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https://doi.org/10.1007/s11750-025-00701-6

ORIGINAL PAPER

Semicontinuity conditions for set-valued equilibrium problems and applications to set optimization problems

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Received: 19 November 2024 / Accepted: 25 June 2025

Abstract
In this paper, we examine parametric set-valued equilibrium problems. We begin by introducing a quasiconvexity property for set-valued maps and exploring its relationship with existing concepts. Next, we analyze the semicontinuity and continuity of approximate solution maps for these problems, without assuming the solid condition of the ordered cone and the compact values of the objective map. Finally, we demonstrate an application of the main results to set optimization problems involving a possibly less order relation.

Keywords Approximate solution · Equilibrium problems · Semicontinuity · Set optimization problems

Mathematics Subject Classification 49K40 · 90C31 · 91B50

1 Introduction

Optimization theory is one of the rapidly developing and promising fields of Mathematics, with numerous practical applications across nearly all aspects of life and society (Mordukhovich 2018; Goeleven 2017; Kinderlehrer and Stampacchia 2000). Vector optimization theory, in particular, has found numerous applications in decision-making problems in physics, medicine, economics, engineering, transportation, and chemistry (Geering 2007; Lenhart and Workman 2007; Bigi et al. 2019). Furthermore, many practical situations have led to the development of generalized models of vector optimization problems. One of the extended directions in this area is the study of set-valued equilibrium problems (Kassay and Rădulescu 2018) and set optimization problems (Khan et al. 2015).

Many important and interesting results have been achieved in various topics related to these problems, including existence conditions (TN 2022; Alleche and Rădulescu 2017; Hernández and Rodríguez-Marín 2007), optimality conditions (Tung 2022;

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Hernández and Rodríguez-Marín 2007; Alonso and Rodríguez-Marín 2009), stability (Xu and Li 2014, 2016; Li et al. 2016), solution methods (Shehu et al. 2023; Zhao et al. 2025) and well-posedness (Gutiérrez et al. 2012; Khoshkhabar-amiranloo and Khorram 2015). The semicontinuity of the solution maps to parametric equilibrium problems and parametric set optimization problems has been extensively studied by the authors. For the parametric equilibrium problems, we start with the paper (Anh et al. 2020a), sufficient conditions for the Hausdorff continuity of approximate solutions in two models associated with weakly set-valued equilibrium problems were examined, with linear scalarization techniques for sets and the concavity of the objective maps being utilized. In a separate study (Anh et al. 2021), the authors investigated the Hausdorff continuity of solution maps of two models related to strongly set-valued equilibrium problems by applying relaxed concavity conditions. In Anh et al. (2024), based on the scalarization method, along with generalized concavity of set-valued maps, the authors achieved the Hausdorff continuity of the approximate solution maps to set-valued equilibrium problems. For the parametric set optimization problems, Xu and Li (2014) investigated upper semicontinuity, lower semicontinuity, and the closedness of the solution maps based on the upper order relation using constraint conditions named upper-property, converse upper-property, and the continuity of the set-valued objective map. Subsequently, Xu and Li (2016) weakened and modified the assumptions in Xu and Li (2014), introducing a level map and removing the upper-property. Then, Han and Huang (2017) used the strictly convexity of objective maps to study the semicontinuity of the solution maps to the set optimization problem. Recently, in Anh et al. (2020b), using scalarization methods and cone convexity (or concavity) assumptions, the authors achieve the Hausdorff continuity of the ideal solution maps to set optimization problems. In Han and Yu (2022), the authors used the oriented distance function to obtain the upper and lower semicontinuity of approximate weak minimal solution maps to parametric set optimization problems. Very recently, the authors in Anh et al. (2024) have used new nonlinear scalarization functions to study sufficient conditions for the Hausdorff continuity of approximate solution maps to parametric set optimization problems involving set less order relations.

The compactness assumption on the values of set-valued objective maps constitutes an important condition in the study of the semicontinuity of solution maps (Han and Yu 2022; Anh et al. 2024; Xu and Li 2016). Nevertheless, this assumption also restricts the applicability of the corresponding results. Thus, it becomes necessary to replace the compactness assumption with an alternative condition or a weaker form of compactness. Fortunately, recent works (Durea and Florea 2024a, b) have explored the possibility of obtaining a sequential characterization of the compactness of a set with respect to a cone. In Durea and Florea (2024a), the authors have investigate the potential for obtaining a sequential characterization of the compactness of a set relative to a cone. Building on this foundation, they consider several set-valued equilibrium problems and employ the previously established notion of generalized compactness to analyze the existence of solutions to these problems. In Durea and Florea (2024b), the authors study optimality conditions for a special kind of solution to set optimization problems, with the most important one being the sequential compactness with respect to a cone.

Inspired by the observations mentioned above, this article aims to explore sufficient conditions for the semicontinuity of approximate solution maps to set-valued equilibrium problems, without assuming the solid condition of the ordered cone and the compact values of the objective map. Subsequently, we investigate the continuity of ideally approximate solution maps to set optimization problems.

The rest of the paper is structured as follows: Sect. 2 reviews essential concepts and their properties that will be necessary for the subsequent discussions. Section 3 outlines sufficient conditions for the semicontinuity of solution maps to parametric set-valued equilibrium problems. In Sect. 4, we concentrate on the continuity of solution maps to set optimization problems.

