mathbf{u}}_n + \varrho(1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}_n)\big)\right\|_{\mathrm{L}^p(0,T;\mathrm{H}^{-1}(\Omega))} \leq C,
$$

where C is a positive constant, independent of n. Consequently, by the Banach–Alaoglu theorem, there exists a subsequence $(\mathbf{u}_{n_\ell})_{\ell=1}^\infty$ such that

$$
\frac{\partial}{\partial t}\big(\tau^\alpha\varrho\dot{\mathbf{u}}_{n_\ell} + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}_{n_\ell})\big) \rightharpoonup \frac{\partial}{\partial t}\big(\tau^\alpha\varrho\dot{\mathbf{u}} + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}})\big),
$$

weakly in $\mathrm{L}^p(0,T;[\mathrm{H}^{-1}(\Omega)]^3)$ for any $p\in[1,2]$ such that $p<\frac{1}{1-\alpha}$. As

$$
\tau^\alpha\varrho\dot{\mathbf{u}} + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}) \in \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^3)
$$

and

$$\frac{\partial}{\partial t}\big(\tau^\alpha \varrho \dot{\mathbf{u}} + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}})\big) \in \mathrm{L}^1(0,T;[\mathrm{H}^{-1}(\Omega)]^3),$$

it once again follows, thanks to the continuous embedding of $[\mathrm{L}^2(\Omega)]^3$ into $[\mathrm{H}^{-1}(\Omega)]^3$, that

$$\tau^\alpha \varrho \dot{\mathbf{u}} + (1-\tau^\alpha)(-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}}) \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^3). \tag{4.20}$$

However, as $\varrho\dot{\mathbf{u}} \in \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^3)$, we have that $t \in [0,T] \mapsto (\varrho\dot{\mathbf{u}}(t),\mathbf{w})$ belongs to $\mathrm{L}^\infty(0,T)$ for each $\mathbf{w} \in [\mathrm{L}^2(\Omega)]^3$, and therefore, thanks to the smoothing property of the convolution, the function $t \in [0,T] \mapsto -\dot{e}_{\alpha,1}(t) *_t (\varrho\dot{\mathbf{u}}(t),\mathbf{w})$ belongs to $\mathrm{C}([0,T])$. Consequently,

$$t \in [0,T] \mapsto (-\dot{e}_{\alpha,1}(t) *_t \varrho\dot{\mathbf{u}}(t),\mathbf{w}) \in \mathrm{C}([0,T]) \qquad \forall\, \mathbf{w} \in [\mathrm{L}^2(\Omega)]^3,$$

meaning that $(1-\tau^\alpha)(-\dot{e}_{\alpha,1}) *_t \varrho\dot{\mathbf{u}} \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^3)$, and therefore by (4.20), also $\varrho\dot{\mathbf{u}} \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^3)$. Because $\varrho_0 \le \varrho(\mathbf{x}) \le \varrho_1$ a.e. on Ω, it then follows that

$$\dot{\mathbf{u}} \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^3).$$

That completes the proof of the existence of a weak solution.

Step 2: Proof of the energy inequality. Next we prove that weak solutions whose existence we have thus proved satisfy the energy inequality in the statement of the theorem. Our starting point is (4.12). By Lemma 3.1, we have that, for all $t \in (0,T]$,

$$\begin{aligned}&\int_0^t \int_\Omega \frac{\partial}{\partial s}(-\dot{e}_{\alpha,1} *_s \sqrt{\varrho}\dot{\mathbf{u}}_n)(s)\cdot\sqrt{\varrho}\dot{\mathbf{u}}_n(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\\ &\quad\ge \frac{1}{2}(-\dot{e}_{\alpha,1}(\cdot) *_t \|\sqrt{\varrho}\dot{\mathbf{u}}_n(\cdot)\|^2_{\mathrm{L}^2(\Omega)})(t) + \frac{1}{2}\int_0^t -\dot{e}_{\alpha,1}(s)\|\sqrt{\varrho}\dot{\mathbf{u}}_n(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s,\end{aligned}$$

and each of the two terms on the right-hand side is nonnegative. By omitting the first term from the right-hand side of this equality, and substituting the resulting inequality into (4.12) we have that, for all $t \in (0,T]$,

$$\begin{aligned}&\frac{\tau^\alpha}{2}\|\dot{\mathbf{u}}_n(t)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}_n(t))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_n(t)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\\ &\quad+\frac{1-\tau^\alpha}{2}\int_0^t -\dot{e}_{\alpha,1}(s)\|\dot{\mathbf{u}}_n(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s \le A(t)\exp(t+1-e_{\alpha,1}(t)).\end{aligned} \tag{4.21}$$

As $\dot{\mathbf{u}}_{n_\ell} \rightharpoonup \dot{\mathbf{u}}$ weakly* in $\mathrm{L}^\infty(0,T;[\mathrm{L}^2_\varrho(\Omega)]^3)$, the weak lower-semicontinuity of the norm function and (4.21) imply that

$$\begin{aligned}\|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)} \le \|\dot{\mathbf{u}}\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2_\varrho(\Omega))} &\le \liminf_{\ell\to\infty}\|\dot{\mathbf{u}}_{n_\ell}\|^2_{\mathrm{L}^\infty(0,t;\mathrm{L}^2_\varrho(\Omega))}\\ &\le A(t)\exp(t+1-e_{\alpha,1}(t))\end{aligned} \tag{4.22}$$

for all $t \in (0,T]$ and a.e. $s \in (0,t]$. Similarly, because $\mathbf{u}_{n_\ell} \rightharpoonup \mathbf{u}$ weakly* in $\mathrm{L}^\infty(0,T;[\mathrm{H}^1_0(\Omega)]^3)$,

