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ORIGINAL ARTICLE

Stability with Minuscule Structure for Chromatic Thresholds

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Abstract

The chromatic threshold $\delta_\chi(H)$ of a graph H is the infimum of $d > 0$ such that the chromatic number of every n-vertex H-free graph with minimum degree at least dn is bounded by a constant depending only on H and d. Allen, Böttcher, Griffiths, Kohayakawa, and Morris determined the chromatic threshold for every H; in particular, they showed that if $\chi(H) = r \geq 3$, then $\delta_\chi(H) \in \{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\}$. While the chromatic thresholds have been completely determined, rather surprisingly the structural behaviors of extremal graphs near the threshold remain unexplored. In this paper, we establish the stability theorems for chromatic threshold problems. We prove that every n-vertex H-free graph G with $\delta(G) \geq (\delta_\chi(H) - o(1))n$ and $\chi(G) = \omega(1)$ must be structurally close to one of the extremal configurations. Furthermore, we give a stronger stability result when H is a clique, showing that G admits a partition into independent sets and a small subgraph on a sublinear number of vertices. We show that this small subgraph has fractional chromatic number $2 + o(1)$ and is homomorphic to a Kneser graph defined by subsets of a logarithmic size set; both these two bounds are best possible. This is the first stability result that captures the lower-order structural features of extremal graphs. We also study two variations of chromatic thresholds. Replacing chromatic number by its fractional counterpart, we determine the fractional chromatic thresholds for all graphs. Another variation is the bounded-VC chromatic thresholds, which was introduced by Liu, Shangguan, Skokan, and Xu very recently. Extending work of Łuczak and Thomassé on the triangle case, we determine the bounded-VC chromatic thresholds for all cliques.

Keywords Chromatic thresholds · Stability · Kneser graphs

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1 Introduction

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to color its vertices so that adjacent vertices do not get the same color. The chromatic number is a central concept in graph theory. In the 1940s, Zykov [37] and Tutte (see Erdős [8]) first constructed triangle-free graphs with unbounded chromatic number. Later, Erdős in 1959 [8] extended this result, showing that for every $k, \ell \in \mathbb{N}$, there exist graphs with girth at least ℓ and chromatic number at least k by a surprising probabilistic construction. However, all these constructions are sparse graphs with low minimum degrees relative to the number of vertices.

In 1973, Erdős and Simonovits [11] raised the problem of bounding the chromatic number of triangle-free graphs with large minimum degree. They introduced what is now known as the *chromatic threshold* of a graph H:

$$\begin{aligned}\delta_\chi(H) := \inf\{d : \exists\, C = C(H,d) \text{ such that if } G \text{ is a graph on } n \text{ vertices,}\\ \text{with } \delta(G) \geq dn \text{ and } H \nsubseteq G, \text{ then } \chi(G) \leq C\}.\end{aligned}$$

Based on a construction by Hajnal, Erdős and Simonovits [11] conjectured that $\delta_\chi(K_3) = 1/3$. Over the past 50 years, the chromatic threshold problem has been one of the central topics in extremal graph theory. The conjecture of Erdős and Simonovits was eventually confirmed in 2002 by Thomassen [33]. Later, Goddard and Lyle [14] and Nikiforov [29] independently extended the result of Thomassen by showing that $\delta_\chi(K_r) = \frac{2r-5}{2r-3}$ for every $r \geq 3$, and Thomassen [34] showed that the chromatic threshold of every odd cycle of length at least five is 0. Further work on this topic can be found in [5, 6, 15, 20, 27, 28]. Building upon ideas and tools developed by previous papers, in a remarkable work, Allen, Böttcher, Griffiths, Kohayakawa, and Morris [1] completely determined the chromatic threshold for every H. In particular, they showed that for any graph H with $\chi(H) = r \geq 3$, $\delta_\chi(H) \in \{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\}$. We refer the interested readers to also [4, 18, 25] for more recent work related to this topic.

1.1 Stability for Chromatic Thresholds

The main concern of this paper is to study the *stability* of chromatic thresholds. Stability results demonstrate that graphs with nearly maximum size must be structurally close to extremal graphs. An archetype example of stability result is the one for Turán problem. Let $T_{n,r-1}$ be the *Turán graph*, which is the complete balanced $(r-1)$-partite graph on n vertices. Turán theorem [36] shows that among all n-vertex K_r-free graphs, $T_{n,r-1}$ has the maximum number of edges, and it is the unique graph with this property. In 1966, Erdős and Simonovits [10, 32] famously proved that for any graph H with $\chi(H) = r$, any n-vertex H-free graph G with $e(G) \geq \left(\frac{r-2}{r-1} - o(1)\right)\binom{n}{2}$ can be obtained from $T_{n,r-1}$ by adding or deleting $o(n^2)$ edges. Originating from the work of Erdős and Simonovits, stability has become a key topic and important tool in extremal graph theory and it has been extensively studied over the years; see [3, 13, 19, 22, 24, 30] for some recent examples. See also [35] for an example in additive combinatorics.

In [1], Allen et al. introduced three families of extremal graphs (see Sect. 2.1), each corresponding to one value of $\delta_\chi(H)$ listed above (cf. Theorem 2.1). Motivated by these highly structured constructions, we initiate a systematic study of the stability phenomena for chromatic thresholds. Although the chromatic threshold $\delta_\chi(H)$ has been completely determined, surprisingly little is known about the structure of extremal graphs near the threshold. Note that $\delta(G) \geq (\frac{r-2}{r-1} - o(1))n$ implies that $e(G) \geq \big(\frac{r-2}{r-1} - o(1)\big)\binom{n}{2}$, hence the stability theorem for the family of graphs H with $\delta_\chi(H) = \frac{r-2}{r-1}$ follows from the Erdős–Simonovits stability theorem. The main concern of this paper is to establish the stability theorems for graphs H with $\delta_\chi(H) \in \{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}\}$, as detailed below.

Let us begin with the most classical case of cliques. To discuss the extremal graphs for $\delta_\chi(K_r)$, we need to introduce the Kneser graphs. Let $n, m \in \mathbb{N}$. The *Kneser graph* $\mathrm{KN}(n, m)$ is defined on the vertex set $\binom{[n]}{m}$, where two vertices $A, B \in \binom{[n]}{m}$ are adjacent if and only if $A \cap B = \varnothing$. Roughly speaking, the extremal graphs for $\delta_\chi(K_r)$ consist of two parts:

- (*Micro*) one part is a Kneser graph $\mathrm{KN}(m, (\frac{1}{2} - o(1))m)$ on $o(n)$ vertices that contributes to the high chromatic number of G;
- (*Macro*) the other is a complete $(r-1)$-partite graph with part sizes in the ratio $(\frac{2}{2r-3}, \dots, \frac{2}{2r-3}, \frac{1}{2r-3})$ that establishes the required minimum degree of G.

Notice that although Kneser graphs play a decisive role in establishing the lower bound of $\delta_\chi(K_r)$, they only account for a negligible $o(n)$ vertices in the construction (see Construction 2.4 for more details).

Since the construction of Hajnal in the 1960 s, no other tight constructions are known. One can replace the use of Kneser graph in Hajnal's construction with an appropriate Borsuk graph. However, this is essentially the same construction as such Borsuk graph has independence number almost half of its order and admits a homomorphism to a Kneser graph $\mathrm{KN}(m, (\frac{1}{2} - o(1))m)$ for some m. It is not inconceivable that there could be other types of extremal graphs for cliques.

Our main result reads as follows, showing that all *almost* extremal graphs must essentially look like the one in Hajnal's construction, thereby establishing the stability for $\delta_\chi(K_r)$. The fractional chromatic number χ_f is defined in Sect. 2. Let $K_t(n_1, \dots, n_t)$ denote the complete t-partite graph with parts of sizes $n_1, \dots, n_t$.

Theorem 1.1 (Stability for $\delta_\chi(K_r)$) *Let $r \geq 3$ and G be a K_r-free graph on n vertices with $\delta(G) \geq \big(\frac{2r-5}{2r-3} - o(1)\big)n$ and $\chi(G) = \omega(1)$. Then,*

$$\Big|E(G) \triangle E\Big(K_{r-1}\Big(\frac{2n}{2r-3}, \dots, \frac{2n}{2r-3}, \frac{n}{2r-3}\Big)\Big)\Big| = o(n^2). \tag{1}$$

Furthermore, G admits a vertex partition $V(G) = A^* \cup B_1^* \cup \dots \cup B_{r-1}^*$ with the following properties:

(i) $|A^*| = o(n)$, $|B_i^*| = \left(\frac{2}{2r-3} \pm o(1)\right)n$ *for each* $i \in [r-2]$, *and* $|B_{r-1}^*| = \left(\frac{1}{2r-3} \pm o(1)\right)n$;
(ii) *for each* $i \in [r-1]$, B_i^* *is an independent set;*
(iii) $\chi(G[A^*]) = \omega(1)$ *and* $\chi_f(G[A^*]) \leq 2 + o(1)$; *moreover,* $G[A^*]$ *admits a homomorphism to* $\mathrm{KN}(m, (\frac{1}{2} - o(1))m)$, *where* $m = O(\log n)$.

Several remarks are in order. First of all, (1) and parts (i) and (ii) determine the macro structure of almost extremal graphs similar to other stability results. The interesting part of Theorem 1.1 is that it also captures their micro structure as shown by part (iii). This is the *first* stability result that witnesses such lower scale structure. Moreover, in part (iii), the bound on the fractional chromatic number on part A^* is best possible. More importantly, the bound on m is optimal. Indeed, by taking $m \geq \frac{1}{2}\log_2 n$ with $\binom{m}{(\frac{1}{2}-o(1))m} \leq \sqrt{n}$, $G[A^*]$ could be a Kneser graph $\mathrm{KN}(m, (\frac{1}{2} - o(1))m)$. To achieve this optimal bound on m, we utilize random projections akin to the Johnson–Lindenstrauss lemma [21] for low-distortion embedding of high-dimensional point sets.

We also establish the stability result for general graphs H with $\chi(H) = r \geq 3$ and $\delta_\chi(H) = \frac{2r-5}{2r-3}$. For a graph H with $\chi(H) = r \geq 3$, the *decomposition family* $\mathcal{M}(H)$ consists of all bipartite graphs that can be obtained from H by deleting $r-2$ color classes in an r-coloring of H.

Theorem 1.2 (Stability for $\delta_\chi(H) = \frac{2r-5}{2r-3}$) *Let* H *be a graph with* $\chi(H) = r \geq 3$ *and* $\delta_\chi(H) = \frac{2r-5}{2r-3}$. *Every* H*-free graph* G *on* n *vertices with* $\delta(G) \geq \left(\frac{2r-5}{2r-3} - o(1)\right)n$ *and* $\chi(G) = \omega(1)$ *satisfies*

$$\left|E(G)\triangle E\left(K_{r-1}\left(\frac{2n}{2r-3}, \ldots, \frac{2n}{2r-3}, \frac{n}{2r-3}\right)\right)\right| = o(n^2).$$

Furthermore, G *admits a vertex partition* $V(G) = A \cup B_1 \cup \cdots \cup B_{r-1}$ *with the following properties:*

(i) $|A| = o(n)$, $|B_i| = \left(\frac{2}{2r-3} \pm o(1)\right)n$ *for each* $i \in [r-2]$, *and* $|B_{r-1}| = \left(\frac{1}{2r-3} \pm o(1)\right)n$;
(ii) *for every forest* $F \in \mathcal{M}(H)$ *and* $i \in [r-1]$, $G[B_i]$ *is* F*-free;*
(iii) $\chi(G[A]) = \omega(1)$ *but* $\chi_f(G[A]) = O(1)$.

It would be interesting to see if the fractional chromatic number in part (iii) can be improved to $2 + o(1)$ as in the clique case.

Finally, we present the stability result for the graphs H with $\delta_\chi(H) = \frac{r-3}{r-2}$.

Theorem 1.3 (Stability for $\delta_\chi(H) = \frac{r-3}{r-2}$) *Let* H *be a graph with* $\chi(H) = r \geq 4$ *and* $\delta_\chi(H) = \frac{r-3}{r-2}$. *Every* H*-free graph* G *on* n *vertices with* $\delta(G) \geq \left(\frac{r-3}{r-2} - o(1)\right)n$ *and* $\chi(G) = \omega(1)$ *contains a vertex subset* $S \subseteq V(G)$ *of size* $\frac{n}{r-2}$ *such that the following holds. Let* $\hat{G}$ *be the spanning subgraph of* G *obtained by deleting all edges induced by* S, *then*

$$|E(\hat{G})\triangle E(T_{n,r-2})| = o(n^2).$$

In comparison with Theorem 1.1 and Theorem 1.2 where we nail down the location of $(1 - o(1))$-fraction of edges, here in Theorem 1.3 we have no control on the edges induced by the special part S. Indeed, in the extremal graphs, the induced subgraph on part S could be any 'locally bipartite' graph with edge density between 0 and $\frac{1}{2}$; we refer readers to Remark 2.6 for more details.

1.2 Other Extensions of Chromatic Thresholds

The chromatic threshold problem has been extended to what is now known as the *homomorphism threshold*, as initially discussed in [34]. Further extensions have been discussed in [7, 14, 18]. This section delves into two additional extensions of chromatic thresholds.

In Theorems 1.1 and 1.2, while the chromatic number of graph G can be arbitrarily large, the fractional chromatic number remains bounded. This leads to the following natural question: can we reduce the minimum degree requirement of an H-free graph G if only asking for bounded fractional chromatic number? More precisely, we define the *fractional chromatic threshold* for a graph H as follows:

$$\begin{aligned}\delta_{\chi_f}(H) := \inf\{d : \exists\, C = C(H, d) \text{ such that if } G \text{ is a graph on } n \text{ vertices,}\\ \text{with } \delta(G) \geq dn \text{ and } H \not\subseteq G, \text{ then } \chi_f(G) \leq C\}.\end{aligned}$$

In contrast to the usual chromatic thresholds (cf. Theorem 2.1), Kneser graph is not an obstruction anymore for cliques in the fractional counterpart. One shall expect a lower threshold in the fractional setting. Our next result confirms this intuition, determining the fractional chromatic threshold for all graphs.

Theorem 1.4 *Let H be a graph with $\chi(H) = r \geq 3$. Then*

$$\delta_{\chi_f}(H) = \begin{cases} \frac{r-3}{r-2} & \textit{if } \mathcal{M}(H) \textit{ contains a forest,} \\ \frac{r-2}{r-1} & \textit{otherwise.} \end{cases}$$

While being interesting on its own, Theorem 1.4 is also a key ingredient in proving Theorem 1.2 (iii) when we need to bound the fractional chromatic number of the high chromatic part A.