2 Preliminaries

In this paper, let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{W}$ be normed spaces. Let $\mathcal{C} \subset \mathbb{Y}$ be a nontrivial, pointed, closed, convex cone with possibly empty interior. $\mathbb{B}$ stands for the unit ball in both $\mathbb{X}$ and $\mathbb{Y}$. Let $\mathbb{R}_+$, $\mathbb{R}_{++}$ be the set of all nonnegative and positive real numbers, respectively.

The family of all nonempty subsets of $\mathbb{Y}$ is denoted by $\mathbf{P}(\mathbb{Y})$. Let Ω be a nonempty convex subset of $\mathbb{X}$, Λ be a nonempty subset of $\mathbb{W}$. We consider the following parametric set-valued equilibrium problem.

(SEP): Find $\bar{x} \in K(\lambda)$ such that

$$F(\bar{x}, y, \lambda) \cap \mathcal{C} \neq \emptyset \quad \forall y \in K(\lambda), \tag{1}$$

where $F : \Omega \times \Omega \times \Lambda \rightrightarrows \mathbb{Y}$ and $K : \Lambda \rightrightarrows \Omega$ are set-valued maps with nonempty values.

We now recall an approximate solution concept introduced in Anh et al. (2021), under the assumption that the cone $\mathcal{C}$ has a nonempty interior, i.e., $\operatorname{int}\mathcal{C} \neq \emptyset$. For $(\varepsilon, \lambda) \in \mathbb{R}_+ \times \Lambda$ and $e \in \operatorname{int}\mathcal{C}$, the ε-approximate solution set with respect to e of (SEP) is defined by

$$\{x \in K(\lambda) \mid (F(x, y, \lambda) + \varepsilon e) \cap \mathcal{C} \neq \emptyset \quad \forall y \in K(\lambda)\},$$

or equivalently,

$$\{x \in K(\lambda) \mid 0 \in F(x, y, \lambda) + \varepsilon e - \mathcal{C} \quad \forall y \in K(\lambda)\}.$$

Obviously, the above concept depends on the choice of a vector e in the interior of the ordering cone. From this observation, we introduce the ε-solutions to (SEP) as follows:

Definition 1 Let $(\varepsilon, \lambda) \in \mathbb{R}_+ \times \Lambda$. A vector $x \in K(\lambda)$ is said to be ε-solution to (SEP), written as $x \in \mathrm{Sol}_+(\mathrm{SEP})(\varepsilon, \lambda)$ if

$$0 \in F(x, y, \lambda) + \varepsilon\mathbb{B}_+ - \mathcal{C} \quad \forall y \in K(\lambda),$$

where $\mathbb{B}_+ = \mathbb{B} \cap \mathcal{C}$.

Remark 1 (a) It is evident that the set $\mathrm{Sol}_+(\mathrm{SEP})(\varepsilon,\lambda)$ does not depend on the choice of a vector e from the interior of the ordering cone. Moreover, we do not require the cone to have an empty interior.

(b) Let $\mathbb{S}$ be the unit sphere and $\mathbb{S}_+ := \mathbb{S} \cap \mathcal{C}$. Then, we have

$$\varepsilon\mathbb{B}_+ - \mathcal{C} = \varepsilon\mathbb{S}_+ - \mathcal{C}.$$

Indeed, we only present for $\varepsilon\mathbb{B}_+ - \mathcal{C} \subset \varepsilon\mathbb{S}_+ - \mathcal{C}$, as the reverse inclusion is trivial. We first prove that $\mathbb{B}_+ \subset \mathbb{S}_+ - \mathcal{C}$. For any $b \in \mathbb{B}_+$, there exists $b' \in (b + \mathcal{C}) \cap \mathbb{S}_+$. This means that, $b' \in \mathbb{S}_+$ and $b \in b' - \mathcal{C} \subset \mathbb{S}_+ - \mathcal{C}$, and hence $\mathbb{B}_+ \subset \mathbb{S}_+ - \mathcal{C}$. Since $\mathcal{C}$ is a cone, for all $\varepsilon > 0$, we have $\varepsilon\mathbb{B}_+ \subset \varepsilon\mathbb{S}_+ - \mathcal{C}$. By the convexity of $\mathcal{C}$, one has $\varepsilon\mathbb{B}_+ - \mathcal{C} \subset \varepsilon\mathbb{S}_+ - \mathcal{C}$.

From Remark 1(b), we can equivalently represent the solution map $\mathrm{Sol}_+(\mathrm{SEP})$ as follows.

$$\mathrm{Sol}_+(\mathrm{SEP})(\varepsilon,\lambda) := \{x \in K(\lambda) \mid 0 \in F(x,y,\lambda) + \varepsilon\mathbb{S}_+ - \mathcal{C} \quad \forall y \in K(\lambda)\}.$$

Remark 2 It is clear that, for each vector $(\varepsilon,\lambda) \in \mathbb{R}_+ \times \Lambda$,

$$\mathrm{Sol}_+(\mathrm{SEP})(0,\lambda) \subset \mathrm{Sol}_+(\mathrm{SEP})(\varepsilon,\lambda).$$

We now recall some concepts used in what follows. Let Q be a set-valued map from $\mathbb{X}$ to $\mathbb{Y}$ with nonempty values on $\mathbb{X}$ and $x_0 \in \mathbb{X}$.