$$\begin{aligned}
&\left[\frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}(s))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\right] \\
&\quad \le \operatorname{ess.sup}_{s\in(0,t]}\left[\frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}(s))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\right] \\
&\quad \le \liminf_{\ell\to\infty}\left\{\operatorname{ess.sup}_{s\in(0,t]}\left[\frac{\mu}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}_{n_\ell}(s))\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}_{n_\ell}(s)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\right]\right\} \\
&\quad \le A(t)\exp(t+1-e_{\alpha,1}(t)) \qquad (4.23)
\end{aligned}$$

for all $t \in (0,T]$ and a.e. $s \in (0,t]$. Finally, because $(-e_{\alpha,1})^{\frac{1}{2}}\dot{\mathbf{u}}_{n_\ell} \rightharpoonup (-e_{\alpha,1})^{\frac{1}{2}}\dot{\mathbf{u}}$ weakly in the function space $\mathrm{L}^2(0,T;[\mathrm{L}^2_\varrho(\Omega)]^3)$, we have that

$$\begin{aligned}
\int_0^t -\dot{e}_{\alpha,1}(s)\|\dot{\mathbf{u}}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s &\le \liminf_{\ell\to\infty}\int_0^t -\dot{e}_{\alpha,1}(s)\|\dot{\mathbf{u}}_{n_\ell}(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s \\
&\le A(t)\exp(t+1-e_{\alpha,1}(t))
\end{aligned} \qquad (4.24)$$

for all $t \in (0,T]$. Summing (4.22)–(4.24) we deduce the asserted energy inequality (4.7).

Step 3: Attainment of the initial conditions for $\mathbf{u}$ *and* $\dot{\mathbf{u}}$. Next, we shall prove that the initial condition $\mathbf{u}(0,\cdot) = \mathbf{g}(\cdot)$ is satisfied in the sense of continuous functions from $[0,T]$ into $[\mathrm{L}^2(\Omega)]^3$. To this end, we note that

$$\|\mathbf{u}(0,\cdot) - \mathbf{u}_{n_\ell}(0,\cdot)\|_{\mathrm{L}^2(\Omega)} \le \|\mathbf{u} - \mathbf{u}_{n_\ell}\|_{\mathrm{C}([0,T];\mathrm{L}^2(\Omega))} \to 0 \qquad \text{as } \ell\to\infty,$$

thanks to (4.17). Since $\mathbf{u}_{n_\ell}(0,\cdot) = P_{n_\ell}\mathbf{g}(\cdot) \to \mathbf{g}(\cdot)$ strongly in $[\mathrm{L}^2_\varrho(\Omega)]^3 \simeq [\mathrm{L}^2(\Omega)]^3$ as $\ell\to\infty$, we finally deduce by the triangle inequality that $\mathbf{u}(0,\cdot) - \mathbf{g}(\cdot) = 0$. Therefore, $\mathbf{u}(0,\cdot) = \mathbf{g}(\cdot)$, with $\mathbf{u} \in \mathrm{C}([0,T];[\mathrm{L}^2(\Omega)]^3)$.

To show that the initial condition, $\dot{\mathbf{u}}(0,\cdot) = \mathbf{h}(\cdot)$ is satisfied we note that, thanks to $(4.1)_1$ and $(4.1)_2$, we have $\mathbf{u} \in \mathrm{W}^{1,\infty}(0,T;[\mathrm{L}^2(\Omega)]^3) = \mathrm{C}^{0,1}([0,T];[\mathrm{L}^2(\Omega)]^3)$, so we can perform partial integration with respect to t in the first term on the left-hand side of (4.2), resulting in

$$\begin{aligned}
&-\tau^\alpha(\varrho\mathbf{u}(0,\cdot),\dot{\mathbf{v}}(0,\cdot)) - \tau^\alpha\int_0^T(\varrho\dot{\mathbf{u}}(s,\cdot),\dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s \\
&-(1-\tau^\alpha)\int_0^T((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot),\dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s \\
&+\int_0^T\big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)))\mathbf{I}\,,\boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s \\
&= -\tau^\alpha(\varrho\mathbf{g},\dot{\mathbf{v}}(0,\cdot)) + \tau^\alpha(\varrho\mathbf{h},\mathbf{v}(0,\cdot)) + \int_0^T\langle\mathbf{b}(s,\cdot),\mathbf{v}(s,\cdot)\rangle\,\mathrm{d}s
\end{aligned}$$

for all $\mathbf{v} \in \mathrm{W}^{2,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}^1_0(\Omega)]^3)$ with $\mathbf{v}(T,\cdot) = 0$ and $\dot{\mathbf{v}}(T,\cdot) = 0$. As $\mathbf{u}(0,\cdot) = \mathbf{g}(\cdot)$, the first term on the left-hand side and the first term on the right-hand side cancel, whereby

$$\begin{aligned}
&-\tau^\alpha \int_0^T (\varrho\dot{\mathbf{u}}(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s - (1-\tau^\alpha)\int_0^T ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s \\
&+ \int_0^T \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)))\mathbf{I}\,, \boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s \\
&= \tau^\alpha(\varrho\mathbf{h}, \mathbf{v}(0,\cdot)) + \int_0^T \langle \mathbf{b}(s,\cdot), \mathbf{v}(s,\cdot)\rangle\,\mathrm{d}s
\end{aligned}$$

for all $\mathbf{v} \in \mathrm{W}^{2,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}^1_0(\Omega)]^3)$ with $\mathbf{v}(T,\cdot) = 0$ and $\dot{\mathbf{v}}(T,\cdot) = 0$. As the set of all such $\mathbf{v}$ is dense in the set of all $\mathbf{v} \in \mathrm{W}^{1,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}^1_0(\Omega)]^3)$ with $\mathbf{v}(T,\cdot) = 0$, it follows that

$$\begin{aligned}
&-\tau^\alpha \int_0^T (\varrho\dot{\mathbf{u}}(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s - (1-\tau^\alpha)\int_0^T ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d} \\
&\qquad + \int_0^T \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)))\mathbf{I}\,, \boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s \qquad (4.25)\\
&\qquad = \tau^\alpha(\varrho\mathbf{h}, \mathbf{v}(0,\cdot)) + \int_0^T \langle \mathbf{b}(s,\cdot), \mathbf{v}(s,\cdot)\rangle\,\mathrm{d}s
\end{aligned}$$

holds for all $\mathbf{v} \in \mathrm{W}^{1,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}^1_0(\Omega)]^3)$ with $\mathbf{v}(T,\cdot) = 0$.