Another extension is to combine with the notion of *VC-dimension* (see Sect. 7 for definition). The study of the use of VC-dimension in chromatic thresholds began in 2010, when Łuczak and Thomassé [27] provided an intriguing new proof of $\delta_\chi(K_3) = \frac{1}{3}$ using the theory of VC-dimension. Moreover, they showed that for graphs with bounded VC-dimension, the minimum degree condition can be relaxed to $\delta(G) \geq cn$ for any constant $c > 0$. Recently, Liu, Shangguan, Skokan, and Xu [25] gave a new short proof of Łuczak and Thomassé's result and introduced the concept of *bounded-VC chromatic thresholds*, as described below:

$$\begin{aligned}\delta_\chi^{\mathrm{VC}}(H) := \inf\{c \geq 0 : \forall d \in \mathbb{N},\ \exists\, C = C(c, d, H)\\ \text{such that if } G \text{ is a graph on } n \text{ vertices,}\\ \text{with } \mathrm{VC}(G) \leq d, \delta(G) \geq cn \text{ and } H \not\subseteq G, \text{ then } \chi(G) \leq C\}.\end{aligned}$$

In this formulation, Łuczak and Thomassé's result implies $\delta_\chi^{\mathrm{VC}}(K_3) = 0$. We extend their result by determining the bounded-VC chromatic thresholds for all complete graphs.

Theorem 1.5 *For every $r \geq 3$, $\delta_\chi^{\mathrm{VC}}(K_r) = \frac{r-3}{r-2}$.*

We note that Theorem 1.5 has already found applications; it is used in very recent work of Huang, Liu, Rong and Xu [18] where they introduced an asymmetric version of homomorphism thresholds that interpolates the bounded-VC chromatic and homomorphism threshold problems.

Structure of this paper. In Sect. 2, we present some preliminaries including the construction of the extremal graphs for chromatic thresholds. We first give the proof of Theorem 1.2 in Sect. 3 and then build on it to prove Theorem 1.1 in Sect. 4. The proof of Theorem 1.3 is given in Sect. 5. Finally, we will determine the fractional chromatic thresholds in Sect. 6 and the bounded-VC chromatic thresholds of cliques in Sect. 7.

2 Preliminaries

Notation For positive integers $k \leq n$, let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the family of all k-element subsets of $[n]$. Given two finite sets X, Y, their *symmetric difference* is denoted as $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. For reals a, b, c, we write $a = (b \pm c)$ if $a \in [b - c, b + c]$. For the sake of presentation, we will omit floors and ceilings whenever they are not important.

Let $G = (V, E)$ be a graph, where V denotes the set of vertices and E denotes the set of edges. The number of vertices of G is given by $|G| := |V(G)|$, and the number of edges by $e(G) := |E(G)|$. For a vertex $v \in V(G)$, let $N(v)$ be the set of neighbors of v, and let $d(v) = |N(v)|$ be the *degree* of v. The *minimum degree* of G is $\delta(G) = \min_{v \in V(G)} d(v)$. Given a subset $U \subseteq V(G)$, let $N(U) = \bigcap_{u \in U} N(u)$ denote the *common neighborhood* of U. The *induced subgraph* $G[U]$ consists of the vertices in U and all edges in G that have both endpoints in U. For disjoint vertex sets $U_1, \ldots, U_s$, let $G[U_1, \ldots, U_s]$ denote the subgraph induced by the edges with endpoints in distinct sets U_i and U_j for $1 \leq i < j \leq s$.

For two graphs G and H, the *join* $G \vee H$ is formed by taking disjoint copies of G and H, and adding all edges between $V(G)$ and $V(H)$. The s-blowup of a graph G, denoted by $G[s]$, is obtained by replacing each vertex $v \in V(G)$ with an independent set I_v of size s, and connecting two sets I_u and I_v with a complete bipartite graph whenever $uv \in E(G)$.

Given two graphs G and F, we say that G is *homomorphic* to F, denoted by $G \xrightarrow{\text{hom}} F$, if there exists a homomorphism $\phi : V(G) \to V(F)$ that preserves adjacencies, that is, for every $uv \in E(G)$, we have $\phi(u)\phi(v) \in E(F)$.

For positive integers a, b, a *b-fold coloring* of a graph G is an assignment of sets of size b to its vertices such that adjacent vertices receive disjoint sets; an *$a : b$-coloring* is a b-fold coloring where each assigned set is chosen from $\binom{[a]}{b}$. Note that G has an $a : b$-coloring if and only if $G \xrightarrow{\text{hom}} \mathrm{KN}(a, b)$. The *$b$-fold chromatic number* $\chi_b(G)$ of G is the smallest a such that there exists an $a : b$-coloring of G. The *fractional chromatic number* $\chi_f(G)$ of G is defined to be $\chi_f(G) = \lim_{b\to\infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}$. It is clear that we always have $\chi_f(G) \le \chi(G)$, and if $G \xrightarrow{\text{hom}} \mathrm{KN}(a, b)$, then $\chi_f(G) \le \frac{a}{b}$.

2.1 Extremal Graphs for Chromatic Thresholds

In this subsection, we review the three families of graphs constructed in [1], which establish the lower bounds for chromatic thresholds. Readers familiar with these constructions can skip this subsection.

In [27], Łuczak and Thomassé defined a graph H to be *near-acyclic* if $\chi(H) = 3$ and H admits a partition into a forest F and an independent set S such that every odd cycle of H intersects S in at least two vertices. Equivalently, for each tree T in F with color classes $V_1(T)$ and $V_2(T)$, no vertex in S has neighbors in both $V_1(T)$ and $V_2(T)$.[1] We say that H is *r-near-acyclic* if $\chi(H) = r \ge 3$ and there exist $r - 3$ independent sets in H whose removal results in a near-acyclic graph.

Theorem 2.1 ([1]) *Let H be a graph with $\chi(H) = r \ge 3$. Then*

$$\delta_\chi(H) \in \left\{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\right\}.$$

Moreover, $\delta_\chi(H) \ne \frac{r-2}{r-1}$ if and only if H has a forest in its decomposition family, and $\delta_\chi(H) = \frac{r-3}{r-2}$ if and only if H is r-near-acyclic.

In [1], the authors constructed three families of graphs, each corresponding to one of the values $\delta_\chi(H) \in \{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\}$, such that for every $C \in \mathbb{N}$ there exists an infinite sequence of n-vertex H-free graphs G_n with minimum degree $\delta(G_n) \ge (\delta_\chi(H) - o(1)) \cdot n$ and chromatic number $\chi(G_n) \ge C$, where $o(1) \to 0$ as $n \to \infty$. We refer to these graphs as *extremal graphs* for chromatic thresholds.

Extremal graphs for $\delta_\chi(H) = \frac{r-2}{r-1}$. The *girth* of a graph G is defined as the length of its shortest cycle. The original proof by Erdős [8] showed the existence of graphs G in which both the girth and the ratio $\frac{|G|}{\alpha(G)}$ are arbitrarily large. Consequently, it follows from $\chi(G) \ge \chi_f(G) \ge \frac{|G|}{\alpha(G)}$ [31] that the girth and the fractional chromatic number can also be made arbitrarily large simultaneously.

For each $k, \ell \in \mathbb{N}$, we shall call a graph G a (k, ℓ)-*Erdős graph* if it has fractional chromatic number at least k, and girth at least ℓ.

Construction 2.2 *(Extremal graphs for $\delta_\chi(H) = \frac{r-2}{r-1}$)* For every such H and every $C \in \mathbb{N}$, let G' be a $(C, |H| + 1)$-Erdős graph. Let $G = G' \vee K_{r-2}[|G'|]$. In other words, G is obtained from placing a copy of G' into one part of a balanced complete $(r - 1)$-partite graph.

Note that G is H-free as there is no forest in the decomposition family of H and the girth of G' is larger than the order of H, and G satisfies $\delta(G) = \frac{r-2}{r-1} \cdot |G|$ and $\chi(G) \ge C$, thus confirming the lower bound $\delta_\chi(H) \ge \frac{r-2}{r-1}$.

[1] For example, C_5 is near-acyclic.

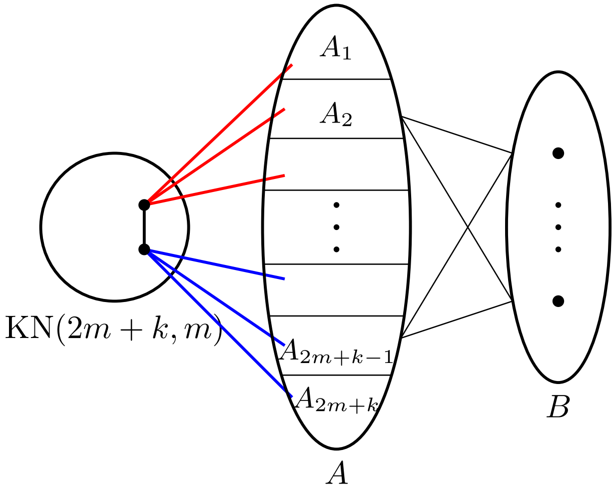


Fig. 1 The Hajnal graph $H(k, \ell, m)$

Extremal graphs for $\delta_\chi(H) = \frac{2r-5}{2r-3}$. To establish a lower bound for the family of graphs H with $\chi(H) = r \geq 3$ and $\delta_\chi(H) = \frac{2r-5}{2r-3}$, the authors of [1] introduced the *r-Borsuk–Hajnal graph*, which is an extension of the *r-Hajnal graph* (see Construction 2.4). We utilize only the r-Hajnal graph. We recommend that readers refer to [1] for more details.

Definition 2.3 *(Hajnal graphs)* Let $k, \ell, m \in \mathbb{N}$ and $2m + k \mid \ell$. The *Hajnal graph* $H(k, \ell, m)$ is a graph on $\binom{2m+k}{m} + 3\ell$ vertices obtained as follows (see Fig. 1 above for an illustration).

- Take vertex disjoint copies of a Kneser graph $\mathrm{KN}(2m + k, m)$ and a complete bipartite graph $K_{2\ell,\ell}$ with vertex set $A \cup B$, where $|A| = 2\ell$, and $|B| = \ell$;
- Partition A into $2m + k$ pieces $A_1, \ldots, A_{2m+k}$ of equal size;
- For $S \in \binom{[2m+k]}{m} = V(\mathrm{KN}(2m + k, m))$, $j \in [2m + k]$ and $y \in A_j$, add an edge between S and y whenever $j \in S$ (so every $S \in \binom{[2m+k]}{m}$ is connected to precisely $\frac{m|A|}{2m+k}$ vertices of A).

Lovász [26] famously proved that $\chi(\mathrm{KN}(n, m)) = n - 2m + 2$. Clearly, $\chi(H(k, \ell, m)) \geq \chi(\mathrm{KN}(2m + k, m)) = k + 2$ and $H(k, \ell, m)$ is triangle-free provided $m = \omega(k)$. Therefore, taking $\ell = \omega(\binom{2m+k}{m})$ and $k \geq C - 2$, $G = H(k, \ell, m)$ is a $(3 + o(1))\ell$-vertex triangle-free graph with $\delta(G) \geq \ell \geq \left(\frac{1}{3} - o(1)\right) \cdot |G|$ and $\chi(G) \geq C$, verifying the lower bound $\delta_\chi(K_3) \geq \frac{1}{3}$.

Construction 2.4 *(r-Hajnal graphs, extremal graphs for $\delta_\chi(K_r)$)* Given $r \geq 3$, let $H(k, \ell, m)$ be the Hajnal graph defined in Definition 2.3. The r-Hajnal graph is defined to be $H_r(k, \ell, m) := H(k, \ell, m) \vee K_{r-3}[2\ell]$.

Since $H(k, \ell, m)$ is K_3-free, $H_r(k, \ell, m)$ is K_r-free. Setting $\ell = \omega(\binom{2m+k}{m})$ and $k \geq C - 2$, it is not hard to see that $G = H_r(k, \ell, m)$ is a $(2r - 3 + o(1))\ell$-vertex K_r-free graph with $\delta(G) \geq (2r - 5)\ell \geq \left(\frac{2r-5}{2r-3} - o(1)\right)|G|$ and $\chi(G) \geq C$.

We remark that the *r-Borsuk–Hajnal graph* differs from the r-Hajnal graph only at the Kneser graph part and the edges incident to it.

Extremal graphs for $\delta_\chi(H) = \frac{r-3}{r-2}$. A slight modification of Construction 2.2 shows that every graph H with $\chi(H) = r \geq 3$ satisfies $\delta_\chi(H) \geq \frac{r-3}{r-2}$ as follows.

Construction 2.5 *(Extremal graph for* $\delta_\chi(H) = \frac{r-3}{r-2}$*)* For every H with $\chi(H) = r \geq 3$ and every $C \in \mathbb{N}$, let G' be a $(C, |H| + 1)$-Erdős graph. Let $G = G' \vee K_{r-3}[|G'|]$.

By construction, every induced subgraph of G' with at most $|H|$ vertices is a forest and hence 2-colorable. Consequently, every induced subgraph of G with at most $|H|$ vertices is $(r-1)$-colorable. Therefore, G is an H-free graph with $\delta(G) = \frac{r-3}{r-2} \cdot |G|$ and $\chi(G) \geq C$, thus proving the lower bound $\delta_\chi(H) \geq \frac{r-3}{r-2}$.

Remark 2.6 Note that if we replace the graph G in Construction 2.5 with $G^* := (G' \cup K_{|G'|,|G'|}) \vee K_{r-3}[3|G'|]$, then G^* is also an H-free graph with $\delta(G^*) = \frac{r-3}{r-2} \cdot |G^*|$ and $\chi(G^*) \geq C$. Here, $e(G^*) = e(T_{|G^*|,r-2}) + \Omega(|G^*|^2)$. More generally, we can replace the $(G' \cup K_{|G'|,|G'|})$ part with any graph in which any set of $|H|$ vertices induces a bipartite graph; such a graph could have any edge density in $[0, \frac{1}{2}]$. Thus, we essentially have no control over the structure of the high chromatic number part of G^*.

2.2 Auxiliary Results

In this section, we present several auxiliary results that will be used throughout the paper.

First we introduce the famous regularity lemma. Let (A, B) be a pair of subsets of vertices of G. Let $e(A, B)$ denote the number of edges with one endpoint in A and the other in B. Define the *density* of the pair (A, B) as $d(A, B) = \frac{e(A,B)}{|A||B|}$. For any $\varepsilon > 0$, the pair (A, B) is said to be ε*-regular* if $|d(A, B) - d(X, Y)| < \varepsilon$ for every $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$. Moreover, given $0 < d < 1$, we say that (A, B) is (ε, d)*-regular* if it is ε-regular and has density at least d.