Definition 2 *(See* (Göpfert et al. 2003, Definitions 2.5.1 and 2.5.16)*)* Q is said to be

(a) *upper semicontinuous* (usc) at $x_0 \in \mathbb{X}$ if for any neighborhood $\mathcal{V}$ of $Q(x_0)$, there exists some neighborhood $\mathcal{U}$ of x_0 such that $Q(\mathcal{U}) \subset \mathcal{V}$;

(b) *lower semicontinuous* (lsc) at $x_0 \in \mathbb{X}$ if for any open subset $\mathcal{V}$ of $\mathbb{Y}$ with $Q(x_0) \cap \mathcal{V} \neq \emptyset$, there exists some neighborhood $\mathcal{U}$ of x_0 such that

$$Q(x) \cap \mathcal{V} \neq \emptyset \quad \forall x \in \mathcal{U};$$

(c) *continuous* at $x_0 \in \mathbb{X}$ if it is both usc and lsc at x_0;

(d) *$\mathcal{C}$-upper semicontinuous* ($\mathcal{C}$-usc) at $x_0 \in \mathbb{X}$, if for any for any neighborhood $\mathcal{N}$ of $Q(x_0)$, there is a neighborhood $\mathcal{U}$ of x_0 such that

$$Q(x) \subset \mathcal{N} + \mathcal{C} \quad \forall x \in \mathcal{U};$$

(e) *$\mathcal{C}$-lower semicontinuous* ($\mathcal{C}$-lsc) at $x_0 \in \mathbb{X}$, if for any open subset $\mathcal{N}$ of $\mathbb{Y}$ with $Q(x_0) \cap \mathcal{N} \neq \emptyset$ there is a neighborhood $\mathcal{U}$ of x_0 such that

$$Q(x) \cap (\mathcal{N} - \mathcal{C}) \neq \emptyset \quad \forall x \in \mathcal{U};$$

(f) *$\mathcal{C}$-continuous* at $x_0 \in \mathbb{X}$, if it is both $\mathcal{C}$-usc and $\mathcal{C}$-lsc at x_0.

Lemma 1 (See (Hu and Papageorgiou 1997, p. 37))

(a) *Q is lsc at x_0 if for all $x_n \to x_0$ and $y_0 \in Q(x_0)$, then there exists $y_n \in Q(x_n)$ such that $y_n \to y_0$.*
(b) *Q is lsc at x_0 if for all $x_n \to x_0$, then one has $Q(x_0) \subset \liminf Q(x_n)$, where*

$$\liminf Q(x_n) := \{y_0 \in \mathbb{Y} \mid \exists \{y_n\} \text{ with } y_n \in Q(x_n), \{y_n\} \to y_0\}.$$

Lemma 2 (See (Hu and Papageorgiou 1997, p. 41)) *If $Q(x_0)$ is compact, then Q is usc at x_0 if and only if for any sequence $\{x_n\}$ converging to x_0 and $y_n \in Q(x_n)$, there is a subsequence $\{y_{n_k}\}$ converging to some $y_0 \in Q(x_0)$.*

Definition 3 *(See* (Durea and Florea 2024a, Definition 2.4)*).* A set $\mathcal{D} \in \mathbf{P}(\mathbb{Y})$ is called *$\mathcal{C}$-sequentially compact* if for any sequence $\{d_n\} \subset \mathcal{D}$, there is a sequence $\{c_n\} \subset \mathcal{C}$ such that the sequence $\{d_n - c_n\}$ has a convergent subsequence towards an element of $\mathcal{D}$.

Remark 3 *(See* (Durea and Florea 2024a, Theorem 2.7)*)* If $\mathcal{D}$ is $\mathcal{C}$-compact, then $\mathcal{D}$ is $\mathcal{C}$-sequentially compact. Conversely, it follows from (Durea and Florea 2024a, Theorem 2.11) that if $\mathcal{D}$ is $\mathcal{C}$-sequentially compact and separable, then $\mathcal{D}$ is $\mathcal{C}$-compact.

Lemma 3 (See (Han 2025, Theorem 3.1)) *If Q is $\mathcal{C}$-usc with $\mathcal{C}$-sequentially compact values at $x_0 \in \mathbb{X}$, then for any sequence $\{x_n\} \subset \mathbb{X}$ converging to x_0 and for any $y_n \in Q(x_n)$, there exist $c_n \in \mathcal{C}$ and $y_0 \in Q(x_0)$ such that the sequence $\{y_n - c_n\}$ has a subsequence $\left\{y_{n_k} - c_{n_k}\right\}$ converging to y_0.*

Lemma 4 (See (Han 2025, Theorem 3.2)) *If Q is $\mathcal{C}$-lsc at $x_0 \in \mathbb{X}$, then for any sequence $\{x_n\} \subset \mathbb{X}$ converging to x_0 and for any $y_0 \in Q(x_0)$, there exists $y_n \in Q(x_n)$ and $c_n \in \mathcal{C}$ such that the sequence $\{y_n + c_n\}$ converges to y_0.*

Definition 4 *(See* (Kuroiwa et al. 1997, Definition 3.2)*)* Let Ω be a nonempty convex subset of $\mathbb{X}$. Q is said to be *$\mathcal{C}$-convex* on Ω if for any $x_1, x_2 \in \Omega$ and $t \in [0, 1]$,

$$tQ(x_2) + (1-t)Q(x_1) \subset Q(tx_2 + (1-t)x_1) + \mathcal{C}.$$

Motivated by Anh et al. (2024), we propose generalized concepts related to Definition 4.