We fix a $t_0 \in (0,T)$ and for $\varepsilon \in (0, T-t_0)$ we define

$$\varphi_\varepsilon(t) := \begin{cases} 1 & \text{for } 0 \le t \le t_0, \\ 1 - \frac{1}{\varepsilon}(t-t_0) & \text{for } t_0 < t < t_0+\varepsilon, \\ 0 & \text{for } t_0+\varepsilon \le t \le T. \end{cases}$$

Clearly, $\varphi_\varepsilon \in \mathrm{C}^{0,1}([0,T])$, the weak derivative of φ_ε is $\varphi'_\varepsilon = -\frac{1}{\varepsilon}\chi_{(t_0,t_0+\varepsilon)}$, and $\varphi_\varepsilon(T) = 0$. Hence, for any $\mathbf{w} \in [\mathrm{H}^1(\Omega)]^3$, and taking $\mathbf{v} = \varphi_\varepsilon\mathbf{w}$ in (4.25), we have that

$$\begin{aligned}
&\tau^\alpha \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} (\varrho\dot{\mathbf{u}}(s,\cdot), \mathbf{w}(\cdot))\,\mathrm{d}s \\
&\quad + (1-\tau^\alpha)\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot), \mathbf{w}(\cdot))\,\mathrm{d}s \\
&\quad + \int_0^{t_0+\varepsilon} \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)))\mathbf{I}\,, \varphi_\varepsilon(s)\,\boldsymbol{\varepsilon}(\mathbf{w}(\cdot))\big)\,\mathrm{d}s \qquad (4.26)\\
&= \tau^\alpha(\varrho\mathbf{h}, \mathbf{w}) + \int_0^{t_0+\varepsilon} \langle \mathbf{b}(s,\cdot), \varphi_\varepsilon(s)\mathbf{w}(\cdot)\rangle\,\mathrm{d}s
\end{aligned}$$

for all $\mathbf{w} \in [\mathrm{H}^1_0(\Omega)]^3$. As $\varrho\dot{\mathbf{u}} \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^3)$ and $(-\dot{e}_{\alpha,1}) *_t \varrho\dot{\mathbf{u}} \in \mathrm{C}([0,T];[\mathrm{L}^2(\Omega)]^3)$ (cf. the end of STEP 1), we can pass to the limit $\varepsilon \to 0_+$ in (4.26), with $t_0 \in (0,T)$ fixed, to deduce by applying Lebesgue's differentiation theorem to the first and the second integral on the left-hand side of (4.26), recalling the continuity of the integrands in those integrals as functions of the integration variable s, for $\mathbf{w} \in [\mathrm{H}^1_0(\Omega)]^3$ fixed, and using the continuity of the integral with respect to its (upper) limit in the third integral on the left-hand side of (4.26) and the second term on the right-hand side of (4.26), that

$$\begin{aligned}\tau^\alpha\,(\varrho\dot{\mathbf{u}}(t_0,\cdot),\mathbf{w}(\cdot)) &+ (1-\tau^\alpha)\,((-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}})(t_0,\cdot),\mathbf{w}(\cdot))\\ &+ \int_0^{t_0} \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)))\mathbf{I}\,,\boldsymbol{\varepsilon}(\mathbf{w}(\cdot))\big)\,\mathrm{d}s\\ &= \tau^\alpha(\varrho\mathbf{h},\mathbf{w}) + \int_0^{t_0} \langle \mathbf{b}(s,\cdot),\mathbf{w}(\cdot)\rangle\,\mathrm{d}s\end{aligned} \tag{4.27}$$

for all $\mathbf{w} \in [\mathrm{H}^1_0(\Omega)]^3$ and all $t_0 \in (0,T)$. Next, with $\mathbf{w} \in [\mathrm{H}^1_0(\Omega)]^3$ fixed, we pass to the limit $t_0 \to 0_+$ in (4.27), noting that the third term on the left-hand side and the second term on the right-hand side both vanish in this limit thanks to the continuity of these integrals as functions of t_0, and that, for the same reason and by Fubini's theorem, also

$$\begin{aligned}&\lim_{t_0\to 0_+} ((-\dot{e}_{\alpha,1} *_t \varrho\dot{\mathbf{u}})(t_0,\cdot),\mathbf{w}(\cdot))\\ &\quad = -\lim_{t_0\to 0_+} \int_0^{t_0} \dot{e}_{\alpha,1}(t)\,((\varrho\dot{\mathbf{u}})(t-t_0,\cdot),\mathbf{w}(\cdot))\,\mathrm{d}t = 0.\end{aligned}$$

Consequently, upon passage to the limit $t_0 \to 0_+$, the equality (4.27) collapses to

$$\tau^\alpha\,(\varrho\dot{\mathbf{u}}(0,\cdot),\mathbf{w}(\cdot)) = \tau^\alpha(\varrho\mathbf{h},\mathbf{w}) \qquad \forall\,\mathbf{w} \in [\mathrm{H}^1_0(\Omega)]^3.$$

As $\tau \in (0,1]$ and $[\mathrm{H}^1_0(\Omega)]^3$ is dense in $[\mathrm{L}^2(\Omega)]^3$ it then follows that

$$(\varrho\dot{\mathbf{u}}(0,\cdot),\mathbf{w}(\cdot)) = (\varrho\mathbf{h},\mathbf{w}) \qquad \forall\,\mathbf{w} \in [\mathrm{L}^2(\Omega)]^3.$$

Because $\varrho \in \mathrm{L}^\infty(\Omega)$ and $\varrho(x) \geq \rho_0 > 0$ a.e. in Ω (cf. (1.5)), we finally have that

$$(\dot{\mathbf{u}}(0,\cdot),\mathbf{w}(\cdot)) = (\mathbf{h},\mathbf{w}) \qquad \forall\,\mathbf{w} \in [\mathrm{L}^2(\Omega)]^3,$$

and therefore $\dot{\mathbf{u}}(0,\cdot) = \mathbf{h}(\cdot)$, as an equality in $\mathrm{C}_w([0,T],[\mathrm{L}^2(\Omega)]^3)$.