A partition $V_0 \cup V_1 \cup \cdots \cup V_k$ of $V(G)$ is called an ε*-regular partition* if $|V_0| \leq \varepsilon n$, $|V_1| = \cdots = |V_k|$, and all but at most εk^2 of the pairs (V_i, V_j), where $1 \leq i, j \leq k$, are ε-regular. Given an ε-regular partition $V_0 \cup V_1 \cup \cdots \cup V_k$ of $V(G)$ and $0 < d < 1$, we define a graph R, called the (ε, d)*-reduced graph* of G, as follows: the vertex set is $V(R) = [k]$ and an edge ij is included in $E(R)$ if and only if (V_i, V_j) is an (ε, d)-regular pair. The partition classes $V_1, \ldots, V_k$ are called the *clusters* of G. For brevity, for each $I \subseteq [k]$ we write $V_I = \bigcup_{i \in I} V_i$.

We will use the following minimum degree form of the regularity lemma.

Theorem 2.7 ([23, Theorem 1.10]) *Let* $0 < \varepsilon < d < \delta < 1$*, and let* $k_0 \in \mathbb{N}$*. There exists a constant* $k_1 = k_1(k_0, \varepsilon, \delta, d)$ *such that the following holds. Every graph G on* $n > k_1$ *vertices, with minimum degree* $\delta(G) \geq \delta n$*, has an* (ε, d)*-reduced graph R on k vertices, with* $k_0 \leq k \leq k_1$ *and* $\delta(R) \geq (\delta - d - \varepsilon)k$*.*

We will also make use of the counting lemma.

Theorem 2.8 (Counting lemma, [23, Theorem 3.1]) *Let G be a graph with (ε, d)-reduced graph R whose clusters contain m vertices each, and suppose that there is a homomorphism $\phi : V(H) \to V(R)$. Then G contains at least*

$$\frac{1}{|\operatorname{Aut}(H)|}(d - \varepsilon|H|)^{e(H)} m^{|H|}$$

copies of H, each with the property that every vertex $x \in V(H)$ lies in the cluster corresponding to the vertex $\phi(x)$ of R.

We state one more useful fact about subpairs of (ε, d)-regular pairs.

Fact 2.9 *(Slicing lemma, see e.g.,* [23, Fact 1.5]*)* Let $\varepsilon < \alpha \leq \frac{1}{2}$. Let (U, W) be an (ε, d)-regular pair and suppose that $U' \subseteq U$, $W' \subseteq W$ satisfy $|U'| \geq \alpha|U|$ and $|W'| \geq \alpha|W|$. Then (U', W') is $(\frac{\varepsilon}{\alpha}, d - \varepsilon)$-regular.

The following two lemmas are basic facts of reduced graphs.

Lemma 2.10 *Let G be a graph with an (ε, d)-reduced graph R on k vertices, where $\frac{1}{k} \leq 2\varepsilon$. Let I be an independent set of R and let $V_I = \bigcup_{i \in I} V_i$. Then $e(G[V_I]) < (2\varepsilon + \frac{d}{2})n^2$.*

Proof Since I is an independent set in R, the edges of $G[V_I]$ can be categorized into the following three types:

- edges with both endpoints in V_i for some $i \in I$. There are at most $k\binom{n/k}{2}$ such edges;
- edges between distinct clusters V_i and V_j, for some distinct $i, j \in I$ for which (V_i, V_j) is not ε-regular. There are at most $\varepsilon k^2 \cdot \frac{n^2}{k^2}$ such edges;
- edges between distinct clusters V_i and V_j, for some distinct $i, j \in I$ for which (V_i, V_j) is ε-regular but $d(V_i, V_j) < d$. There are at most $d\binom{k}{2}\frac{n^2}{k^2}$ such edges.

Summing up the edges from these three types yields the desired bound. □

Lemma 2.11 *Let $r \geq 3$ be an integer and $\varepsilon, \beta, \mu, d > 0$ with $\varepsilon \leq d \leq \frac{\beta}{20r}$. Let G be an n-vertex graph with minimum degree $\delta(G) \geq (\frac{r-1}{r} - \beta)n$. Suppose that $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ is the ε-regular partition of G where $\frac{1}{k} \leq 2\varepsilon$ and R is the (ε, d)-reduced graph of G from this partition. If $|E(R) \triangle E(T_{k,r})| \leq \mu k^2$, then $|E(G) \triangle E(T_{n,r})| \leq (\beta + 2\mu)n^2$.*

Proof Since $|E(R) \triangle E(T_{k,r})| \leq \mu k^2$, there exists a balanced r-partition of $V(R)$, say $[k] = U_1 \cup \cdots \cup U_r$, such that $\sum_{i=1}^{r} e(R[U_i]) \leq \mu k^2$. For each $i \in [r]$, let $W_i = \bigcup_{j \in U_i} V_j$. By Lemma 2.10,

$$\sum_{i=1}^{r} e(G[W_i]) \leq r\left(2\varepsilon + \frac{d}{2}\right)n^2 + \mu k^2 \frac{n^2}{k^2} < \left(\frac{\beta}{8} + \mu\right)n^2.$$

Since $e(G) \geq \frac{\delta(G)n}{2}$ and the vertices in V_0 are incident with at most εn^2 edges, the r-partite subgraph of G defined on the vertex set $W_1 \cup \cdots \cup W_r$ satisfies

$$\begin{aligned}
e(G[W_1, \ldots, W_r]) &\geq \frac{\delta(G)n}{2} - \varepsilon n^2 - \sum_{i=1}^{r} e(G[W_i]) \\
&\geq \left(\frac{r-1}{2r} - \frac{\beta}{2}\right)n^2 - \varepsilon n^2 - \left(\frac{\beta}{8} + \mu\right)n^2 \\
&= \left(\frac{r-1}{2r} - \frac{5\beta}{8} - \mu - \varepsilon\right)n^2.
\end{aligned}$$

Since $e(T_{n,r}) \leq \frac{r-1}{2r} \cdot n^2$ and $\varepsilon \leq \frac{\beta}{60}$, we have

$$\begin{aligned}
|E(G) \triangle E(T_{n,r})| &= |E(G) \setminus E(T_{n,r})| + |E(T_{n,r}) \setminus E(G)| \\
&\leq \left(\varepsilon + \frac{\beta}{8} + \mu\right)n^2 + \left(\frac{5\beta}{8} + \mu + \varepsilon\right)n^2 \leq (\beta + 2\mu)n^2,
\end{aligned}$$

as needed. □

The following two lemmas from [1] describe some useful properties of graphs.

Lemma 2.12 ([1, Lemma 9]) *Let $\alpha, \delta > 0$ and $r, t \in \mathbb{N}$. Let G be a graph on n vertices, and $X \subseteq V(G)$.*

(a) *If $\delta(G) \geq \delta n$ and $|X| \geq \left(\alpha^{\frac{1}{r-2}}(r-2) + (1-\delta)(r-3)\right)n$, then $G[X]$ contains at least αn^{r-2} copies of K_{r-2}.*
(b) *For any graph H with $\chi(H) \leq r$, if $G[N(x)]$ contains at least $\alpha n^{(r-1)|H|}$ copies of $K_{r-1}[|H|]$ for every $x \in X$, then we have either $H \subseteq G$ or $|X| \leq \frac{|H|}{\alpha}$.*

Although [1, Lemma 9] is stated for graphs H satisfying $H \subseteq F \vee K_{r-2}[t]$ for some forest F, this condition is not used in the proof. Hence the above lemma also holds.

Lemma 2.13 ([1, Lemma 10]) *Let $\alpha, \delta > 0$ and $r, t \in \mathbb{N}$, let F be a forest, and suppose that $H \subseteq F \vee K_{r-2}[t]$. Let G be an H-free graph on n vertices, and let $X \subseteq V(G)$ be such that every edge $xy \in E(G[X])$ is contained in at least αn^{r-2} copies of K_r in G. Then $\chi(G[X]) \leq (\frac{2|F|}{\alpha'}) + 1$ for some $\alpha' = \alpha'(\alpha, r, t)$.*

The following two well-known facts are useful.

Fact 2.14 Let F be a forest and G be a graph on $n \geq 1$ vertices. If $e(G) \geq |F|n$, then $F \subseteq G$.

Fact 2.15 Let F be a forest and G be an F-free graph, then $\chi(G) \leq |F|$.

We will utilize the classic result of Andrásfai, Erdős, and Sós [2].

Theorem 2.16 ([2]) *Let $r \geq 3$ and let G be a K_r-free graph on n vertices such that $\delta(G) > \frac{3r-7}{3r-4} \cdot n$. Then $\chi(G) \leq r - 1$.*

3 Stability for $\delta_\chi(H) = \frac{2r-5}{2r-3}$

In this section we prove the following theorem, which implies Theorem 1.2.

Theorem 3.1 *Let $r \geq 3$ and H be a graph with $\chi(H) = r$ and $\delta_\chi(H) = \frac{2r-5}{2r-3}$. For every $0 < \beta < (40r|H|)^{-2}$, there exist $C = C(H, \beta)$ and $C' = C'(H)$ such that the following hold.*

For every n-vertex H-free graph G with $\delta(G) \geq \left(\frac{2r-5}{2r-3} - \beta\right)n$ and $\chi(G) \geq C$, we have

$$\left|E(G) \triangle E\left(K_{r-1}\left(\frac{2n}{2r-3}, \ldots, \frac{2n}{2r-3}, \frac{n}{2r-3}\right)\right)\right| \leq 20r|H|\sqrt{\beta}n^2. \qquad (2)$$

Moreover, G admits a vertex partition $V(G) = A \cup B_1 \cup \cdots \cup B_{r-1}$ with the following properties:

(i) *$|A| \leq 7r|H|\sqrt{\beta}n$, $|B_i| = \left(\frac{2}{2r-3} \pm 2\sqrt{\beta}\right)n$ for each $i \in [r-2]$, and $|B_{r-1}| = \left(\frac{1}{2r-3} \pm 5r|H|\sqrt{\beta}\right)n$. Moreover, for every $i \in [r-2]$ and $v \in B_i$, v has at most $5\sqrt{\beta}n$ non-neighbors outside B_i;*
(ii) *for every forest $F \in \mathcal{M}(H)$ and $i \in [r-1]$, $G[B_i]$ is F-free;*
(iii) *$\chi(G[A]) \geq C - (r-1)|H|$ but $\chi_f(G[A]) \leq C'$.*

Proof Let C be a sufficiently large integer whose value will be determined later. Let G be an H-free graph on n vertices with $\delta(G) \geq \left(\frac{2r-5}{2r-3} - \beta\right)n$ and $\chi(G) > C$.[2] Applying Theorem 2.7 to G with $d := \frac{\beta}{4}$, $\varepsilon := \frac{d^2}{2d+2|H|}$ and $k_0 = \lceil \frac{1}{2\varepsilon} \rceil$, we obtain a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ of G, where $k_0 \leq k \leq k_1(k_0, \varepsilon, \delta, d) = k_1(H, \beta)$, together with an (ε, d)-reduced graph R on the vertex set $V(R) = [k]$, such that $\delta(R) \geq \left(\frac{2r-5}{2r-3} - 2\beta\right)k$.

Define a new partition of $V(G)$ by setting, for every subset $I \subseteq [k]$,

$$X_I := \{x \in V(G) : i \in I \Leftrightarrow |N(x) \cap V_i| \geq d|V_i|\}.$$

That is, X_I consists of all vertices having linear degree to V_i when and only when $i \in I$.

First, we show that if $|I|$ is not large, then $\chi(G[X_I])$ is necessarily bounded.

Claim 3.2 There exists $C_1 = C_1(H, \beta)$ such that for every $I \subseteq V(R)$ with $|I| \leq (\frac{2r-4}{2r-3} - (2r-1)\beta)k$, we have $\chi(G[X_I]) \leq C_1$.

Proof For any edge $xy \in E(G[X_I])$, let $X = N(x) \cap N(y) \cap V_I$. Note that

$$\min\{|N(x) \cap V_I|, |N(y) \cap V_I|\} \geq \delta(G) - |V_0| - d(n - |V_0|) \geq \left(\frac{2r-5}{2r-3} - 2\beta\right)n,$$

which implies that

$$|X| \geq 2 \cdot \left(\frac{2r-5}{2r-3} - 2\beta\right)n - |V_I| \geq \left(\frac{2r-6}{2r-3} + (2r-5)\beta\right)n.$$

[2] As C is sufficiently large, we can assume $n \geq \chi(G) > C$ is also sufficiently large.

Applying Lemma 2.12 (a) with $\alpha = \beta^{r-2}, \delta = \frac{2r-5}{2r-3} - \beta$, and $X = N(\{x, y\})$, $G[N(\{x, y\})]$ contains at least $\beta^{r-2}n^{r-2}$ copies of K_{r-2}. Since G is H-free, Lemma 2.13 shows that $\chi(G[X_I])$ is bounded by some constant $C_1 = C_1(H, \beta)$. $\square$

Next we show that if $R[I]$ contains a K_{r-1}, then $\chi(G[X_I])$ is also bounded.

Claim 3.3 There exists $C_2 = C_2(H, \beta)$ such that for every $I \subseteq V(R)$ with $K_{r-1} \subseteq R[I]$, we have $\chi(G[X_I]) \leq C_2$.

Proof Suppose that $R[I]$ contains a copy of K_{r-1} with vertex set $[r-1] \subseteq I$. For any $x \in X_I$, we have $|N(x) \cap V_i| \geq d|V_i|$ for all $i \in [r-1]$. By Fact 2.9, it follows that for every pair $\{i, j\} \in \binom{[r-1]}{2}$, the pair $(N(x) \cap V_i, N(x) \cap V_j)$ is $(\frac{\varepsilon}{d}, d-\varepsilon)$-regular. Hence, by Theorem 2.8, $G[N(x)]$ contains at least $\alpha n^{(r-1)|H|}$ copies of $K_{r-1}[|H|]$ for some $\alpha = \alpha(H, \beta) > 0$. Since G is H-free, it follows from Lemma 2.12 (b) that $\chi(G[X_I]) \leq |X_I| \leq \frac{|H|}{\alpha}$. By letting $C_2(H, \beta) = \frac{|H|}{\alpha}$, this proves the claim. $\square$

Let $C = 3^k \max\{C_1, C_2\}$. As $\sum_{I \subseteq [k]} \chi(G[X_I]) \geq \chi(G) \geq C$, it follows from the pigeonhole principle that there exists a set $I \subseteq [k]$ such that $\chi(G[X_I]) \geq \frac{C}{2^k} > \max\{C_1, C_2\}$. Fix such a set I. Then by Claim 3.2 and Claim 3.3, $|I| \geq \big(\frac{2r-4}{2r-3} - (2r-1)\beta\big)k$ and $R[I]$ is K_{r-1}-free. Moreover, for $r \geq 4$,

$$\delta(R[I]) \geq \delta(R) - (k - |I|) \geq |I| - \left(\frac{2}{2r-3} + 2\beta\right)k > \frac{3r-10}{3r-7}|I|.$$

It follows from Theorem 2.16 that $\chi(R[I]) \leq r-2$, and this holds for $r = 3$ as well since $R[I]$ is an independent set in this case.