Definition 5 Let Ω and Γ be nonempty convex subsets of $\mathbb{X}$ and $\mathbb{R}$, respectively. For any $(x_1, r_1), (x_2, r_2) \in \Omega \times \Gamma$, we set $(x_t, r_t) := t(x_2, r_2) + (1-t)(x_1, r_1)$ for all $t \in]0, 1[$. The map Q is said to be *$(\mathcal{C}, \Gamma)$-quasiconvex* on Ω if

$$\left[0 \in (Q(x_1) + r_1\mathbb{B}_+ - \mathcal{C}) \cap (Q(x_2) + r_2(\operatorname{int}\mathbb{B}_+) - \mathcal{C})\right] \\ \Longrightarrow \left[0 \in Q(x_t) + r_t(\operatorname{int}\mathbb{B}_+) - \mathcal{C}\right].$$

The relationship of $\mathcal{C}$-convexity and $(\mathcal{C}, \Gamma)$-quasiconvexity is presented in the following result.

Lemma 5 *Let Ω be a nonempty convex subset of $\mathbb{X}$. If the map Q is $(-\mathcal{C})$-convex on Ω, then it is $(\mathcal{C}, \Gamma)$-quasiconvex on Ω for any nonempty convex subset Γ of $\mathbb{R}_+$.*

Proof For all $(x_1, r_1), (x_2, r_2) \in \Omega \times \Gamma$, we assume that

$$\left[0 \in (Q(x_1) + r_1\mathbb{B}_+ - \mathcal{C}) \cap (Q(x_2) + r_2(\operatorname{int}\mathbb{B}_+) - \mathcal{C})\right],$$

or equivalently

$$(Q(x_1) + r_1\mathbb{B}_+) \cap \mathcal{C} \neq \emptyset \text{ and } (Q(x_2) + r_2(\operatorname{int}\mathbb{B}_+)) \cap \mathcal{C} \neq \emptyset.$$

Let $z_1 \in Q(x_1)$, $b_1 \in \mathbb{B}_+$, $b_2 \in \operatorname{int}\mathbb{B}_+$ and $z_2 \in Q(x_2)$ such that

$$z_1 + r_1 b_1 \in \mathcal{C} \text{ and } z_2 + r_2 b_2 \in \mathcal{C}.$$

By the $(-\mathcal{C})$-convexity of Q, for any $t \in]0, 1[$, there exist $z_t \in Q(tx_2 + (1-t)x_1)$ and $c \in \mathcal{C}$ such that

$$z_t = tz_2 + (1-t)z_1 + c.$$

Consequently,

$$z_t + (1-t)r_1 b_1 + tr_2 b_2 = (1-t)(z_1 + r_1 b_1) + t(z_2 + r_2 b_2) + \mathcal{C} \in \mathcal{C}.$$

This leads to

$$\left[Q(tx_2 + (1-t)x_1) + (tr_2 + (1-t)r_1)(\operatorname{int}\mathbb{B}_+)\right] \cap \mathcal{C} \neq \emptyset \quad \forall t \in]0, 1[,$$

that is,

$$0 \in Q(tx_2 + (1-t)x_1) + (tr_2 + (1-t)r_1)(\operatorname{int}\mathbb{B}_+) - \mathcal{C}.$$

Therefore, Q is $(\mathcal{C}, \Gamma)$-quasiconvex on Ω. □

The following example illustrates that the converse statement of Lemma 5 are not true.

Example 1 Let $\mathbb{X} = \Omega = \mathbb{R}$, $\mathbb{Y} = \mathbb{R}^2$, $\mathcal{C} = [0, +\infty[\times\{0\}$ and $\mathbb{B} = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 \leq 1\}$. Then, $\mathbb{B}_+ = [0, 1] \times \{0\}$. A map $Q : \mathbb{X} \rightrightarrows \mathbb{Y}$ is defined by

$$Q(x) = \begin{cases} \{-2\} \times [-2, 0], & \text{if } x = 0, \\ [-3 - x^2, -2] \times [0, 2], & \text{if } x \neq 0. \end{cases}$$

We show that Q is $(\mathcal{C}, \mathbb{R}_+)$-quasiconvex on Ω. Indeed, let $x_1, x_2 \in \Omega$ and $r_1, r_2 \in \mathbb{R}_+$ such that

$$0 \in Q(x_1) + r_1\mathbb{B}_+ - \mathcal{C} \text{ and } 0 \in Q(x_2) + r_2(\operatorname{int}\mathbb{B}_+) - \mathcal{C},$$

then we have $r_1 \geq 2$ and $r_2 > 2$. Consequently,

$$0 \in Q(x_t) + r_t(\operatorname{int}\mathbb{B}_+) - \mathcal{C} \quad \forall t \in]0,1[,$$

where $(x_t, r_t) := t(x_2, r_2) + (1-t)(x_1, r_1)$.