Step 4. Uniqueness of the solution. Having shown, for initial data $\mathbf{g}$, $\mathbf{h}$, $\mathbf{S}$ and the source term $\mathbf{f}$ satisfying (3.11), and for any $\tau \in (0,1]$, $\varrho,\mu,\lambda \in \mathrm{L}^\infty(\Omega)$, with $\varrho(\mathbf{x}) \geq \varrho_0 > 0$, $\mu(\mathbf{x}) \geq \mu_0 > 0$ and $\lambda(\mathbf{x}) \geq 0$ for a.e. $\mathbf{x} \in \Omega$, and $\alpha \in (0,1)$, the existence of a weak solution (4.1) to the problem (1.2), (1.3), (2.5) satisfying the equality (4.2) with (4.3) we now to turn to the proof of uniqueness of weak solutions. Suppose that $\mathbf{u}_1$ and $\mathbf{u}_2$ are two weak solutions to (1.2), (1.3), (2.5) subject to the same initial data and source term. Then, they both satisfy (4.25), and therefore, thanks to the linearity of the problem their difference $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$ satisfies

$$-\tau^\alpha \int_0^T (\varrho\dot{\mathbf{u}}(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s - (1-\tau^\alpha)\int_0^T ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot), \dot{\mathbf{v}}(s,\cdot))\,\mathrm{d}s$$
$$+ \int_0^T \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(s,\cdot)))\mathbf{I}\,, \boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s = 0 \tag{4.28}$$

for all $\mathbf{v} \in \mathrm{W}^{1,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}^1_0(\Omega)]^3)$ with $\mathbf{v}(T,\cdot) = 0$. We fix a $t_0 \in (0,T)$, and let

$$\mathbf{v}(t,\mathbf{x}) := \begin{cases} -\int_t^{t_0} \mathbf{u}(s,\mathbf{x})\,\mathrm{d}s & \text{for } 0 < t \le t_0, \\ \mathbf{0} & \text{for } t_0 < t < T. \end{cases}$$

Clearly, $\mathbf{v} \in \mathrm{W}^{1,\infty}(0,T;[\mathrm{H}^1_0(\Omega)]^3)$ with $\mathbf{v}(T,\cdot) = \mathbf{0}$, and hence the function $\mathbf{v}$, thus defined, is an admissible test function. We therefore have from (4.28) that

$$-\tau^\alpha \int_0^{t_0} (\varrho\dot{\mathbf{u}}(s,\cdot), \mathbf{u}(s,\cdot))\,\mathrm{d}s - (1-\tau^\alpha)\int_0^{t_0} ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot), \mathbf{u}(s,\cdot))\,\mathrm{d}s$$
$$+ \int_0^{t_0} \big(2\mu\boldsymbol{\varepsilon}(\dot{\mathbf{v}}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\dot{\mathbf{v}}(s,\cdot)))\mathbf{I}\,, \boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\big)\,\mathrm{d}s = 0. \tag{4.29}$$

Focusing in particular on the first and the third term on the left-hand side of (4.29) we then have that

$$-\frac{1}{2}\tau^\alpha \int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{u}(s,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s - (1-\tau^\alpha)\int_0^{t_0} ((-\dot{e}_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot), \mathbf{u}(s,\cdot))\,\mathrm{d}s$$
$$+ \int_0^{t_0} \left(\frac{\mathrm{d}}{\mathrm{d}s}\|\boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\right)\mathrm{d}s = 0. \tag{4.30}$$

As

$$\begin{aligned}(-\dot{e}_{\alpha,1} *_s \dot{\mathbf{u}})(s,\cdot) &= \frac{\mathrm{d}}{\mathrm{d}s}(-e_{\alpha,1} *_s \dot{\mathbf{u}})(s,\cdot) - (-e_{\alpha,1}(0))\dot{\mathbf{u}}(s,\cdot) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}(-e_{\alpha,1} *_s \dot{\mathbf{u}})(s,\cdot) + \dot{\mathbf{u}}(s,\cdot),\end{aligned}$$

inserting this into the second term on the left-hand side of (4.30) yields

$$\begin{aligned}&-\frac{1}{2}\tau^\alpha \int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{u}(s,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s \\ &-(1-\tau^\alpha)\int_0^{t_0} \left(\frac{\mathrm{d}}{\mathrm{d}s}(-e_{\alpha,1} *_s \varrho\dot{\mathbf{u}})(s,\cdot) + \varrho\dot{\mathbf{u}}(s,\cdot), \mathbf{u}(s,\cdot)\right)\mathrm{d}s \\ &+ \int_0^{t_0} \left(\frac{\mathrm{d}}{\mathrm{d}s}\|\boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{v}(s,\cdot)))\|^2_{\mathrm{L}^2_\lambda(\Omega)}\right)\mathrm{d}s = 0.\end{aligned}$$