Now, we describe the process of constructing the partition. Recall that any vertex in R has at most $(\frac{2}{2r-3} + 2\beta)k$ non-neighbors. Consequently, any independent set in R has size at most $(\frac{2}{2r-3} + 2\beta)k$.

- **Step 1.** As $\chi(R[I]) \leq r-2$, I admits a partition $I = \bigcup_{i=1}^{r-2} Z_i$, where each Z_i is an independent set in R.
- **Step 2.** For each $i \in [r-2]$, let $V_{Z_i} = \bigcup_{j \in Z_i} V_j$, $B_i = \{v \in V_{Z_i} : |N(v) \cap V_{Z_i}| < 2\sqrt{\beta}n\} \subseteq V_{Z_i}$.
- **Step 3.** Select sets $T_1, \ldots, T_{r-2}$ in order, each of size $|H|$, such that

$$T_1 \subseteq B_1,\ T_2 \subseteq B_2 \cap N(T_1), \ldots, T_{r-2} \subseteq B_{r-2} \cap N\left(\bigcup_{i=1}^{r-3} T_i\right).$$

- **Step 4.** Let

$$B_{r-1} = N\left(\bigcup_{i=1}^{r-2} T_i\right) \setminus \bigcup_{i=1}^{r-2} B_i, \qquad A = V(G) \setminus \left(\bigcup_{i=1}^{r-1} B_i\right).$$

We first prove that this process can be executed. To demonstrate this, it suffices to show that we can select $T_1, \dots, T_{r-2}$ in Step 3. To this end, we need to bound the size of each B_i.

Claim 3.4 For each $i \in [r-2]$, $\left(\frac{2}{2r-3} - 2\sqrt{\beta}\right)n < |B_i| \le \left(\frac{2}{2r-3} + 2\beta\right)n$.

Proof Since Z_i is an independent set with $|Z_i| \le (\frac{2}{2r-3} + 2\beta)k$, we have

$$|Z_i| \ge |I| - (r-3)\left(\frac{2}{2r-3} + 2\beta\right)k > \left(\frac{2}{2r-3} - 4r\beta\right)k,$$

which concludes that $\left(\frac{2}{2r-3} - \sqrt{\beta}\right)n \le |V_{Z_i}| \le \left(\frac{2}{2r-3} + 2\beta\right)n$ where the lower bound follows from $\beta < \frac{1}{25r^2}$. Moreover, by Lemma 2.10 we have $e(G[V_{Z_i}]) < \left(2\varepsilon + \frac{d}{2}\right)n^2 < \beta n^2$. Hence,

$$\left(\frac{2}{2r-3} - 2\sqrt{\beta}\right)n < |V_{Z_i}| - \frac{2e(G[V_{Z_i}])}{2\sqrt{\beta}n} \le |B_i| \le |V_{Z_i}| \le \left(\frac{2}{2r-3} + 2\beta\right)n,$$

as claimed. □

The next claim shows that each vertex in B_i has only a few non-neighbors outside B_i.

Claim 3.5 For every $i \in [r-2]$ and $v \in B_i$, v has at most $5\sqrt{\beta}n$ non-neighbors outside B_i, that is, $|(V(G) \setminus B_i) \setminus N(v)| \le 5\sqrt{\beta}n$.

Proof As $\delta(G) \ge \left(\frac{2r-5}{2r-3} - \beta\right)n$, v is non-adjacent to at most $\left(\frac{2}{2r-3} + \beta\right)n$ vertices in $V(G)$. Moreover, by the definition of B_i, we have $|N(v) \cap B_i| \le |N(v) \cap V_{Z_i}| < 2\sqrt{\beta}n$. Thus, by Claim 3.4, v is non-adjacent to at least $|B_i| - 2\sqrt{\beta}n \ge \left(\frac{2}{2r-3} - 4\sqrt{\beta}\right)n$ vertices in B_i. It follows that v is non-adjacent to at most $(4\sqrt{\beta} + \beta) < 5\sqrt{\beta}n$ vertices in $V(G) \setminus B_i$. □

Now, we can show that the process in Step 3 is well-defined. Assume that we have chosen $T_1, \dots, T_{i-1}$. Then, by Claim 3.4 and Claim 3.5, we obtain that

$$\begin{aligned}\left|B_i \cap N\left(\bigcup_{j=1}^{i-1} T_j\right)\right| &\ge |B_i| - 5(i-1)|H|\sqrt{\beta}n \\ &\ge \left(\frac{2}{2r-3} - 2\sqrt{\beta}\right)n - 5(r-3)|H|\sqrt{\beta}n \\ &> |H|.\end{aligned}$$

Thus we can choose the desired T_i in B_i for each $i \in [r-2]$ in Step 3.

Next we show that the partition $V(G) = A \cup B_1 \cup \dots \cup B_{r-1}$ satisfies the properties listed in Theorem 3.1. Let $B = B_1 \cup \dots \cup B_{r-2}$. By Claim 3.4, we have $\left(\frac{2r-4}{2r-3} - 2r\sqrt{\beta}\right)n \le |B| \le \left(\frac{2r-4}{2r-3} + 2r\beta\right)n$. Thus, it follows from Claim 3.5 that

$$\begin{aligned}\left(\frac{1}{2r-3}+2r\sqrt{\beta}\right)n &\geq n-|B| \geq |B_{r-1}| \\ &= \left|N\left(\bigcup_{i=1}^{r-2} T_i\right)\cap (V(G)\setminus B)\right| \\ &\geq |V(G)\setminus B| - (r-2)|H|5\sqrt{\beta}n \\ &\geq \left(\frac{1}{2r-3}-5r|H|\sqrt{\beta}\right)n.\end{aligned}$$

Finally, since $|A| = n-|B|-|B_{r-1}| \leq 7r|H|\sqrt{\beta}n$, we complete the proof of Theorem 3.1 (i).

Now, we prove Theorem 3.1 (ii). By our construction, it is clear that $G[T_1,\ldots,T_{r-2},B_{r-1}]$ is a complete $(r-1)$-partite graph and $G[T_1,\ldots,T_{r-2}] = K_{r-2}[|H|]$. Hence, B_{r-1} is F-free for every $F\in\mathcal{M}(H)$; otherwise, G contains a copy of $F\vee K_{r-2}[|H|]\supseteq H$, a contradiction. Suppose that $G[B_1]$ contains a copy of $F\in\mathcal{M}(H)$. Applying the same process with T_1 containing the vertex set of this copy of F, it follows that $N(T_1)$ also contains a copy of $K_{r-2}[|H|]$, which is again a contradiction. Similarly, one can show that $G[B_i]$ is F-free for each $i\in[r-2]$, as desired.

We now proceed to prove Theorem 3.1 (iii). By Fact 2.15, $\chi(G[B\cup B_{r-1}])\leq (r-1)|F|\leq (r-1)|H|$. Thus, $\chi(G[A])\geq C-(r-1)|H|$. Moreover, noting that $\delta(G) > (\frac{2r-5}{2r-3}-\beta)n > (\frac{r-3}{r-2}+\frac{1}{r^3})n$, it follows from Theorem 6.2 (applied to G) that $\chi_f(G[A])\leq \chi_f(G)\leq C' = C'(H)$, which verifies Theorem 3.1 (iii).

Finally, we prove (2). Since for every $i\in[r-2]$ and $v\in B_i$, v is non-adjacent to at most $5\sqrt{\beta}n$ vertices outside B_i, the total number of missing edges among all distinct pairs of B_i, B_j is at most $\frac{5\sqrt{\beta}n^2}{2} < 3\sqrt{\beta}n^2$. Moreover, since for every forest $F\in\mathcal{M}(H)$ and every $i\in[r-1]$, $G[B_i]$ is F-free, by Fact 2.14, the number of edges within the B_i's is at most $(r-1)|H|n < \sqrt{\beta}n^2$. Hence,

$$\begin{aligned}&\left|E(G)\triangle E\left(K_{r-1}\left(\frac{n}{2r-3},\frac{2n}{2r-3},\ldots,\frac{2n}{2r-3}\right)\right)\right| \\ &\leq |A|n+\sqrt{\beta}n^2+3\sqrt{\beta}n^2+n\cdot\left(\sum_{i=1}^{r-2}\left||B_i|-\frac{2n}{2r-3}\right|+\left||B_{r-1}|-\frac{n}{2r-3}\right|\right) \\ &\leq 20r|H|\sqrt{\beta}n^2,\end{aligned}$$

as needed. □

4 Stability for $\delta_\chi(K_r)$

In this section we prove the following theorem, which implies Theorem 1.1.

Theorem 4.1 *For every $r\geq 3$ and $0<\beta<40^{-2}r^{-4}$, there exists $C = C(r,\beta)$ such that the following hold. For every n-vertex K_r-free graph G with $\delta(G)\geq \left(\frac{2r-5}{2r-3}-\beta\right)n$*

and $\chi(G) \geq C$, G admits a vertex partition $V(G) = A^* \cup B_1^* \cup \cdots \cup B_{r-1}^*$ with the following properties:

(i) $|A^*| \leq 7r^2\sqrt{\beta}n$, $|B_i^*| = \big(\frac{2}{2r-3} \pm 2\sqrt{\beta}\big)n$ *for each* $i \in [r-2]$, *and* $|B_{r-1}^*| = \big(\frac{1}{2r-3} \pm 5r^2\sqrt{\beta}\big)n$;
(ii) *for each* $i \in [r-1]$, B_i^* *is an independent set;*
(iii) $\chi(G[A^*]) \geq C - r + 1$ *and* $\chi_f(G[A^*]) \leq 2 + 50r^3\sqrt{\beta}$; *in particular,* $G[A^*]$ *is homomorphic to* $\mathrm{KN}(m, (\frac{1}{2} - \varepsilon)m)$, *where* $m = \ln \frac{n}{\beta}$ *and* $\varepsilon = 100r^3\sqrt{\beta}$.

Proof Let C be $C(K_r, \beta)$ given from Theorem 3.1. Suppose that G is a K_r-free graph with $\delta(G) \geq \big(\frac{2r-5}{2r-3} - \beta\big)n$ and $\chi(G) \geq C$. Applying Theorem 3.1, let $V(G) = A \cup B_1 \cup \cdots \cup B_{r-1}$ be a partition satisfying the conclusion of Theorem 3.1.

We first note that $G[B_1, \ldots, B_{r-1}]$ is rich in K_{r-1}.

Claim 4.2 Let $T_1 \subseteq B_1, \ldots, T_{r-1} \subseteq B_{r-1}$ be subsets with $|T_i| \geq 5r\sqrt{\beta}n$ for each $i \in [r-1]$. Then $K_{r-1} \subseteq G[T_1, \ldots, T_{r-1}]$.

Proof We can greedily pick $w_1 \in T_1, w_2 \in N(w_1) \cap T_2, \ldots, w_{r-1} \in N(\{w_1, \ldots, w_{r-2}\}) \cap T_{r-1}$, so that $\{w_1, \ldots, w_{r-1}\}$ forms a copy of K_{r-1}. To see this, for every $i \in [r-2]$, by Theorem 3.1 (i), we have $|T_{i+1} \setminus N(w_j)| \leq 5\sqrt{\beta}n$ for each $j \in [i]$. Hence

$$|N(\{w_1, \ldots, w_i\}) \cap T_{i+1}| \geq |T_{i+1}| - \sum_{j=1}^{i} |T_{i+1} \setminus N(w_j)| \geq 5r\sqrt{\beta}n - 5i\sqrt{\beta}n > 0.$$

Therefore, the vertices $w_1, \ldots, w_{r-1}$ are well-defined. □

To obtain the desired sets B_i^* in Theorem 4.1, we need a refined analysis on $G[A]$.

Claim 4.3 For every vertex $x \in A$, there exists exactly one $i \in [r-1]$ such that $|N(x) \cap B_i| < 5r\sqrt{\beta}n$.

Proof Fix an arbitrary $x \in A$. Since $|N(x)| \geq \big(\frac{2r-5}{2r-3} - \beta\big)n$, $|B_{r-1}| \geq \big(\frac{1}{2r-3} - 5r^2\sqrt{\beta}\big)n$ and $|B_i| > \big(\frac{2}{2r-3} - 2\sqrt{\beta}\big)n$ for $i \in [r-2]$, there is at most one $i \in [r-1]$ such that $|N(x) \cap B_i| < 5r\sqrt{\beta}n$. On the other hand, if $|N(x) \cap B_i| \geq 5r\sqrt{\beta}n$ for every $i \in [r-1]$, then applying Claim 4.2 with $T_i := N(x) \cap B_i$, it follows that $K_{r-1} \subseteq G[T_1, \ldots, T_{r-1}] \subseteq G[N(x)]$, which implies that $K_r \subseteq G$, a contradiction. □

For each $i \in [r-1]$, let

$$R_i = \{x \in A : |N(x) \cap B_i| < 5r\sqrt{\beta}n\}.$$

Then by Claim 4.3, $R_1, \ldots, R_{r-1}$ are pairwise disjoint and form a partition of A. Set

$$B_{r-1}^* := B_{r-1}, \quad B_i^* := B_i \cup R_i \text{ for } i \in [r-2],$$
$$A^* = V(G) \setminus \left(\bigcup_{i=1}^{r-1} B_i^* \right) = R_{r-1}.$$

Next we demonstrate that these B_i^*'s satisfy the conclusions (i)–(iii) of Theorem 4.1.