However, Q is not $-\mathcal{C}$-convex on Ω. To see this, let $x_1 = -1$, $x_2 = 1$ and $t = \frac{1}{2}$, we have

$$\frac{1}{2}Q(-1) + \frac{1}{2}Q(1) = [-4,-2]\times[0,2] \not\subseteq \{-2\}\times[-2,0] - \mathcal{C} = Q\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) - \mathcal{C}.$$

3 Semicontinuity of solution maps to parametric set-valued equilibrium problems

To our knowledge, studies on the continuity and semicontinuity of solution maps to (SEP) have typically employed assumptions related to convexity and compactness, and required the ordering cone to have a nonempty interior, see e.g, Anh et al. (2021), Ansari et al. (2024) and the references therein. Motivated by these studies, we will investigate (semi)continuity for $\mathrm{Sol}_+(\mathrm{SEP})(\cdot,\cdot)$ without requiring the ordering cone to have a nonempty interior, while also using weaker assumptions than those in Anh et al. (2021) and Ansari et al. (2024).

Since the existence of solutions for equilibrium problems has been extensively investigated (Eslamizadeh and Naraghirad 2020; Jafari et al. 2017; Alleche and Rădulescu 2017; Durea 2007; Durea and Florea 2024a), we assume in this paper that all types of solution sets for the reference problems are nonempty. We now present the lower semicontinuity of $\mathrm{Sol}_+(\mathrm{SEP})(\cdot,\cdot)$.

Theorem 1 *Assume that*

(i) *K is continuous with convex and compact values on Λ;*
(ii) *F is $(-\mathcal{C})$-lower semicontinuous on $K(\Lambda)\times K(\Lambda)\times\Lambda$;*
(iii) *for $\lambda \in \Lambda$, $y \in K(\lambda)$, $F(\cdot, y, \lambda)$ is $(\mathcal{C}, \mathbb{R}_{++})$-quasiconvex on $K(\lambda)$.*

Then, $\mathrm{Sol}_+(\mathrm{SEP})(\cdot,\cdot)$ *is lower semicontinuous on* $\mathbb{R}_{++}\times\Lambda$.

Proof Let $S : \mathbb{R}_{++}\times\Lambda \rightrightarrows \Omega$ be defined by

$$S(\varepsilon,\lambda) := \{x \in K(\lambda) \mid 0 \in F(x,y,\lambda) + \varepsilon(\operatorname{int}\mathbb{B}_+) - \mathcal{C} \quad \forall y \in K(\lambda)\},$$

for all $\varepsilon > 0$ and $\lambda \in \Lambda$. Let $(\varepsilon_0, \lambda_0) \in \mathbb{R}_{++}\times\Lambda$ be arbitrary. We now show that S is lower semicontinuous at $(\varepsilon_0,\lambda_0)$. If S is not lsc at $(\varepsilon_0,\lambda_0)$, then we can find $x_0 \in S(\varepsilon_0,\lambda_0)$ and a sequence $\{(\varepsilon_n,\lambda_n)\} \subset \mathbb{R}_{++}\times\Lambda$ converging to $(\varepsilon_0,\lambda_0)$ such that for all $x_n \in S(\varepsilon_n,\lambda_n)$, $\{x_n\}$ does not converge to x_0. Thanks to the lower semicontinuity of K at λ_0 and Lemma 1(a), we can find $\bar{x}_n \in K(\lambda_n)$ satisfying $\bar{x}_n \to x_0$. By the above contradiction assumption, there is a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ such that for all n_k, $\bar{x}_{n_k} \notin S(\varepsilon_{n_k},\lambda_{n_k})$, i.e., there exists $y_{n_k} \in K(\lambda_{n_k})$,

$$0 \notin F(x_{n_k}, y_{n_k}, \lambda_{n_k}) + \varepsilon_{n_k}(\operatorname{int}\mathbb{B}_+) - \mathcal{C}.$$

Consequently,

$$z_{n_k} \notin -\varepsilon_{n_k}(\operatorname{int}\mathbb{B}_+) + \mathcal{C} \quad \forall z_{n_k} \in F(x_{n_k}, y_{n_k}, \lambda_{n_k}). \tag{2}$$

Since K is upper semicontinuous with compact values at λ_0 and $y_{n_k} \in K(\lambda_{n_k})$, by Lemma 2, we assume that the sequence $\{y_{n_k}\}$ converges to $y_0 \in K(\lambda_0)$ (take a subsequence if necessary). In view of $x_0 \in S(\varepsilon_0, \lambda_0)$, we have

$$0 \in F(x_0, y_0, \lambda_0) + \varepsilon_0(\operatorname{int}\mathbb{B}_+) - \mathcal{C}.$$

Then, we can find some $z_0 \in F(x_0, y_0, \lambda_0)$ such that

$$z_0 \in -\varepsilon_0(\operatorname{int}\mathbb{B}_+) + \mathcal{C}.$$

By (ii) and Lemma 4, there exist $\hat{z}_{n_k} \in F(x_{n_k}, y_{n_k}, \lambda_{n_k})$ and $c_{n_k} \in (-\mathcal{C})$ such that

$$\{\hat{z}_{n_k} + c_{n_k}\} \to z_0.$$

Suppose that $\hat{z}_{n_k} + c_{n_k} \in -\varepsilon_{n_k}(\operatorname{int}\mathbb{B}_+) + \mathcal{C}$, then we have

$$\hat{z}_{n_k} \in -c_{n_k} - \varepsilon_{n_k}(\operatorname{int}\mathbb{B}_+) + \mathcal{C}.$$