Hence, by performing partial integration in the second integral on the left-hand side, and because $\mathbf{v}(t_0,\cdot)=0$, it follows that

$$\begin{aligned}
&-\frac{1}{2}\tau^{\alpha}\left(\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}-\|\mathbf{u}(0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}\right)\\
&-\frac{1}{2}(1-\tau^{\alpha})\left(\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}-\|\mathbf{u}(0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}\right)\\
&-(1-\tau^{\alpha})\int_0^{t_0}\left((e_{\alpha,1}*_s\sqrt{\varrho}\dot{\mathbf{u}})(s,\cdot),\sqrt{\varrho}\dot{\mathbf{u}}(s,\cdot)\right)\,\mathrm{d}s\\
&+(1-\tau^{\alpha})\left((e_{\alpha,1}*_s\varrho\dot{\mathbf{u}})(s,\cdot),\mathbf{u}(s,\cdot)\right)\big|_{s=0}^{s=t_0}\\
&-\|\boldsymbol{\varepsilon}(\mathbf{v}(0,\cdot))\|^2_{\mathrm{L}^2_{\mu}(\Omega)}-\frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{v}(0,\cdot)))\|^2_{\mathrm{L}^2_{\lambda}(\Omega)}=0.
\end{aligned}$$

Again, because $\mathbf{u}\in\mathrm{C}([0,T];[\mathrm{L}^2(\Omega)]^3)$ satisfies $\mathbf{u}(0,\mathbf{x})=\mathbf{0}$ for a.e. $\mathbf{x}\in\Omega$, rearrangement yields

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}+(1-\tau^{\alpha})\int_0^{t_0}\left((e_{\alpha,1}*_s\sqrt{\varrho}\dot{\mathbf{u}})(s,\cdot),\sqrt{\varrho}\dot{\mathbf{u}}(s,\cdot)\right)\,\mathrm{d}s\\
&\quad+\|\boldsymbol{\varepsilon}(\mathbf{v}(0,\cdot))\|^2_{\mathrm{L}^2_{\mu}(\Omega)}+\frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{v}(0,\cdot)))\|^2_{\mathrm{L}^2_{\lambda}(\Omega)}\\
&\qquad=(1-\tau^{\alpha})\left((e_{\alpha,1}*_t\varrho\dot{\mathbf{u}})(t_0,\cdot),\mathbf{u}(t_0,\cdot)\right).
\end{aligned}\tag{4.31}$$

Thus, thanks to Lemma 3.1 the second term on the left-hand side of (4.31) can be bounded below, yielding

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}\\
&\quad+\frac{1}{2}(1-\tau^{\alpha})\left[(e_{\alpha,1}*_t\|\dot{\mathbf{u}}(\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)})(t_0)+\int_0^{t_0}e_{\alpha,1}(s)\|\dot{\mathbf{u}}(s,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}\,\mathrm{d}s\right]\\
&\quad+\|\boldsymbol{\varepsilon}(\mathbf{v}(0,\cdot))\|^2_{\mathrm{L}^2_{\mu}(\Omega)}+\frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{v}(0,\cdot)))\|^2_{\mathrm{L}^2_{\lambda}(\Omega)}\\
&\qquad\le(1-\tau^{\alpha})\left((e_{\alpha,1}*_t\varrho\dot{\mathbf{u}})(t_0,\cdot),\mathbf{u}(t_0,\cdot)\right).
\end{aligned}\tag{4.32}$$

Next, we will show that for any $t_0>0$ such that $t_0\le\min(T,1)$ the term on the right-hand side of (4.32) can be completely absorbed into the left-hand side of the inequality. Indeed, by Young's inequality, Minkowski's integral inequality, and the Cauchy–Schwarz inequality,

$$\begin{aligned}
&(1-\tau^{\alpha})\left((e_{\alpha,1}*_t\varrho\dot{\mathbf{u}})(t_0,\cdot),\mathbf{u}(t_0,\cdot)\right)\\
&\le\frac{1}{2}(1-\tau^{\alpha})\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}+\frac{1}{2}(1-\tau^{\alpha})\,\|(e_{\alpha,1}*_t\dot{\mathbf{u}})(t_0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}\\
&\le\frac{1}{2}(1-\tau^{\alpha})\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_{\varrho}(\Omega)}+\frac{1}{2}(1-\tau^{\alpha})\left[(e_{\alpha,1}*_t\|\dot{\mathbf{u}}\|_{\mathrm{L}^2_{\varrho}(\Omega)})(t_0)\right]^2
\end{aligned}$$

$$\le \frac{1}{2}(1-\tau^\alpha)\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}(1-\tau^\alpha)\left[\int_0^{t_0} e_{\alpha,1}(t_0-s)\,\mathrm{d}s\right]\left[(e_{\alpha,1} *_t \|\dot{\mathbf{u}}\|^2_{\mathrm{L}^2_\varrho(\Omega)})(t_0)\right].$$

As $e_{\alpha,1}(0)=1$ and $t\in[0,\infty)\mapsto e_{\alpha,1}(t)$ is positive and monotonic decreasing, it follows that

$$(1-\tau^\alpha)\left((e_{\alpha,1} *_t \varrho\dot{\mathbf{u}})(t_0,\cdot),\mathbf{u}(t_0,\cdot)\right) \le \frac{1}{2}(1-\tau^\alpha)\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}(1-\tau^\alpha)t_0\left[(e_{\alpha,1} *_t \|\dot{\mathbf{u}}\|^2_{\mathrm{L}^2_\varrho(\Omega)})(t_0)\right].$$

Substituting this into the right-hand side of (4.32) and, because $\tau\in(0,1]$ and $t_0\le 1$, yields

$$\frac{1}{2}\tau^\alpha\|\mathbf{u}(t_0,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}(1-\tau^\alpha)\left[\int_0^{t_0} e_{\alpha,1}(s)\|\dot{\mathbf{u}}(s,\cdot)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s\right] + \|\boldsymbol{\varepsilon}(\mathbf{v}(0,\cdot))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{v}(0,\cdot)))\|^2_{\mathrm{L}^2_\lambda(\Omega)} \le 0. \tag{4.33}$$

Thus we deduce that $\mathbf{u}(t,\cdot)=\mathbf{0}$ for all $t\in[0,t_0]$, for any $t_0>0$ such that $t_0\le\min(T,1)$. If $t_0<T$, then having shown that $\mathbf{u}(t,\cdot)=\mathbf{0}$ for all $t\in[0,t_0]$ we repeat the argument on successive time intervals $[kt_0,\min(T,(k+1)t_0)]$, $k=1,2,\ldots,K$, with initial data $\mathbf{u}(kt_0,\cdot)=\mathbf{0}$, $\mathbf{u}_t(kt_0,\cdot)=\mathbf{0}$, where K is the (unique) positive integer such that $Kt_0<T$ and $(K+1)t_0\ge T$. Hence, $\mathbf{u}(t,\cdot)=\mathbf{0}$ for all $t\in[0,T]$. Thus we have shown the uniqueness of the weak solution.