We first prove part (ii). Notice first that as $\mathcal{M}(K_r) = \{K_2\}$, it follows immediately from Theorem 3.1 (ii) that each $B_i, i \in [r-1]$, is an independent set. As $B_{r-1}^* := B_{r-1}$ and $B_i^*, i \in [r-2]$, are symmetric, it suffices to show that B_1^* is independent. By the definition of R_1 and that B_1 is independent, every vertex $v \in B_1^*$ satisfies

$$|N(v) \cap B_1^*| \le |R_1| + |N(v) \cap B_1| \le |A| + 5r\sqrt{\beta}n \le 10r^2\sqrt{\beta}n.$$

As $A \cup B_1 \cup \cdots \cup B_{r-1}$ satisfies Theorem 3.1 (i) and $|B_1^*| > \left(\frac{2}{2r-3} - 2\sqrt{\beta}\right)n$, v is non-adjacent to at least $\left(\frac{2}{2r-3} - 12r^2\sqrt{\beta}\right)n$ vertices in B_1^*. Moreover, since v is non-adjacent to at most $\left(\frac{2}{2r-3} + \beta\right)n$ vertices in $V(G)$, we see that v is non-adjacent to at most $13r^2\sqrt{\beta}n$ vertices in $V(G)\backslash B_1^*$. To show that B_1^* is independent, suppose to the contrary that $xy \in E(G)$ for some distinct $x, y \in B_1^*$. Let $T_j := N(x) \cap N(y) \cap B_j^*$ for $j \ne 1$. Thus, we have $|T_j| \ge |B_j^*| - 26r^2\sqrt{\beta}n \ge 5r\sqrt{\beta}n$ for each $j \ne 1$. Applying Claim 4.2 with T_j for $2 \le j \le r-1$ gives that $K_{r-2} \subseteq G[T_2, \ldots, T_{r-1}] \subseteq G[N(x) \cap N(y)]$, a contradiction. This proves part (ii).

Now, part (i) is obvious since $|A^*| \le |A| \le 7r^2\sqrt{\beta}n$, $|B_{r-1}^*| = |B_{r-1}|$ and by part (ii), for $i \in [r-2]$, we have $|B_i| \le |B_i^*| \le n - \delta(G) \le \left(\frac{2}{2r-3} + \beta\right)n$.

Finally, we prove part (iii). It is clear that $\chi(G - A^*) = r - 1$ and $\chi(G[A^*]) \ge C - r + 1$.

Claim 4.4 For any edge $xy \in E(G[A^*])$, there exists a unique $i \in [r-2]$ such that $N(\{x, y\}) \cap B_i^* = \varnothing$.

Proof Let $B^* := \bigcup_{i=1}^{r-2} B_i^*$, then $|B^*| \le n - |B_{r-1}^*| \le \left(\frac{2r-4}{2r-3} + 5r^2\sqrt{\beta}\right)n$. For each $v \in A^* = R_{r-1}$, we have

$$|N(v) \cap B^*| \ge \left(\frac{2r-5}{2r-3} - \beta\right)n - |A^*| - |N(v) \cap B_{r-1}^*| \ge \left(\frac{2r-5}{2r-3} - 13r^2\sqrt{\beta}\right)n.$$

Fixing an edge $xy \in E(G[A^*])$, we then get

$$|N(\{x, y\}) \cap B^*| \ge |N(x) \cap B^*| + |N(y) \cap B^*| - |B^*| \ge \left(\frac{2r-6}{2r-3} - 31r^2\sqrt{\beta}\right)n. \tag{3}$$

Notice that there is a unique $i \in [r-2]$ such that $|N(\{x, y\}) \cap B_i^*| < 5r\sqrt{\beta}n$. Indeed, if for each $i \in [r-2]$, $|N(\{x, y\}) \cap B_i^*| \ge 5r\sqrt{\beta}n$, then we can apply Claim 4.2 with $T_j = N(\{x, y\}) \cap B_j^*$ for $j \in [r-2]$ to obtain a copy of K_{r-2} in $G[N(x) \cap N(y)]$, a contradiction. But if there are at least two such indices, then as $|B_j^*| \ge \left(\frac{2}{2r-3} - 2\sqrt{\beta}\right)n$, x and y would have too few common neighbors in B^*, contradicting (3).

We shall show that this unique index $i \in [r-2]$ is the desired one. Suppose for contradiction that there exists a vertex $w \in B_i^*$ such that $w \in N(\{x, y\})$. Since for each $j \in [r-2] \setminus \{i\}$, $|B_j^*| \le |B_j| + |A| \le \left(\frac{2}{2r-3} + 9r^2\sqrt{\beta}\right)n$, we have

$$
\begin{aligned}
|N(\{x,y\}) \cap B_j^*| &\ge |N(\{x,y\}) \cap B^*| - |N(\{x,y\}) \cap B_i^*| - \sum_{\ell \in [r-2]\setminus\{i,j\}} |B_\ell^*| \\
&\ge \left(\frac{2}{2r-3} - 45r^3\sqrt{\beta}\right)n.
\end{aligned}
$$

Since $w \in B_i^*$, by Theorem 3.1 (i), for each $j \in [r-2] \setminus \{i\}$, $|N(w) \cap B_j^*| \ge |B_j^*| - 5\sqrt{\beta}n \ge \left(\frac{2}{2r-3} - 7\sqrt{\beta}\right)n$. Therefore, $|N(\{x,y,w\}) \cap B_j^*| \ge 5r\sqrt{\beta}n$ for each $j \in [r-2] \setminus \{i\}$, and by Claim 4.2 we have $K_{r-3} \subseteq G[N(\{x,y,w\})]$, a contradiction. □

Let $a := \left(\frac{2}{2r-3} + \beta\right)n$ and $b := \left(\frac{1}{2r-3} - 12r^2\sqrt{\beta}\right)n$. We aim to show that $G[A^*] \xrightarrow{\text{hom}} \mathrm{KN}(a,b)$. It suffices to construct a map $\phi : A' \to \binom{[a]}{b}$ for each connected component $A' \subseteq A^*$ such that if $uv \in E(G[A'])$, then $\phi(u) \cap \phi(v) = \varnothing$. Fix a connected component A' in A^*. We next observe that each vertex in A' is adjacent to at least almost half of each B_j^*, $j \in [r-2]$.

Claim 4.5 For any $v \in A'$ and any $j \in [r-2]$, $|N(v) \cap B_j^*| \ge \left(\frac{1}{2r-3} - 12r^2\sqrt{\beta}\right)n$.

Proof Note that

$$
|N(v) \cap B_j^*| \ge |N(v)| - |A^*| - |N(v) \cap B_{r-1}^*| - \sum_{\ell \in [r-2]\setminus\{j\}} |B_\ell^*|.
$$

Recall that $B_{r-1}^* = B_{r-1}$ and $A^* = R_{r-1}$, and so $|N(v) \cap B_{r-1}^*| \le 5r\sqrt{\beta}n$. Since B_ℓ^* is an independent set, $|B_\ell^*| \le n - \delta(G) \le a$. Therefore,

$$
\begin{aligned}
|N(v) \cap B_j^*| &\ge \left(\frac{2r-5}{2r-3} - \beta\right)n - 5r^2\sqrt{\beta}n - 5r\sqrt{\beta}n - \left(\frac{2r-6}{2r-3} + r\beta\right)n \\
&\ge \left(\frac{1}{2r-3} - 12r^2\sqrt{\beta}\right)n,
\end{aligned}
$$

as claimed. □

We need to strengthen Claim 4.4 as follows.

Claim 4.6 For every connected component $A' \subseteq A^*$, there exists a unique index $i_{A'} \in [r-2]$ such that for every edge $xy \in E(G[A'])$, $N(\{x,y\}) \cap B_{i_{A'}}^* = \varnothing$.

Proof Fix a component A' in A^*, we are done by Claim 4.4 if A' is a single edge. We may then assume that there is a 3-vertex path xyz in A'. Let i_{xy} and i_{yz} be the indices obtained from Claim 4.4 such that $N(\{x,y\}) \cap B_{i_{xy}}^* = \varnothing$ and $N(\{y,z\}) \cap B_{i_{yz}}^* = \varnothing$. It suffices to prove that $i_{xy} = i_{yz} = i_{A'}$ as then we can take a spanning tree of A' and propagate the index $i_{A'}$.

Suppose for a contradiction that $i_{xy} \neq i_{yz}$. As $N(\{x, y\}) \cap B^*_{i_{xy}} = \varnothing$, we see that y has at least $|N(x) \cap B^*_{i_{xy}}| \geq \left(\frac{1}{2r-3} - 12r^2\sqrt{\beta}\right)n$ non-neighbors in $B^*_{i_{xy}}$ by Claim 4.5. Similarly, y has at least $|N(z) \cap B^*_{i_{yz}}| \geq \left(\frac{1}{2r-3} - 12r^2\sqrt{\beta}\right)n$ non-neighbors in $B^*_{i_{yz}}$. On the other hand, $y \in A^* = R_{r-1}$, and so $|N(y) \cap B^*_{r-1}| \leq 5r\sqrt{\beta}n$. Altogether, y has at least $\left(\frac{3}{2r-3} - 50r^2\sqrt{\beta}\right)n$ non-neighbors, contradicting the minimum degree condition of G. □

Fix the unique index $i_{A'} \in [r-2]$ for A' guaranteed by Claim 4.6. For every $v \in A'$, we can define $\phi(v)$ to be an arbitrary b-subset in $N(v) \cap B^*_{i_{A'}}$. Therefore, it follows from Claim 4.6 that for every edge $uv \in E(G[A'])$, $\phi(u) \cap \phi(v) = \varnothing$ as desired, hence ϕ is a b-fold coloring on $B^*_{i_{A'}}$.

Notice that $|B^*_j| \leq a$ for any $j \in [r-2]$, so for every connected component $A' \subseteq A^*$, we have $\chi_f(G[A']) \leq \frac{a}{b}$. Consequently, for sufficiently large n, $\chi_f(G[A^*]) \leq \frac{a}{b} \leq 2 + 50r^3\sqrt{\beta}$, which proves the first half of part (iii) and implies that $G[A^*] \xrightarrow{\text{hom}} \mathrm{KN}(a, b)$. The second half follows from the lemma below. Indeed, applying Lemma 4.7 with $\delta = \sqrt{\beta}$ and $m = \ln\frac{n}{\beta}$, we have $\frac{b}{a} - \delta \geq \frac{1}{2} - \varepsilon$ and so that $G[A^*] \xrightarrow{\text{hom}} \mathrm{KN}(m, (\frac{1}{2} - \varepsilon)m)$ as desired.

This completes the proof of Theorem 4.1. □

Lemma 4.7 *Let G be a graph on n vertices. If $G \xrightarrow{\text{hom}} \mathrm{KN}(a, b)$, then for every $\delta \in (0, \frac{b}{a})$ and $m > \ln\frac{n}{2\delta^2}$ we have $G \xrightarrow{\text{hom}} \mathrm{KN}(m, (\frac{b}{a} - \delta)m)$.*

To prove Lemma 4.7, we recall the hypergeometric distributions. Suppose that there is a finite set containing n elements, exactly $k \leq n$ of which are marked, and suppose that we sample $s \leq n$ elements uniformly at random without replacement. Let X be the random variable that counts the number of marked elements in the sample. Then we say that X has the *hypergeometric distribution* with parameters (n, k, s). We will need the following standard tail bound on hypergeometric distributions.

Lemma 4.8 ([17]) *Let X have the hypergeometric distribution with parameters (n, k, s). Then for $\delta \in (0, \frac{k}{n})$ the following inequality holds*

$$\Pr\left[X \leq \left(\frac{k}{n} - \delta\right)s\right] \leq e^{-2\delta^2 s}.$$

Proof of Lemma 4.7 Let $\phi : V(G) \to \binom{[a]}{b}$ be a homomorphism from G to $\mathrm{KN}(a, b)$. Let S be a uniformly chosen random m-subset of $[a]$. For each vertex $v \in V(G)$, define a random variable $X_v = |\phi(v) \cap S|$. Clearly, X_v follows the hypergeometric distribution with parameters (a, b, m). Then, by Lemma 4.8, we have that for every $v \in V(G)$,

$$\Pr\left[X_v \leq \left(\frac{b}{a} - \delta\right)m\right] \leq e^{-2\delta^2 m} < \frac{1}{n}.$$

By the union bound, there exists a choice of $S \in \binom{[a]}{m}$ such that for all $v \in V(G)$,

$$\left(\frac{b}{a} - \delta\right)m < X_v = |\phi(v) \cap S| \leq |S| = m.$$

For each $v \in V(G)$, define $\psi(v)$ to be an arbitrary $((\frac{b}{a} - \delta)m)$-subset of $\phi(v) \cap S$. By the previous discussion, $\psi(v)$ is well-defined. Moreover, for every $uv \in E(G)$, we have

$$\psi(u) \cap \psi(v) \subseteq (\phi(u) \cap S) \cap (\phi(v) \cap S) \subseteq \phi(u) \cap \phi(v) = \varnothing,$$

where the last equality holds by the property of ϕ. Thus, $\psi : V(G) \to \binom{S}{(b/a-\delta)m}$ is indeed a homomorphism from G to $\mathrm{KN}(m, (\frac{b}{a} - \delta)m)$, as needed. □

5 Stability for $\delta_\chi(H) = \frac{r-3}{r-2}$

In this section, we prove the following theorem.

Theorem 5.1 *Let $r \geq 4$, $0 < \beta < \frac{1}{30r^3}$ and H be a graph with $\chi(H) = r$ and $\delta_\chi(H) = \frac{r-3}{r-2}$. Then there exists $C = C(H, \beta)$ such that the following holds. For every n-vertex H-free graph G with $\delta(G) \geq \left(\frac{r-3}{r-2} - \beta\right)n$ and $\chi(G) \geq C$ there exists an induced subgraph $G' \subseteq G$ on $n' = \frac{r-3}{r-2}n$ vertices such that*

$$|E(G')\Delta E(T_{n',r-3})| \leq 30r^3\beta n'^2. \tag{4}$$

Note that Theorem 5.1 implies Theorem 1.3. Indeed, let $S = V(G) \setminus V(G')$, then $|S| = \frac{n}{r-2}$, and we can derive from (4) and the min-degree condition $\delta(G) \geq \left(\frac{r-3}{r-2} - \beta\right)n$ that every vertex in G' is almost complete to S. Hence, letting $\hat{G}$ be the spanning subgraph of G obtained by deleting all edges induced by S, we see that $\hat{G}$ is in edit-distance $o(n^2)$-close to $T_{n,r-2}$ as desired.

We begin by stating an embedding lemma, which is a consequence of [1, Lemmas 24, 25 and Propositions 26, 36]. A formal proof is included in the appendix for completeness.

Lemma 5.2 ([1]) *Let H be an r-near-acyclic graph, and let $\varepsilon, \beta, d > 0$ be real numbers. Suppose that G is a graph and that $X, Y, Z_1, \ldots, Z_{r-3}$ are pairwise disjoint subsets of $V(G)$ such that $|Y| = |Z_j|$ for each $j \in [r-3]$. Assume that the pairs (Y, Z_j) and (Z_i, Z_j) are (ε, d)-regular for all distinct i, j, and*

$$|N(x) \cap Y| \geq \beta|Y| \quad \textit{and} \quad |N(x) \cap Z_j| \geq \left(\frac{1}{2} + \beta\right)|Z_j|$$

for every $x \in X$ and every $j \in [r-3]$. If $\chi(G[X]) \geq C$ for some sufficiently large constant $C = C(H, \varepsilon, \beta, d)$, then $H \subseteq G$.