It follows from $-c_{n_k} \in \mathcal{C}$ that

$$\hat{z}_{n_k} \in \mathcal{C} - \varepsilon_{n_k}(\operatorname{int}\mathbb{B}_+) + \mathcal{C} \subset -\varepsilon_{n_k}(\operatorname{int}\mathbb{B}_+) + \mathcal{C},$$

which contradicts (2). Thus, $\hat{z}_{n_k} + c_{n_k} \notin -\varepsilon_{n_k}(\operatorname{int}\mathbb{B}_+) + \mathcal{C}$ which together with $\varepsilon_{n_k} > 0$ implies that

$$\frac{\hat{z}_{n_k} + c_{n_k}}{\varepsilon_{n_k}} \notin (-\operatorname{int}\mathbb{B}_+) + \mathcal{C}.$$

Since $(-\operatorname{int}\mathbb{B}_+) + \mathcal{C}$ is open and $\left\{\frac{\hat{z}_{n_k} + c_{n_k}}{\varepsilon_{n_k}}\right\}$ converges to $\frac{z_0}{\varepsilon_0}$, we obtain

$$\frac{z_0}{\varepsilon_0} \notin (-\operatorname{int}\mathbb{B}_+) + \mathcal{C}, \text{ or equivalently } z_0 \notin -\varepsilon_0(\operatorname{int}\mathbb{B}_+) + \mathcal{C}.$$

This contradicts the fact that $z_0 \in -\varepsilon_0(\operatorname{int}\mathbb{B}_+) + \mathcal{C}$, and hence S is lsc at $(\varepsilon_0, \lambda_0)$.

We next claim that

$$\mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_0, \lambda_0) \subset \mathrm{cl}S(\varepsilon_0, \lambda_0), \tag{3}$$

where "cl" is the closure. For $\bar{x} \in \mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_0, \lambda_0)$ and $x_1 \in S(\varepsilon_0, \lambda_0)$, one has

$$0 \in (F(\bar{x}, y, \lambda_0) + \varepsilon_0\mathbb{B}_+ - \mathcal{C}) \cap (F(x_1, y, \lambda_0) + \varepsilon_0(\operatorname{int}\mathbb{B}_+) - \mathcal{C}) \quad \forall y \in K(\lambda_0).$$

Since $F(\cdot, y, \lambda_0)$ is $(\mathcal{C}, \mathbb{R}_{++})$-quasiconvex on $K(\lambda_0)$, for any $t \in]0, 1[$, one has

$$0 \in F(tx_1 + (1-t)\bar{x}, y, \lambda_0) + \varepsilon_0(\operatorname{int} \mathbb{B}_+) - \mathcal{C}.$$

Thus, $x_t := tx_1 + (1-t)\bar{x}$ belongs to $S(\varepsilon_0, \lambda_0)$. Because $x_t \to \bar{x}$ when $t \to 0$, we have $\bar{x} \in \operatorname{cl} S(\varepsilon_0, \lambda_0)$, and hence (3) follows. By Lemma 1(b), the lower semicontinuity of S at $(\varepsilon_0, \lambda_0)$ leads to

$$S(\varepsilon_0, \lambda_0) \subset \liminf S(\varepsilon_n, \lambda_n),$$

for all $(\varepsilon_n, \lambda_n) \to (\varepsilon_0, \lambda_0)$. Combining this with (3), we have

$$\mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_0, \lambda_0) \subset \operatorname{cl} S(\varepsilon_0, \lambda_0) \subset \operatorname{cl} \liminf S(\varepsilon_n, \lambda_n). \tag{4}$$

In view of (Aubin and Frankowska 1990, Definition 1.4.6), one has

$$\operatorname{cl} \liminf S(\varepsilon_n, \lambda_n) = \liminf S(\varepsilon_n, \lambda_n).$$

This together with (4) implies that

$$\mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_0, \lambda_0) \subset \liminf \mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_n, \lambda_n),$$

as $S(\varepsilon_n, \lambda_n) \subset \mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_n, \lambda_n)$. Consequently, $\mathrm{Sol}_+(\mathrm{SEP})(\cdot, \cdot)$ is lsc at $(\varepsilon_0, \lambda_0)$. Since $(\varepsilon_0, \lambda_0)$ is an arbitrary element in $\mathbb{R}_{++} \times \Lambda$, the proof is complete. □

Passing the upper semicontinuity of the solution map $\mathrm{Sol}_+(\mathrm{SEP})(\cdot, \cdot)$, we have the following result.

Theorem 2 *Assume that*

(i) *K is continuous with compact values on Λ;*
(ii) *F is $(-\mathcal{C})$-usc with $(-\mathcal{C})$-sequentially compact values on $K(\Lambda) \times K(\Lambda) \times \Lambda$.*

Then, $\mathrm{Sol}_+(\mathrm{SEP})(\cdot, \cdot)$ *is upper semicontinuous on* $\mathbb{R}_{++} \times \Lambda$.