*S*tep 5. Continuous dependence of the solution on the data. As the problem under consideration is linear, the energy inequality (4.7) implies continuous dependence of weak solutions on the initial data and the load vector.

Step 6. Attainment of the initial condition for $\boldsymbol{\sigma}$. By (4.5) and noting that

$$\mathcal{L}^{-1}\left(\frac{1+p^\alpha}{1+\tau^\alpha p^\alpha}\right) = (\tau^{-\alpha}-1)\,\dot{e}_\alpha(t,\tau^{-\alpha}) + \tau^{-\alpha}\delta, \qquad \text{and}$$

$$\mathcal{L}^{-1}\left(\frac{p^{\alpha-1}}{1+\tau^\alpha p^\alpha}\right) = \tau^{-\alpha}\,e_\alpha(t,\tau^{-\alpha}),$$

we have that

$$\begin{aligned}\tau^\alpha\boldsymbol{\sigma}(t,\cdot) &= (1-\tau^\alpha)\dot{e}_\alpha(t,\tau^{-\alpha}) *_t (2\mu\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)))\mathbf{I})\\ &\quad + (2\mu\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)) + \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)))\mathbf{I})\\ &\quad + e_\alpha(t,\tau^{-\alpha})\,(\tau^\alpha\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I})\\ &=: \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3.\end{aligned} \tag{4.34}$$

We begin by showing that $\boldsymbol{\sigma}$ belongs to $\mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$.

As $e_\alpha(\cdot,\tau^{-\alpha}) \in \mathrm{C}([0,\infty))$ and $\tau^\alpha \mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I} \in [\mathrm{L}^2(\Omega)]^{3\times 3}$ thanks to (3.11), we have that $e_\alpha(\cdot,\tau^{-\alpha})\,(\tau^\alpha \mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I})$ belongs to the function space $\mathrm{C}([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$, and therefore also to the space $\mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$, implying that $\mathbf{A}_3 \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$.

To show that $\mathbf{A}_2 \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$, we begin by recalling that $\mathbf{A}_2 \in \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^{3\times 3})$, because $\mathbf{u}$, as a weak solution, belongs to $\mathrm{L}^\infty(0,T;[\mathrm{H}^1_0(\Omega)]^3)$. Together with the fact that

$$\dot{\mathbf{A}}_2 \in \mathrm{L}^\infty(0,T;[\mathrm{H}^{-1}(\Omega)]^{3\times 3}), \tag{4.35}$$

which we shall now show, and the continuous and dense embedding of $[\mathrm{L}^2(\Omega)]^{3\times 3}$ into $[\mathrm{H}^{-1}(\Omega)]^3$, this will then yield that the function $\mathbf{A}_2$ belongs to $\mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$ (cf., again, eq. (8.49) in Lemma 8.1, Ch. 3 of [13]), as desired. To show that (4.35) holds, we appeal to the following result from the theory of Sobolev spaces of Banach-space-valued functions (cf., for example, Theorem 1.4.40 on p.15 in [7]):

Suppose that X is a reflexive Banach space, I is a nonempty bounded open interval of $\mathbb{R}$, and $u \in \mathrm{L}^p(I;X)$ for some $p \in [1,\infty]$. Then, $u \in \mathrm{W}^{1,p}(I;X)$ if, and only if, there exists a function $g \in \mathrm{L}^p(I;\mathbb{R})$ such that

$$\|u(t)-u(s)\|_X \le \left|\int_s^t g(\tau)\,\mathrm{d}\tau\right|$$

for almost all $s,t \in I$, i.e., for all s,t outside a common null set.

We shall apply this result with $p=\infty$, $X = [\mathrm{H}^{-1}(\Omega)]^{3\times 3}$, and $g(\tau) = \|\dot{\mathbf{u}}(\tau,\cdot)\|_{\mathrm{L}^2(\Omega)}$. Clearly,

$$\begin{aligned}
\|\mathbf{A}_2(t)-\mathbf{A}_2(s)\|_X &= \|2\mu\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)-\mathbf{u}(s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)-\mathbf{u}(s,\cdot)))\mathbf{I}\|_X \\
&= \left\|2\mu\boldsymbol{\varepsilon}\left(\int_s^t \dot{\mathbf{u}}(\tau,\cdot)\,\mathrm{d}\tau\right) + \lambda\,\mathrm{tr}\left(\boldsymbol{\varepsilon}\left(\int_s^t \dot{\mathbf{u}}(\tau,\cdot)\,\mathrm{d}\tau\right)\right)\mathbf{I}\right\|_X \\
&\le 2\mu\left\|\boldsymbol{\varepsilon}\left(\int_s^t \dot{\mathbf{u}}(\tau,\cdot)\,\mathrm{d}\tau\right)\right\|_X + \lambda\left\|\mathrm{tr}\left(\boldsymbol{\varepsilon}\left(\int_s^t \dot{\mathbf{u}}(\tau,\cdot)\,\mathrm{d}\tau\right)\right)\mathbf{I}\right\|_X \\
&\le 2\mu\left\|\int_s^t \dot{\mathbf{u}}(\tau,\cdot)\,\mathrm{d}\tau\right\|_{\mathrm{L}^2(\Omega)} + 3\lambda\left\|\int_s^t \dot{\mathbf{u}}(\tau,\cdot)\,\mathrm{d}\tau\right\|_{\mathrm{L}^2(\Omega)} \\
&\le (2\mu+3\lambda)\left|\int_s^t \|\dot{\mathbf{u}}(\tau,\cdot)\|_{\mathrm{L}^2(\Omega)}\,\mathrm{d}\tau\right| < \infty \qquad \forall\, s,t \in [0,T],
\end{aligned}$$