Proof of Theorem 5.1 Fix $r \geq 4$, $0 < \beta < \frac{1}{30r^3}$ and an r-near-acyclic graph H. Let C be a sufficiently large integer whose value will be determined later. Let G be an H-free graph on n vertices satisfying $\delta(G) > \big(\frac{r-3}{r-2} - \beta\big)n$ and $\chi(G) \geq C$. Set $d := \frac{\beta}{20r}$ and $\varepsilon := \frac{d^2}{2d+2|H|}$. Applying Theorem 2.7 to G with $k_0 = \frac{r}{8\beta}$, we obtain a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$, together with an (ε, d)-reduced graph R with vertex set $[k]$, where $\frac{r}{8\beta} \leq k \leq k_1$ for some $k_1 = k_1(\varepsilon, d, k_0)$. Furthermore, the minimum degree of R satisfies $\delta(R) \geq \big(\frac{r-3}{r-2} - 2\beta\big)k$.

We now define a refined partition of $V(G)$. For each pair of index sets $I_2 \subseteq I_1 \subseteq [k]$, let

$$X(I_1, I_2) := \Big\{ v \in V(G) : i \in I_1 \Leftrightarrow |N(v) \cap V_i| \geq \beta |V_i| \text{ and } i \in I_2 \Leftrightarrow |N(v) \cap V_i| \geq \Big(\frac{1}{2} + \beta\Big)|V_i| \Big\}.$$

That is, $X(I_1, I_2)$ consists of all vertices $v \in V(G)$ such that for each $i \in [k]$, $i \in I_1$ if and only if v has at least $\beta|V_i|$ neighbors in V_i, and $i \in I_2$ if and only if v has at least $(\frac{1}{2} + \beta)|V_i|$ neighbors in V_i.

We first establish a lower bound for $|I_1| + |I_2|$ when $X(I_1, I_2) \neq \varnothing$.

Claim 5.3 If $X(I_1, I_2) \neq \varnothing$, then $|I_1| + |I_2| \geq 2\big(\frac{r-3}{r-2} - 4\beta\big)k$.

Proof For every vertex $x \in X(I_1, I_2)$, since x has at most $(\varepsilon + \beta)n$ neighbors outside V_{I_1},

$$|N(x) \cap V_{I_1}| \geq d(x) - (\varepsilon + \beta)n \geq \Big(\frac{r-3}{r-2} - 3\beta\Big)n.$$

On the other hand,

$$|N(x) \cap V_{I_1}| \leq \frac{n}{k}\left(\Big(\frac{1}{2} + \beta\Big)(|I_1| - |I_2|) + |I_2|\right) \leq \beta n + \frac{n}{2k}(|I_1| + |I_2|).$$

Comparing the upper and lower bounds on $|N(x) \cap V_{I_1}|$ yields the desired inequality. □

Next, for those index sets $I_2 \subseteq I_1 \subseteq [k]$ with $X(I_1, I_2) \neq \varnothing$, we will show that the chromatic number $\chi(G[X(I_1, I_2)])$ is bounded whenever either $R[I_1]$ contains a copy of K_{r-1} or $R[I_2]$ contains a copy of K_{r-2}.

Claim 5.4 There exist $C_1 = C_1(H, \beta)$ and $C_2 = C_2(H, \beta)$ such that for $X(I_1, I_2) \neq \varnothing$, the following hold.

(i) If $R[I_1]$ contains a copy of K_{r-1}, then $\chi(G[X(I_1, I_2)]) \leq C_1$;
(ii) If $R[I_2]$ contains a copy of K_{r-2}, then $\chi(G[X(I_1, I_2)]) \leq C_2$.

Proof Since H is r-near-acyclic, there exists a forest F such that $H \subseteq F \vee K_{r-2}[|H|]$.

For part (i), suppose that $R[I_1]$ contains a copy of K_{r-1}, without loss of generality, assume its vertex set is $[r-1] \subseteq I_1$. For any $x \in X(I_1, I_2)$, we have $|N(x) \cap V_i| \geq d|V_i|$

for all $i \in [r-1]$. By Fact 2.9, it follows that for every pair $\{i, j\} \in \binom{[r-1]}{2}$, the pair $(N(x) \cap V_i, N(x) \cap V_j)$ is $(\frac{\varepsilon}{d}, d-\varepsilon)$-regular. Hence, by Theorem 2.8, $G[N(x)]$ contains at least $\alpha_1 n^{(r-1)|H|}$ copies of $K_{r-1}[|H|]$ for some $\alpha_1 = \alpha_1(H, \beta) > 0$. Since G is H-free, it follows from Lemma 2.12 (b) that $\chi(G[X(I_1, I_2)]) \leq |X(I_1, I_2)| \leq \frac{|H|}{\alpha_1}$. By letting $C_1(H, \beta) = \frac{|H|}{\alpha_1}$, this proves part (i).

For part (ii), suppose that $R[I_2]$ contains a copy of K_{r-2} with vertex set $[r-2] \subseteq I_2$. For any edge $xy \in E(G[X(I_1, I_2)])$, define $V_i' = N(\{x, y\}) \cap V_i$ for each $i \in [r-2]$. Then

$$|V_i'| \geq |N(x) \cap V_i| + |N(y) \cap V_i| - |V_i| \geq 2\beta|V_i|.$$

By Fact 2.9, each pair (V_i', V_j') is $(\frac{\varepsilon}{2\beta}, d-\varepsilon)$-regular for every $\{i, j\} \in \binom{[r-2]}{2}$. Again by Theorem 2.8, $G[\bigcup_{i=1}^{r-2} V_i']$ contains at least $\alpha' n^{r-2}$ copies of K_{r-2} for some $\alpha' = \alpha'(H, \beta) > 0$. Therefore, by Lemma 2.13, there exists a constant $\alpha_2 = \alpha_2(\alpha', H)$ such that $\chi(G[X(I_1, I_2)]) \leq \frac{2|H|}{\alpha_2} + 1$. By letting $C_2(H, \beta) = \frac{2|H|}{\alpha_2} + 1$, part (ii) is thus proved. $\square$

Let C_3 be $C(H, \varepsilon, \beta, d)$ given from Lemma 5.2 and let $C = C(H, \beta) = 5^k \max\{C_1, C_2, 2C_3\}$. Then there exists a pair $I_2 \subseteq I_1 \subseteq [k]$ such that $\chi(G[X(I_1, I_2)]) \geq \frac{C}{4^k} > \max\{C_1, C_2\}$. Fix such a pair (I_1, I_2). By Claim 5.4, we must have that $R[I_1]$ is K_{r-1}-free and $R[I_2]$ is K_{r-2}-free.

We now split the remainder of the proof into two cases, according to whether $R[I_2]$ contains a copy of K_{r-3}. Recall that $\delta(R) \geq \left(\frac{r-3}{r-2} - 2\beta\right)k$, hence any vertex in R has at most $\left(\frac{1}{r-2} + 2\beta\right)k$ non-neighbors. Consequently, any independent set in R has size at most $\left(\frac{1}{r-2} + 2\beta\right)k$.

Case 1: $R[I_2]$ is K_{r-3}-free. In this case, we establish a stronger structural result that

$$|E(G) \triangle E(T_{n,r-2})| \leq 30r^3\beta n'^2.$$

We first claim $\chi(R[I_2]) \leq r-4$. If $r \in \{4, 5\}$, it is clear. Assume $r \geq 6$. By Claim 5.3, $|I_1| + |I_2| \geq 2\left(\frac{r-3}{r-2} - 4\beta\right)k$, hence $|I_2| \geq \left(\frac{r-4}{r-2} - 8\beta\right)k$. Therefore,

$$\delta(R[I_2]) \geq |I_2| - (k - \delta(R)) \geq |I_2| - \left(\frac{1}{r-2} + 2\beta\right)k > \frac{3r-16}{3r-13}|I_2|.$$

Hence, by Theorem 2.16, $R[I_2]$ is $(r-4)$-colorable. Thus, for all $r \geq 4$, I_2 can be partitioned into $r-4$ disjoint independent sets. Hence $|I_2| \leq (r-4)\left(\frac{1}{r-2} + 2\beta\right)k$.

We can then again apply Claim 5.3 to get $|I_1| \geq (1 - 2r\beta)k$, which further implies that

$$\delta(R[I_1]) \geq |I_1| - \left(\frac{1}{r-2} + 2\beta\right)k > \frac{3r-10}{3r-7}|I_1|.$$

Since $R[I_1]$ is K_{r-1}-free, it follows from Theorem 2.16 that $R[I_1]$ is $(r-2)$-colorable. Hence, I_1 can be partitioned as $I_1 = W_1 \cup \cdots \cup W_{r-2}$, where each W_i is an independent set in R. We now estimate $|W_i|$ by observing that

$$\Big(\frac{1}{r-2}+2\beta\Big)k \geq |W_i| \geq |I_1| - (r-3)\Big(\frac{1}{r-2}+2\beta\Big)k \geq \Big(\frac{1}{r-2}-4r\beta\Big)k.$$

Thus, $||W_i| - \frac{k}{r-2}| \leq 4r\beta k$ for each $i \in [r-2]$.

Moreover, as each W_i is independent, each vertex in W_i has at most $\big(\frac{1}{r-2}+2\beta\big)k - |W_i| \leq (4r+2)\beta k$ non-neighbors outside W_i. Thus, the total number of missing edges across all distinct pairs (W_i, W_j) is at most $(2r+1)\beta k^2$. Consequently, we obtain

$$\begin{aligned} |E(R)\triangle E(T_{k,r-2})| &\leq k(k-|I_1|) + (2r+1)\beta k^2 + k \cdot \left(\sum_{i=1}^{r-2} \Big||W_i| - \frac{k}{r-2}\Big|\right) \\ &\leq (4r^2+1)\beta k^2. \end{aligned}$$

It follows from Lemma 2.11 that $|E(G)\triangle E(T_{n,r-2})| \leq (8r^2+3)\beta n^2 \leq 4(8r^2+3)\beta n'^2 \leq 30r^3\beta n'^2$.

Case 2: $R[I_2]$ **contains at least one copy of** K_{r-3}. Suppose that $\{z_1, \ldots, z_{r-3}\} \subseteq I_2$ forms a copy of K_{r-3} in $R[I_2]$. We further divide the remainder of the proof into two subcases, depending on whether $N(\{z_1, \ldots, z_{r-3}\}) \cap I_1 = \varnothing$.

Subcase 2.1: $N(\{z_1, \ldots, z_{r-3}\}) \cap I_1 = \varnothing$. We claim that $R[I_2]$ is $(r-3)$-colorable. As $R[I_2]$ is K_{r-2}-free, this is clear when $r = 4$. We may assume that $r \geq 5$. Note that

$$|N(\{z_1, \ldots, z_{r-3}\})| \geq k - (r-3)(k-\delta(R)) \geq \Big(\frac{1}{r-2} - 2\beta(r-3)\Big)k.$$

Hence $|I_1| \leq \big(\frac{r-3}{r-2} + 2\beta(r-3)\big)k$. By Claim 5.3, we have

$$\Big(\frac{r-3}{r-2} - 3r\beta\Big)k < |I_2| \leq |I_1| \leq \Big(\frac{r-3}{r-2} + 2\beta(r-3)\Big)k.$$

As each vertex in R has at most $\big(\frac{1}{r-2} + 2\beta\big)k$ non-neighbors,

$$\delta(R[I_2]) \geq |I_2| - \Big(\frac{1}{r-2} + 2\beta\Big)k > \frac{3r-13}{3r-10}|I_2|.$$

Then it follows from Theorem 2.16 that $R[I_2]$ is $(r-3)$-colorable as claimed. Therefore, I_2 admits a partition $I_2 = W_1 \cup \cdots \cup W_{r-3}$ where each W_i is an independent set in R for any $i \in [r-3]$. Hence,

$$\Big(\frac{1}{r-2}+2\beta\Big)k \geq |W_i| \geq |I_2| - (r-4)\Big(\frac{1}{r-2}+2\beta\Big)k \geq \Big(\frac{1}{r-2} - (5r-8)\beta\Big)k.$$

Then, for each $i \in [r-3]$, we have $||W_i| - \frac{k}{r-2}| \leq 5r\beta k$.

Moreover, since each vertex in W_i has at most $\left(\frac{1}{r-2}+2\beta\right)k-|W_i| \le (5r-6)\beta k$ non-neighbors outside W_i, the total number of missing edges among all distinct pairs W_i and W_j is at most $\frac{(5r-6)\beta k^2}{2}$. Let $R' = R[I_2]$ and $k' = \frac{r-3}{r-2}k$, $G' = G\left[\bigcup_{i\in I_2} V_i\right]$ and $n' = \frac{r-3}{r-2}n$. We conclude that

$$|E(R')\triangle E(T_{k',r-3})| \le k|k'-|I_2|| + \frac{(5r-6)\beta k^2}{2} + k\cdot\left(\sum_{i=1}^{r-3}\left||W_i| - \frac{k}{r-2}\right|\right)$$
$$\le 5r^2\beta k^2.$$

It follows by Lemma 2.11 that $|E(G')\triangle E(T_{n',r-3})| \le 30r^3\beta n'^2$.

Subcase 2.2: $N(\{z_1,\ldots,z_{r-3}\})\cap I_1 \neq \varnothing$. Pick an arbitrary vertex $y \in N(\{z_1,\ldots,z_{r-3}\})\cap I_1$. Let $I_1' = I_1\setminus\{y,z_1,\ldots,z_{r-3}\}$ and $I_2' = I_2\setminus\{y,z_1,\ldots,z_{r-3}\}$.

If $R[I_2']$ is K_{r-3}-free, the proof is almost identical to that in Case 1 (with only the change of replacing (I_1, I_2) by (I_1', I_2')). Now, suppose that $R[I_2']$ contains a copy of K_{r-3} on the vertices $\{z_1',\ldots,z_{r-3}'\}\subseteq I_2'$. Since G is H-free, the following claim is an immediate consequence of Lemma 5.2.

Claim 5.5 $N(\{z_1',\ldots,z_{r-3}'\})\cap I_1' = \varnothing$.