Proof Suppose that $\mathrm{Sol}_+(\mathrm{SEP})$ is not upper semicontinuous at some $(\varepsilon_0, \lambda_0) \in \mathbb{R}_{++} \times \Lambda$. Then, we can find a neighborhood $\mathcal{N}$ of $\mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_0, \lambda_0)$ and a sequence $\{(\varepsilon_n, \lambda_n)\} \subset \mathbb{R}_{++} \times \Lambda$ converging to $(\varepsilon_0, \lambda_0)$ such that

$$\mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_n, \lambda_n) \not\subset \mathcal{N}.$$

Equivalently, there is $x_n \in \mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_n, \lambda_n) \setminus \mathcal{N}$ for all n. Since K is upper semicontinuous with compact values at λ_0 and $x_n \in K(\lambda_n)$, by Lemma 2, we can assume that the sequence $\{x_n\}$ converges to $x_0 \in K(\lambda_0)$. If $x_0 \notin \mathrm{Sol}_+(\mathrm{SEP})(\varepsilon_0, \lambda_0)$, then there is $y_0 \in K(\lambda_0)$ such that

$$z \notin -\varepsilon_0 \mathbb{B}_+ + \mathcal{C} \quad \forall z \in F(x_0, y_0, \lambda_0). \tag{5}$$

In view of the lower semicontinuity of K and Lemma 1(a), we can find some sequence $\{y_n\}$ with $y_n \in K(\lambda_n)$ converging to y_0. Since $x_n \in \text{Sol}_+(\text{SEP})(\varepsilon_n, \lambda_n)$, there exists $z_n \in F(x_n, y_n, \lambda_n)$ such that

$$z_n \in -\varepsilon_n \mathbb{B}_+ + \mathcal{C}. \tag{6}$$

From $z_n \in F(x_n, y_n, \lambda_n)$ and (ii), by Lemma 3, there are the sequence $\{c_n\} \subset (-\mathcal{C})$ and $z_0 \in F(x_0, y_0, \lambda_0)$ such that the sequence $\{z_n - c_n\}$ has a subsequence $\{z_{n_k} - c_{n_k}\}$ converging to z_0. Thanks to (6) and $c_{n_k} \in (-\mathcal{C})$, we get

$$z_{n_k} - c_{n_k} \in -\varepsilon_{n_k} \mathbb{B}_+ + \mathcal{C}, \text{ and consequently } \frac{z_{n_k} - c_{n_k}}{\varepsilon_{n_k}} \in -\mathbb{B}_+ + \mathcal{C},$$

as $\varepsilon_{n_k} > 0$ for all n_k. Since $(-\mathbb{B}_+ + \mathcal{C})$ is a closed set, we get $z_0 \in -\varepsilon_0 \mathbb{B}_+ + \mathcal{C}$, a contradiction as (5) holds. Hence,

$$x_0 \in \text{Sol}_+(\text{SEP})(\varepsilon_0, \lambda_0) \subset \mathcal{N},$$

which is impossible as $x_n \notin \mathcal{N}$ for all n. Therefore, the map $\text{Sol}_+(\text{SEP})(\cdot, \cdot)$ is upper semicontinuous on $\mathbb{R}_{++} \times \Lambda$. □

To facilitate comparison with existing results, we combine Theorems 1 and 2 to obtain the following result.

Theorem 3 *Assume that*

(i) *K is continuous with compact values on Λ;*
(ii) *F is $(-\mathcal{C})$-continuous with $(-\mathcal{C})$-sequentially compact values on $K(\Lambda) \times K(\Lambda) \times \Lambda$.*
(iii) *for $\lambda \in \Lambda$, $y \in K(\lambda)$, $F(\cdot, y, \lambda)$ is $(\mathcal{C}, \mathbb{R}_{++})$-quasiconvex on $K(\lambda)$.*

Then, $\text{Sol}_+(\text{SEP})(\cdot, \cdot)$ *is continuous on* $\mathbb{R}_{++} \times \Lambda$.

Remark 4 Most studies on the (semi)continuity of approximate solution maps to set-valued equilibrium problems stipulate that the ordered cone must have a nonempty interior (Anh et al. 2021; Han and Huang 2016). Recently, the authors of Anh et al. (2024) have achieved an improved version of the results in Anh et al. (2021) without the requirement of a nonempty interior for the ordered cone. However, in Anh et al. (2024), the image space is required to be a reflexive Banach space. In (Anh et al. 2021, Theorem 3.2), the continuity with compact values of the objective map is key assumptions, while our results given as in Theorem 3, explore sufficient conditions for the continuity of approximate solution maps to set-valued equilibrium problems without this requirement. Therefore, our results improve (Anh et al. 2021, Theorem 3.2).

As mentioned in Remark 4, the compact values of the objective map is one of the essential assumptions used in previous studies when investigating sufficient conditions for the continuity of the solution maps to equilibrium problems. In the following example, we have relaxed this property but still achieve the result as in Theorem 3.

Example 2 Let $\mathbb{X} = \mathbb{W} = \mathbb{R}$, $\mathbb{Y} = \mathbb{R}^2$, $\mathcal{C} = \mathbb{R}^2_+$, $\mathbb{B} = \{\mathbf{u} \in \mathbb{R}^2 \mid \|\mathbf{u}\|_\infty \leq 1\}$, where $\|\mathbf{u}\|_\infty := \max\{|u_1|, |u_2|\}$ for $\mathbf{u} = (u_1, u_2)$, $\Lambda = [2, 4]$, $K(\lambda) = [2, \lambda]$ and

$$F(x, y, \lambda) = (y - x, y^2 - x^2) - \mathbb{R}^2_+.$$

We only focus on verifying the $(\mathcal{C}, \mathbb{R}_{++})$-quasiconvexity of F, as the other assumptions are straightforward.