because $\dot{\mathbf{u}} \in \mathrm{L}^\infty(0,T;[\mathrm{L}^2(\Omega)]^3)$, where we have used the bound $\|\boldsymbol{\varepsilon}(\mathbf{w})\|_X \le \|\mathbf{w}\|_{\mathrm{L}^2(\Omega)}$ with $X = [\mathrm{H}^{-1}(\Omega)]^{3\times 3}$. Therefore, $\mathbf{A}_2 \in \mathrm{W}^{1,\infty}(0,T;[\mathrm{H}^{-1}(\Omega)]^{3\times 3})$, whereby also $\dot{\mathbf{A}}_2 \in \mathrm{L}^\infty(0,T;[\mathrm{H}^{-1}(\Omega)]^{3\times 3})$. Thus we have shown that $\mathbf{A}_2 \in \mathrm{C}_w([0,T];[\mathrm{L}^3(\Omega)]^{3\times 3})$.

Concerning the term $\mathbf{A}_1$, as $\mathbf{A}_1 = (1-\tau^{\alpha})\dot{e}_{\alpha}(t,\tau^{-\alpha}) *_t \mathbf{A}_2$, and $\mathbf{A}_2 \in \mathrm{C}_w([0,T];[\mathrm{L}^3(\Omega)]^{3\times 3})$, also $\mathbf{A}_1 \in \mathrm{C}_w([0,T];[\mathrm{L}^3(\Omega)]^{3\times 3})$.

By summing $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ we thus deduce that $\boldsymbol{\sigma} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$. It remains to prove the attainment of the initial condition $\boldsymbol{\sigma}(0,\cdot) = \mathbf{S}(\cdot)$.

Thanks to Fubini's theorem and the continuity of the integral with respect to its (upper) limit,

$$\lim_{t\to 0_+} (\dot{e}_{\alpha}(t,\tau^{-\alpha}) *_t (2\mu\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)))\mathbf{I}), \boldsymbol{W})$$

$$= \lim_{t\to 0_+} \int_0^t \dot{e}_{\alpha}(s,\tau^{-\alpha})\,((2\mu\boldsymbol{\varepsilon}(\mathbf{u}(t-s,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t-s,\cdot)))\mathbf{I}), \boldsymbol{W})\,\mathrm{d}s = 0.$$

for all $\boldsymbol{W} \in [\mathrm{L}^2(\Omega)]^{3\times 3}$. Hence, by recalling that $\mathbf{A}_2 \in \mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$ and noting that

$$\lim_{t\to 0_+} (2\mu\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}(t,\cdot)))\mathbf{I}, \boldsymbol{W}) = (2\mu\boldsymbol{\varepsilon}(\mathbf{g}(\cdot)) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}(\cdot)))\mathbf{I}, \boldsymbol{W})$$

for all $\boldsymbol{W} \in [\mathrm{L}^2(\Omega)]^{3\times 3}$, and because $e_{\alpha}(0,\tau^{-\alpha}) = 1$, we have from (4.34) that

$$\lim_{t\to 0_+} (\boldsymbol{\sigma}(t,\cdot), \boldsymbol{W}(\cdot)) = (\mathbf{S}(\cdot), \boldsymbol{W}(\cdot)) \qquad \forall\, \boldsymbol{W} \in [\mathrm{L}^2(\Omega)]^{3\times 3}.$$

Therefore, $\boldsymbol{\sigma}(0,\cdot) = \mathbf{S}(\cdot)$ as an equality in $\mathrm{C}_w([0,T];[\mathrm{L}^2(\Omega)]^{3\times 3})$, as required. □

The results of the paper can be straightforwardly extended to initial-boundary-value problems for the fractional Zener wave equation with mixed homogeneous Dirichlet/nonhomogeneous Neumann boundary conditions, i.e., to problems where the domain boundary $\partial\Omega$ is the disjoint union of Γ_{D} and Γ_{N}, with Γ_{D} having positive two-dimensional surface measure,

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_{\mathrm{D}},$$

$$[(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}) \;+\; e_{\alpha,1}\,(\tau^{\alpha}\mathbf{S} - 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) - \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I})]\cdot\boldsymbol{n} = \boldsymbol{s} \text{ on } \Gamma_{\mathrm{N}},$$

where $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$, and the function $\boldsymbol{s} \in \mathrm{L}^{\infty}(0,T;[\mathrm{L}^2(\Gamma_N)]^3)$ is given, at the expense of adding a term of the form

$$\int_0^T \int_{\Gamma_{\mathrm{N}}} \boldsymbol{s}(t,\boldsymbol{\xi})\cdot\mathbf{v}(t,\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}t$$

to the right-hand side of (4.2), replacing the function space $[\mathrm{H}^1_0(\Omega)]^3$ throughout by the function space $[\mathrm{H}^1_{\Gamma_{\mathrm{D}},0}(\Omega)]^3$ consisting of all functions in $[\mathrm{H}^1(\Omega)]^3$ with zero trace on Γ_{D}, and $[\mathrm{H}^{-1}(\Omega)]^3$ signifying the dual space of $[\mathrm{H}^1_{\Gamma_{\mathrm{D}},0}(\Omega)]^3$. In the special case when the initial stress $\mathbf{S}$ is such that $\tau^{\alpha}\mathbf{S} = 2\mu\boldsymbol{\varepsilon}(\mathbf{g}) +$

$\lambda\,\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{g}))\mathbf{I}$, the Neumann boundary condition on Γ_{N} and the source term $\mathbf{b}$ in (4.2), defined by (4.3), are both simplified.