Proof Suppose for contradiction that there exists $y'\in I_1'$ such that $\{y', z_1',\ldots,z_{r-3}'\}$ forms a copy of K_{r-2} in R. Let $Y, Y', Z_1, Z_1',\ldots,Z_{r-3}, Z_{r-3}'$ be the clusters of $V(G)$ corresponding to the vertices $y, y', z_1, z_1',\ldots,z_{r-3}, z_{r-3}'$ in R respectively, and let

$$X = X(I_1,I_2)\cap(Y'\cup Z_1'\cup\cdots\cup Z_{r-3}') \quad\text{and}\quad X' = X(I_1,I_2)\setminus X.$$

It is routine to verify by definition that the two sets $\{X, Y, Z_1,\ldots,Z_{r-3}\}$ and $\{X', Y', Z_1',\ldots,Z_{r-3}'\}$ both satisfy the assumptions of Lemma 5.2. Since G is H-free, it follows from Lemma 5.2 that $\max\{\chi(G[X]),\chi(G[X'])\}\le C_3$, which contradicts the assumption that $\chi(G[X(I_1,I_2)])\ge \frac{C}{4^k} > 2C_3$. □

Then, using essentially the same argument as in Subcase 2.1, with I_2' in place of I_2, one can show that the conclusion of Subcase 2.1 also holds in this case. □

6 Fractional Chromatic Threshold

In this section, we aim to prove Theorem 1.4. First, consider the case when $\mathcal{M}(H)$ contains no forest. As in Construction 2.2, let $G = G'\vee K_{r-2}[|G'|]$, where G' is a $(C, |H|+1)$-Erdős graph. Then, G is an H-free graph with $\delta(G) = \frac{r-2}{r-1}\cdot|G|$ and $\chi_f(G)\ge C$, which implies that $\delta_{\chi_f}(H)\ge\frac{r-2}{r-1}$. On the other hand, $\delta_{\chi_f}(H)$ is trivially upper bounded by the Turán density of H and so by Erdős–Stone Theorem [10], we have $\delta_{\chi_f}(H)\le\frac{r-2}{r-1}$. Thus, we have the following proposition.

Proposition 6.1 *Let H be a graph with $\chi(H) = r\ge 3$. If $\mathcal{M}(H)$ does not contain a forest, then $\delta_{\chi_f}(H) = \frac{r-2}{r-1}$.*

Next, we turn to the case where $\mathcal{M}(H)$ contains a forest. Using Construction 2.5, i.e., let $G = G' \vee K_{r-3}[|G'|]$, where G' is a $(C, |H|+1)$-Erdős graph, we obtain an H-free graph G with $\delta(G) = \frac{r-3}{r-2} \cdot |G|$ and $\chi_f(G) \geq C$, which gives the lower bound $\delta_{\chi_f}(H) \geq \frac{r-3}{r-2}$.

It remains to show the following theorem to obtain the corresponding upper bound.

Theorem 6.2 *Let H be a graph with $\chi(H) = r \geq 3$ and suppose $\mathcal{M}(H)$ contains a forest. Then for any n-vertex H-free graph G with $\delta(G) \geq \left(\frac{r-3}{r-2} + \varepsilon\right)n$, we have $\chi_f(G) = O_{\varepsilon,H}(1)$.*

Proof Suppose that $H \subseteq F \vee K_{r-2}[t]$ for some forest $F \in \mathcal{M}(H)$. Let G be an H-free graph with $\delta(G) \geq \left(\frac{r-3}{r-2} + \varepsilon\right)n$. We shall bound its fractional chromatic number.

If $r \geq 4$, then for any vertex $v \in V(G)$ we have

$$\frac{\delta(G[N(v)])}{d(v)} \geq \frac{d(v) - (n - d(v))}{d(v)} \geq \frac{\left(\frac{r-3}{r-2} + \varepsilon\right)n - \left(\frac{1}{r-2} - \varepsilon\right)n}{\left(\frac{r-3}{r-2} + \varepsilon\right)n} > \frac{r-4}{r-3} + \varepsilon.$$

By Erdős–Simonovits supersaturation theorem [12], there exist at least $\alpha n^{(r-2)t}$ copies of $K_{r-2}[t]$ in $G[N(v)]$ for some constant $\alpha = \alpha(\varepsilon)$. Note that the statement is also true for $r = 3$.

For each copy T of $K_{r-2}[t]$ in G, $G[N(T)]$ must be F-free since G is H-free. By Fact 2.15, $G[N(T)]$ is $|F|$-colorable, as F is a forest. Thus, $N(T)$ can be partitioned into s disjoint independent sets, where $s = \chi(G[N(T)]) \leq |F| \leq |H|$. Label these independent sets as $T(1), T(2), \ldots, T(s)$. Then, for any vertex v and a copy T' of $K_{r-2}[t]$ in $G[N(v)]$, we assign to v the color (T', i) where the independent set $T'(i)$ containing v. Since G contains at most $n^{(r-2)t}$ copies of $K_{r-2}[t]$ and each copy of $K_{r-2}[t]$ defines at most $|H|$ independent sets in its neighborhood, every vertex receives at least $\alpha n^{(r-2)t}$ colors from a palette of size at most $a := |H| n^{(r-2)t}$.

Let $b = \alpha n^{(r-2)t}$. Now we construct an $a : b$-coloring of $V(G)$. For each vertex v, arbitrarily select b colors among those assigned to it. We now show that this is a valid b-fold coloring of G. Indeed, a color is assigned to both vertices u and v if and only if there exists a copy T of $K_{r-2}[t]$ and $1 \leq i \leq |H|$ such that $V(T) \subseteq N(u) \cap N(v)$ and $u, v \in T(i)$, which implies that u and v are non-adjacent. Thus

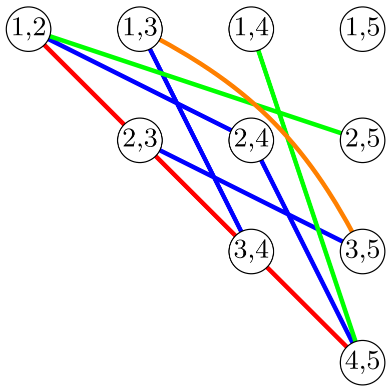
$$\chi_f(G) \leq \frac{a}{b} \leq \frac{|H| n^{(r-2)t}}{\alpha n^{(r-2)t}} = O_{\varepsilon,H}(1),$$

as desired. □

7 Bounded-VC Chromatic Threshold

In this section, we prove Theorem 1.5. Given a set system $\mathcal{F} \subseteq 2^V$, the *Vapnik–Chervonenkis dimension* (*VC-dimension* for short) of $\mathcal{F}$, denoted by $\mathrm{VC}(\mathcal{F})$, is the largest integer d such that there exists a subset $X \subseteq V$ with $|X| = d$ satisfying the following condition: for every subset $Y \subseteq X$, there exists $F \in \mathcal{F}$ with $F \cap X = Y$.

Fig. 2 The shift graph Sh_5^2



In this case, we say that X is *shattered* by $\mathcal{F}$. We define the VC-dimension of a graph G, denoted by $\mathrm{VC}(G)$, to be the VC-dimension of the set system induced by the neighborhoods of its vertices, that is, $\mathrm{VC}(G) := \mathrm{VC}(\mathcal{F})$, where $\mathcal{F} = \{N(v) : v \in V(G)\}$.

7.1 Proof of the Lower Bound

To prove the lower bound $\delta_\chi^{\mathrm{VC}}(K_r) \geq \frac{r-3}{r-2}$, we aim to show that for any integer C, there exists an infinite sequence of n-vertex K_r-free graphs G_n with $\mathrm{VC}(G_n) = 2$, $\delta(G_n) \geq \frac{r-3}{r-2}n$, and $\chi(G_n) \geq C$. For this purpose, we will employ a classical construction due to Erdős and Hajnal [9].

Definition 7.1 *(Shift graph)* For positive integers $m > k \geq 2$, the shift graph Sh_m^k defined on the vertex set $V(\mathrm{Sh}_m^k) = \{(x_1, \dots, x_k) : 1 \leq x_1 < \cdots < x_k \leq m\}$ is a graph in which two vertices $\boldsymbol{x} = (x_1, \dots, x_k)$ and $\boldsymbol{y} = (y_1, \dots, y_k)$ are adjacent if and only if $x_i = y_{i+1}$ for all $i \in [k-1]$.

In [9], Erdős and Hajnal showed that the shift graph Sh_m^2 is K_3-free and has chromatic number $(1 + o(1)) \log_2 m$. We now demonstrate that the VC-dimension of Sh_m^2 equals 2.

Proposition 7.2 $\mathrm{VC}(\mathrm{Sh}_m^2) = 2$.

Proof As depicted in Fig. 2, the 2-set $\{(2, 4), (3, 4)\}$ is shattered which establishes the lower bound. To prove the upper bound $\mathrm{VC}(\mathrm{Sh}_m^2) \leq 2$, suppose, for contradiction, that there exists a subset $\{(i_1, i_2), (j_1, j_2), (k_1, k_2)\} \subseteq V(\mathrm{Sh}_m^2)$ which is shattered by the neighborhoods of vertices in Sh_m^2. Then, there exists $(x_1, x_2) \in V(\mathrm{Sh}_m^2)$ such that all of $(i_1, i_2), (j_1, j_2), (k_1, k_2)$ are neighbors of (x_1, x_2). By definition, every neighbor of (x_1, x_2) must be of the form either (a, x_1) for some $1 \leq a < x_1$ or (x_2, b) for some $x_2 < b \leq m$. By the pigeonhole principle, at least two of the vertices $(i_1, i_2), (j_1, j_2), (k_1, k_2)$ must share a common coordinate structure, say without loss of generality, $(i_1, i_2) = (a, x_1)$ and $(j_1, j_2) = (a', x_1)$ with distinct $a, a' < x_1$. To derive the desired contradiction, it suffices to show that there is no vertex that is adjacent to both (i_1, i_2) and (k_1, k_2), but not to (j_1, j_2). Indeed, if $(k_1, k_2) = (x_2, b)$, then (a, x_1) and (x_2, b) have a unique common neighbor (x_1, x_2), which is also adjacent to (a', x_1). If $(k_1, k_2) = (a'', x_1)$ for some $a'' < x_1$, then the common neighbor of (a, x_1) and (a'', x_1) is of the form (x_1, c) for some $x_1 < c$, which is also adjacent to (a', x_1). $\square$

We now present the construction that establishes the lower bound.

Construction 7.3 Given $m \geq 2$ and $r \geq 4$, for sufficiently large n, let G' be the vertex-disjoint union of Sh_m^2 and $\frac{n}{r-2} - \binom{m}{2}$ isolated vertices. Then let $G = G' \vee K_{r-3}[|G'|]$.

Since G' is K_3-free, G is K_r-free. Moreover, it is not hard to see that any shattered subset of vertices in G intersects at most one vertex part. Thus, we have $\mathrm{VC}(G) = \mathrm{VC}(\mathrm{Sh}_m^2) = 2$. To conclude, G is a K_r-free graph with $\delta(G) \geq \frac{r-3}{r-2} \cdot n$ and $\mathrm{VC}(G) = 2$, and $\chi(G)$ grows arbitrarily large as m tends to infinity.

7.2 Proof of the Upper Bound

The upper bound follows from the theorem below.

Theorem 7.4 *Let $c > 0$, and let G be an n-vertex K_r-free graph with minimum degree $\left(\frac{r-3}{r-2} + c\right)n$ and VC-dimension d. Then $\chi(G) \leq \left(\frac{16d}{c} \cdot \log(\frac{d}{c})\right)^{r-2}$.*

To prove Theorem 7.4, we will make use of the celebrated ε-net theorem of Haussler and Welzl [16]. We will need some definitions before introducing this result.

A *transversal* of a set system $\mathcal{F} \subseteq 2^X$ is a subset $T \subseteq X$ that intersects every set in $\mathcal{F}$. The *transversal number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is the smallest size among all transversals of $\mathcal{F}$. A *fractional transversal* of $\mathcal{F}$ is a weight function $\omega : X \to [0, 1]$ defined on the ground set X such that, for every set $A \in \mathcal{F}$, we have $\omega(A) := \sum_{a \in A} \omega(a) \geq 1$. The *fractional transversal number* of $\mathcal{F}$, denoted by $\tau^*(\mathcal{F})$, is defined as the minimum of $\omega(X) = \sum_{x \in X} \omega(x)$ over all fractional transversals ω of $\mathcal{F}$.

By definitions, $\tau^*(\mathcal{F}) \leq \tau(\mathcal{F})$. However, the gap between them could be arbitrarily large. Haussler and Welzl [16] proved the following lemma, showing that the gap can be bounded if $\mathcal{F}$ has bounded VC-dimension.

Lemma 7.5 ([16]) *Every set system $\mathcal{F}$ with VC-dimension d satisfies $\tau(\mathcal{F}) \leq 16d\tau^*(\mathcal{F}) \log(d\tau^*(\mathcal{F}))$.*

Proof of Theorem 7.4 By assumption, the set system $\mathcal{F} = \{N(v) : v \in G\}$ has VC-dimension d. Since every set in $\mathcal{F}$ contains at least cn elements of $V(G)$, it follows that $\tau^*(\mathcal{F}) \leq \frac{1}{c}$ as demonstrated by the constant fractional transversal $\omega \equiv \frac{1}{cn}$. Therefore, by Lemma 7.5 we can find a transversal $T \subseteq V(G)$ for $\mathcal{F}$ with size $|T| = \tau(\mathcal{F}) \leq \frac{16d}{c} \cdot \log(\frac{d}{c})$.

We prove Theorem 7.4 by induction on r. Observe that by the definition of transversal, $V(G) = \bigcup_{v \in T} N(v)$ and so $\chi(G) \leq \sum_{v \in T} \chi(G[N(v)])$. For the base case $r = 3$, when G is K_3-free, $N(v)$ is an independent set for each $v \in T$. Thus $\chi(G) \leq |T| \leq \frac{16d}{c} \cdot \log(\frac{d}{c})$. Now, assume that Theorem 7.4 holds for every integer $3 \leq r' < r$.

Claim 7.6 For every vertex $v \in V(G)$, $\chi(G[N(v)]) \leq \left(\frac{16d}{c} \cdot \log(\frac{d}{c})\right)^{r-3}$.

Proof Let $v \in V(G)$ and define $G' := G[N(v)]$, then $|G'| = d(v) \geq \left(\frac{r-3}{r-2} + c\right)n$. Since G is K_r-free, it follows that G' is K_{r-1}-free. Thus, for every vertex $w \in V(G')$, we have

$$d_{G'}(w) \geq d_G(w) - (n - |G'|) \geq \left(\frac{r-3}{r-2} + c\right)n - n + |G'| \geq \left(\frac{r-4}{r-3} + c'\right)|G'|,$$

where $c' = \frac{c(r-2)^2}{(r-3)(c(r-2)+r-3)} > c > 0$. Since $\mathrm{VC}(G') \leq \mathrm{VC}(G) \leq d$, by the induction hypothesis, we obtain $\chi(G') \leq \left(\frac{16d}{c} \cdot \log(\frac{d}{c})\right)^{r-3}$. $\square$

Recall that $\chi(G) \leq \sum_{v \in T} \chi(G[N(v)])$. Thus, by Claim 7.6, we have

$$\chi(G) \leq |T| \cdot \left(\frac{16d}{c} \cdot \log\left(\frac{d}{c}\right)\right)^{r-3} \leq \left(\frac{16d}{c} \cdot \log\left(\frac{d}{c}\right)\right)^{r-2},$$

as desired. $\square$

A Appendix: Proof of Lemma 5.2

To embed near-acyclic graphs, we instead embed so-called Zykov graphs which are universal graphs for near-acyclic graphs. Let $K(v, X)$ denote the set of pairs $\{vx : x \in X\}$.

Definition A.1 *(Modified Zykov graphs* [1]*)* Let $T_1, \ldots, T_\ell$ be disjoint trees, where T_j has bipartition $A_j^0 \cup A_j^1$, $j \in [\ell]$. The graph $Z_\ell(T_1, \ldots, T_\ell)$ is constructed as follows:

$$V(Z_\ell(T_1, \ldots, T_\ell)) := \left(\bigcup_{j \in [\ell]} A_j^0 \cup A_j^1\right) \cup \{u_I : I \subseteq [\ell]\}$$

with edge set

$$E(Z_\ell(T_1, \ldots, T_\ell)) := \bigcup_{j=1}^{\ell} \left(E\left(T_j\right) \cup \bigcup_{j \in I \subseteq [\ell]} K(u_I, A_j^0) \cup \bigcup_{j \notin I \subseteq [\ell]} K(u_I, A_j^1)\right).$$

For each $r \geq 3$ and $t \in \mathbb{N}$, the modified Zykov graph $Z_\ell^{r,t}(T_1, \ldots, T_\ell)$ is the graph obtained from $Z_\ell(T_1, \ldots, T_\ell)$ by performing the following two operations:

(a) Add vertices $W = \{w_1, \ldots, w_{r-3}\}$, and all edges with an endpoint in W.
(b) Blow up each vertex u_I with $I \subseteq [\ell]$ to a set S_I of size t and each vertex w_j in W to a set S_j' of size t.

Finally, we shall write $Z_\ell^{r,t}$ for the modified Zykov graph obtained when each tree T_i, $i \in [\ell]$, is a single edge; that is, $Z_\ell^{r,t} = Z_\ell^{r,t}(e_1, \ldots, e_\ell)$.

Observation A.2 ([1]) Let $\chi(H) = r$. Then H is r-near-acyclic if and only if there exist trees $T_1, \dots, T_\ell$ and $t \in \mathbb{N}$ such that H is a subgraph of $Z_\ell^{r,t}(T_1, \dots, T_\ell)$.

Let ℓ be an integer. For each $I \subseteq [\ell]$, let $\boldsymbol{u}(I) = \{u_1, \dots, u_\ell\} \in \{0, 1\}^\ell$ with $u_i = 1$ for $i \in I$ and $u_i = 0$ for $i \notin I$. For convenience, for any $\boldsymbol{u}(I) = \{u_1, \dots, u_\ell\} \in \{0, 1\}^\ell$, let $A^{\boldsymbol{u}(I)} = \bigcup_{i=1}^{\ell} A_i^{u_i}$.

We restate Lemma 5.2 with more details as follows.

Lemma A.3 *Let ℓ, t be two positive integers, and $T = T_1 \cup T_2 \cup \cdots \cup T_\ell$ be a forest, where $V(T_i) = A_i^0 \cup A_i^1$ for $i \in [\ell]$. For any $\beta > 0$, there exist a constant C and sufficiently large integer n such that the following holds: Let G be an n-vertex graph and $X, Y, Z_1, \dots, Z_{r-3}$ be pairwise disjoint subsets of $V(G)$, with $\chi(G[X]) \geq C$ and $|Y| = |Z_j|$ for each $j \in [r-3]$. Suppose that (Y, Z_j) and (Z_i, Z_j) are (ε, d)-regular for each $i \neq j$, and that*

$$|N(x) \cap Y| \geq \beta |Y| \quad \textit{and} \quad |N(x) \cap Z_j| \geq \left(\frac{1}{2} + \beta\right)|Z_j|$$

for every $x \in X$ and $j \in [r-3]$. Then $Z_\ell^{r,t}(T_1, \dots, T_\ell)$ can be embedded into G in the following way: there exist a copy of T in $G[X]$, disjoint subsets S_I of size t for $I \subseteq [\ell]$, and S'_j of size t for $j \in [r-3]$ such that

- *for each $I \subseteq [\ell]$, $S_I \subseteq N(A^{\boldsymbol{u}(I)}) \cap Y$;*
- *for each $j \in [r-3]$, $S'_j \subseteq N(T) \cap Z_j$, and S'_j is completely adjacent to each $S'_{j'}$ with $j \neq j'$ and also to each S_I with $I \subseteq [\ell]$.*

Given a graph G, a set $Y \subseteq V(G)$, and integers $\ell, t \in \mathbb{N}$, define $\mathcal{G}_\ell^{r,t}(Y)$ to be the collection of functions

$$S : 2^{[\ell]} \cup [r-3] \to \binom{Y}{t}.$$

We say that $S = (S_I : I \subseteq [\ell], S'_1, \dots, S'_{r-3}) \in \mathcal{G}_\ell^{r,t}(Y)$ is *proper* if these sets are pairwise disjoint. We shall write $\mathcal{F}_\ell^{r,t}(Y)$ for the collection of proper functions in $\mathcal{G}_\ell^{r,t}(Y)$. For any $S \in \mathcal{F}_\ell^{r,t}(Y)$, it is natural to think of S as a family $\{S_I : I \subseteq [\ell]\} \cup \{S'_j : j \in [r-3]\}$ of disjoint subsets of Y of size t.

For an ordered pair (x, y) of vertices of G, a function $S \in \mathcal{F}_\ell^{r,t}(Y)$, and $i \in [\ell]$, we write $(x, y) \to_i S$, if the following conditions hold:

- $S'_j \subseteq N(x, y)$ for every $j \in [r-3]$;
- $\bigcup_{I : i \in I} S_I \subseteq N(x)$ and $\bigcup_{I : i \notin I} S_I \subseteq N(y)$.

For an edge $e = xy \in E(G)$, we write $e \to_i S$ if either $(x, y) \to_i S$ or $(y, x) \to_i S$. When edges $e_1, \dots, e_\ell$ are given, we write $\mathbf{e}^j$ to denote the j-tuple $(e_1, \dots, e_j)$ for each $j \in [\ell] \cup \{0\}$, with $\mathbf{e}^0$ representing the empty tuple. We define

$$\mathbf{e}^\ell \to S \Leftrightarrow e_i \to_i S \quad \text{for each } i \in [\ell].$$

Note that the graph $Z_\ell^{r,t}$ consists of a set of pairwise disjoint edges $e_1, \dots, e_\ell$ and a function $S \in \mathcal{F}_\ell^{r,t}(Y)$ such that $\mathbf{e}^\ell \to S$.

Recall that for any subset $X \subseteq V(G)$, $E(X)$ denotes the edge set of the subgraph $G[X]$. If $D \subseteq E(G)$, then $\delta(D)$ ($\bar{d}(D)$) represents the minimum degree (average degree, respectively) of the graph induced by D, that is the graph with vertex set $\bigcup_{e \in D} e$ and edge set D.

In the following definition, the sets $D(\mathbf{e}^j)$ should be interpreted as forming a hierarchical structure (or tree) of graphs. Specifically, each edge of $D(\mathbf{e}^j)$ is associated with a distinct graph $D(\mathbf{e}^{j+1})$. This hierarchy satisfies the following key properties:

(a) each graph $D(\mathbf{e}^j)$ has large minimum degree, and
(b) for a positive proportion of the sets $S \in \mathcal{F}_\ell^{r,t}(Y)$, we have $\mathbf{e}^\ell \to S$.

Definition A.4 *((C, α)-rich in copies of $Z_\ell^{r,t}$)* Let X and Y be disjoint vertex sets in a graph G, let $C \in \mathbb{N}$ and $\alpha > 0$, and let $s := 2^\ell t$. We say that (X, Y) is (C, α)-rich in copies of $Z_\ell^{r,t}$ if

$$\begin{aligned}
&\exists D = D(\mathbf{e}^0) \subseteq E(X) \quad \text{s.t.} \\
&\quad \forall e_1 \in D,\ \exists D(\mathbf{e}^1) \subseteq E(X) \quad \text{s.t.} \\
&\qquad\qquad \vdots \\
&\qquad\qquad \forall e_{\ell-1} \in D(\mathbf{e}^{\ell-2}),\ \exists D(\mathbf{e}^{\ell-1}) \subseteq E(X) \quad \text{s.t.} \quad \forall e_\ell \in D(\mathbf{e}^{\ell-1})
\end{aligned}$$

the following properties hold:

(a) $\delta(D), \delta(D(\mathbf{e}^1)), \ldots, \delta(D(\mathbf{e}^{\ell-1})) > C$, and
(b) $|\{S \in \mathcal{F}_\ell^{r,t}(Y) : \mathbf{e}^\ell \to S\}| \geq \alpha |Y|^s$.

Definition A.5 *(Good function, (C, α)-dense, [1])* A function $S \in \mathcal{F}_\ell^{r,t}(Y)$ is (ℓ, t, C, α)-good for a tuple $\mathbf{e}^q$ and (X, Y) if there exist sets

$$E_{q+1}, \ldots, E_\ell \subseteq E(X), \text{ with } \bar{d}(E_j) \geq 2^{-\ell} \alpha C \text{ for each } q+1 \leq j \leq \ell,$$

such that for every $e_{q+1} \in E_{q+1}, \ldots, e_\ell \in E_\ell$, we have $\mathbf{e}^\ell \to S$.

When the constants (ℓ, t, C, α) and the sets (X, Y) are clear from the context, we shall omit them. We shall abbreviate "(ℓ, t, C, α)-good for $\mathbf{e}^0$ and (X, Y)" to "(ℓ, t, C, α)-good for (X, Y)".

The pair (X, Y) is (C, α)-*dense* in copies of $Z_\ell^{r,t}$ if there exist at least $2^{-\ell} \alpha |Y|^s$ families $S \in \mathcal{F}_\ell^{r,t}(Y)$ which are (ℓ, t, C, α)-good for (X, Y).

Proposition A.6 ([1]) *For every $\ell, t \in \mathbb{N}$ and $\beta > 0$, there exists $\alpha > 0$ such that, for every $C \in \mathbb{N}$, there exists $C' \in \mathbb{N}$ such that the following holds. Let G be a graph and let X and Y be disjoint subsets of $V(G)$, such that $|N(x) \cap Y| \geq \beta |Y|$ for every $x \in X$.*

Then either $\chi(G[X]) \leq C'$, or (X, Y) is (C, α)-rich in copies of $Z_\ell^{3,t}$.

Lemma A.7 ([1]) *If (X, Y) is (C, α)-rich in copies of $Z_\ell^{r,t}$, then (X, Y) is (C, α)-dense in copies of $Z_\ell^{r,t}$.*

Combining Proposition A.6 and Lemma A.7, we can substitute the condition "(X, Y) is (C^*, α)-dense in copies of $Z^{3,t^*}_{\ell^*}$" of [1, Proposition 36] with "$|N(x) \cap Y| \geq \beta|Y|$ for every $x \in X$". We will use the proof of [1, Proposition 36] to prove Lemma A.3, omitting the repetitive parts.

Proof of Lemma A.3 Let G be a graph, and let X, Y and $Z_1, \dots, Z_{r-3}$ be disjoint subsets of $V(G)$, with $\chi(G[X]) \geq C$ and $|Y| = |Z_j|$ for each $j \in [r-3]$. Let $Z := Z_1 \cup \dots \cup Z_{r-3}$. Suppose that (Y, Z_j) and (Z_i, Z_j) are (ε, d)-regular for each $i \neq j$, and that

$$|N(x) \cap Y| \geq \beta|Y| \quad \text{and} \quad |N(x) \cap Z_j| \geq \Big(\frac{1}{2} + \beta\Big)|Z_j|$$

for every $x \in X$ and $j \in [r-3]$.

According to [1, Lemma 37 and Claim 39], there exist a copy Q of $K_{r-3}[t]$ with t vertices in Z_j for each $j \in [r-3]$ and a function $S \in \mathcal{F}^{r,t}_\ell(Y \cup Z)$ such that the following hold.

- The sets S_j, $j \in [r-3]$, are the parts of Q;
- for each $I \subseteq [\ell]$, $S_I \subseteq N(Q) \cap Y$ with $|S_I| = t$;
- there exist sets $E_1, \dots, E_\ell \subseteq E(X)$, with $\bar{d}(E_i) \geq 8|T|$ for each $1 \leq i \leq \ell$, such that for every $e_1 \in E_1, \dots, e_\ell \in E_\ell$, we have $\mathbf{e}^\ell \to S$;
- S_j is completely adjacent to each $S_{j'}$ with $j \neq j'$, to each S_I, and to each edge $e \in \bigcup_{i \in [\ell]} E_i$.

Now we can embed $Z^{r,t}_\ell(T_1, \dots, T_\ell)$ via hierarchical selection. For each $i \in [\ell]$ and each edge $e \in E_i$, let $e = xy$ be such that $(x, y) \to_i S$, and orient the edge e from x to y. For each $i \in [\ell]$, by choosing a maximal bipartite subgraph of E_i, and then removing at most half the edges, we can find a set $E'_i \subseteq E_i$ such that,

(a) E'_i is bipartite, with bipartition $A^0_i \cup A^1_i$,
(b) every edge $e \in E'_i$ is oriented from A^0_i to A^1_i, and
(c) $\bar{d}(E'_i) \geq 2|T|$.

Thus, by Fact 2.14, there exists, for each $i \in [\ell]$, a copy T'_i of T_i in the induced subgraph

$$E'_i - (V(T'_1) \cup \dots \cup V(T'_{i-1})),$$

since removing a vertex can only decrease the average degree by at most two. Hence, by the definition of $S \in \mathcal{F}^{r,t}_\ell(Y \cup Z)$, these trees together with S_I for each $I \subseteq [\ell]$ and S_j for each $j \in [r-3]$, form a copy of $Z^{r,t}_\ell(T_1, \dots, T_\ell)$ in G, so we are done. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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