As a possible further, but now nontrivial, extension of the model (1.4), we note that Freed and Diethelm [10] have extended Fung's nonlinear constitutive law for soft biological tissues into a constitutive law involving fractional time-derivatives in the sense of Caputo, first in one space dimension and then in three space-dimensions. The model is derived in a configuration that differs from the current configuration by a rigid-body rotation; it being the polar configuration. Freed and Diethelm introduce mappings for the fractional-order operators of integration and differentiation between the polar and spatial configurations. They then use these mappings in the construction of their proposed viscoelastic model. The mathematical analysis of the associated set of partial differential equations, and the study of wave propagation governed by the associated nonlinear system of nonlocal evolution equations are beyond the scope of the present paper.

We have assumed throughout the paper that $\rho \geq \tau > 0$. When $\tau = 0$, only the first two of the three initial conditions stated in (1.2) need to be imposed. The existence of a unique solution in that case, as well as its continuous dependence on the data can be deduced from existing results in literature (see, for example, Theorem 3.3.1 in [18]). In particular, in the case of $\rho = \tau = 0$ the model collapses to the equations of classical linear elasticity. Thus we have confined ourselves to the case $\rho \geq \tau > 0$. If one were to include the case of $\tau = 0$ in our analysis, the third initial condition in (1.2) would need to be replaced by $\tau^{\alpha}\boldsymbol{\sigma}(0,\mathbf{x}) = \tau^{\alpha}\mathbf{S}(\mathbf{x})$, for $\mathbf{x} \in \Omega$.

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References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces.* Second Ed., Pure and Applied Mathematics (Amsterdam) # 140, Elsevier/Academic Press, Amsterdam (2003).

[2] T.M. Atanacković, A modified Zener model of a viscoelastic body. *Continuum Mech. Thermodyn.* **14** (2002), 137–148; DOI: 10.1007/s001610100056.

[3] R.L. Bagley and R.A. Calico, Fractional order state equations for the control of viscoelastically damped structures. *AIAA J. of Guidance, Control, and Dynamics* **14** (1991), 304–311; DOI: 10.2514/3.20641.

[4] R.L. Bagley and P.J. Torvik, On the fractional calculus model of viscoelastic behavior. *J. of Rheology* **30** (1986), 133–155; 10.1122/1.549887.

[5] J.W. Barrett and E. Süli, Reflections on Dubinskiĭ's nonlinear compact embedding theorem. *Publ. Inst. Math. (Beograd) (N.S.)* **91** (105) (2012), 95–110; DOI: 10.2298/PIM1205095B.

[6] M. Caputo and F. Mainardi, Linear models of dissipation in anelastic solids. *Rivista del Nuovo Cimento* **1** (1971), 161–198; DOI: 10.1007/BF02820620.

[7] T. Cazenave and A. Haraux. *An Introduction to Semilinear Evolution Equations.* Oxford Lecture Ser. in Mathematics and its Applications, Clarendon Press, Oxford (1998).

[8] M. Enelund and B.L. Josefson, Time domain FE-analysis of viscoelastic structures having constitutive relations involving fractional derivatives. In: *Proc. 37th Structures, Structural Dynamics and Materials Conference*, Salt Lake City, UT, USA, AIAA Washington DC (1996), 685–694.

[9] L. Figueroa and E. Süli, Greedy approximation of high-dimensional Ornstein–Uhlenbeck operators. *Foundations of Computational Mathematics* **12** (2012), 573–623; DOI: 10.1007/s10208-012-9122-z.

[10] A.D. Freed and K. Diethelm, Caputo derivatives in viscoelasticity: a non-linear finite-deformation theory for tissue. *Fract. Calc. Appl. Anal.* **10**, No 3 (2007), 219–248.

[11] S. Konjik, Lj. Oparnica, and D. Zorica, Waves in fractional Zener type viscoelastic media. *J. Math. Anal. Appl.* **365**, No 1 (2010), 259–268; DOI: 10.1016/j.jmaa.2009.10.043.

[12] S. Larsson and F. Saedpanah, The continuous Galerkin method for an integrodifferential equation modeling dynamic fractional order viscoelasticity. *IMA J. Numer. Anal.* **30** (2010), 964–986; DOI: 10.1093/imanum/drp014.

[13] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications, Vol. I.* Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York-Heidelberg (1972)

[14] T. Pirtz, Analysis of four-parameter fractional derivative model of real solid materials. *J. of Sound and Vibrations* **195** (1996), 103–115; DOI: 10.1006/jsvi.1996.0406.

[15] F. Seadpanah, Well-posedness of an integro-differential equation with positive type kernels modeling fractional order viscoelasticity.

European J. of Mechanics A / Solids **44** (2014), 201–211; DOI: 10.1016/j.euromechsol.2013.10.014.

[16] K. Šiškova, *Inverse Source Problems in Evolutionary PDE's.* Doctoral Thesis, Ghent University (2018); https://biblio.ugent.be/publication/8583819/file/8583821.pdf.

[17] R. Zacher, Boundedness of weak solutions to evolutionary partial integro-differential equations with discontinuous coefficients. *J. Math. Anal. Appl.* **348** (2008), 137–149; DOI: 10.1016/j.jmaa.2008.06.054.

[18] R. Zacher, *De Giorgi–Nash–Moser Estimates for Evolutionary Partial Integro-Differential Equations.* Habilitationsschrift. Naturwissenschaftlichen Fakultät III der Martin-Luther-Universität Halle-Wittenberg (2010). http://dx.doi.org/10.25673/387.

[19] C. Zener, *Elasticity and Anelasticity of Metals.* University of Chicago Press, Chicago (1948).

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