For each $\lambda \in \Lambda$, $y \in K(\lambda)$, (x_1, r_1), $(x_2, r_2) \in \Omega \times \Gamma$, we set $(x_t, r_t) := t(x_2, r_2) + (1-t)(x_1, r_1)$ for all $t \in]0, 1[$. Assume that

$$0 \in F(x_1, y, \lambda) + r_1\mathbb{B}_+ - \mathcal{C} \text{ and } 0 \in F(x_2, y, \lambda) + r_2 \operatorname{int} \mathbb{B}_+ - \mathcal{C}, \tag{7}$$

we show that

$$0 \in F(x_t, y, \lambda) + r_t \operatorname{int} \mathbb{B}_+ - \mathcal{C}. \tag{8}$$

By the definition of F and (7), we have

$$0 \in (y-x_1, y^2-x_1^2)-\mathbb{R}^2_+ + r_1\mathbb{B}_+ - \mathbb{R}^2_+ \text{ and } 0 \in (y-x_2, y^2-x_2^2)-\mathbb{R}^2_+ + r_2 \operatorname{int} \mathbb{B}_+ - \mathbb{R}^2_+.$$

Hence,

$$(x_1 - y, x_1^2 - y^2) \in r_1\mathbb{B}_+ - \mathbb{R}^2_+ \text{ and } (x_2 - y, x_2^2 - y^2) \in r_2 \operatorname{int} \mathbb{B}_+ - \mathbb{R}^2_+,$$

or equivalently,

$$\begin{cases} x_1 - y \leq r_1 \\ x_1^2 - y^2 \leq r_1 \end{cases} \text{ and } \begin{cases} x_2 - y < r_2 \\ x_2^2 - y^2 < r_2. \end{cases}$$

Then,

$$\begin{cases} (1-t)x_1 - (1-t)y \leq (1-t)r_1 \\ (1-t)x_1^2 - (1-t)y^2 \leq (1-t)r_1 \end{cases} \text{ and } \begin{cases} tx_2 - ty < tr_2 \\ tx_2^2 - ty^2 < tr_2, \end{cases}$$

which implies that

$$\begin{cases} x_t - y < r_t \\ x_t^2 - y^2 < (1-t)x_1^2 + tx_2^2 - y^2 < r_t. \end{cases}$$

Thus,

$$(x_t - y, x_t^2 - y^2) \in r_t \operatorname{int} \mathbb{B}_+ - \mathbb{R}^2_+,$$

i.e., (8) follows.

It follows from direct computation that

$$\mathrm{Sol}_+(\mathrm{SEP})(\varepsilon, \lambda) = \left[-\sqrt{\varepsilon + \lambda^2}, \sqrt{\varepsilon + \lambda^2}\right].$$

However, the result in (Anh et al. 2021, Theorems 3.2) is not applicable as the compact values of F is violated.

4 Application to set optimization problems

Let $\mathbf{P}(\mathbb{Y})$, Ω, Λ, and K be defined as in Sect. 2. We consider parametric set optimization problems defined by for $\lambda \in \Lambda$,

$$\text{(SOP)} \quad \min \quad Q(x, \lambda) \quad \text{subject to} \quad x \in K(\lambda),$$

where $Q : \Omega \times \Lambda \rightrightarrows \mathbb{Y}$ is nonempty set-valued map with $(-\mathcal{C})$-convex values.

In Jahn and Ha (2011), the authors introduce several order relations on the sets, one of which, the possibly less order relation is presented as follows.

Definition 6 *(See* (Jahn and Ha 2011, Definition 3.4)*)* Let $A, B \in \mathbf{P}(\mathbb{Y})$ be arbitrarily chosen sets. Then, the *possibly less order relation* $\preccurlyeq_p$ is defined by

$$A \preccurlyeq_p B :\Leftrightarrow (\exists a \in A \;\; \exists b \in B : a \leq b).$$

It follows from (Jahn and Ha 2011, Proposition 3.3) that for arbitrary sets $A, B \in \mathbf{P}(\mathbb{Y})$, we have

$$A \preccurlyeq_p B \Leftrightarrow 0 \in B - A - \mathcal{C}.$$

Motivated by the works (Han 2019; Anh et al. 2020b), we propose the concept of an ideally approximate solution to parametric set optimization problem.

Definition 7 Let $\lambda \in \Lambda$ and $\varepsilon > 0$. An element $\bar{x}$ of $K(\lambda)$ is called an *ideally approximate solution* to (SOP), written as $\bar{x} \in \mathrm{Sol}_+(\mathrm{SOP})(\varepsilon, \lambda)$ if

$$0 \in Q(y, \lambda) - Q(\bar{x}, \lambda) + \varepsilon \mathbb{B}_+ - \mathcal{C} \quad \forall y \in K(\lambda).$$

The focus of this section is primarily on examining the continuity of ideally approximate solution maps to parametric set optimization problems.

Definition 8 *(See* (Seto et al. 2018, Definition 3.2)*)* Let Ω be a nonempty convex subset of $\mathbb{X}$. A set-valued map $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be $\mathcal{C}$*-quasiconvex* on Ω if for any convex subset A of $\mathbb{Y}$, $x_1, x_2 \in \Omega$ and $t \in]0, 1[$,

$$0 \in tG(x_2) + (1-t)G(x_1) + A + \mathcal{C}$$

implies

$$0 \in G((1-t)x_1 + tx_2) + A + \mathcal{C}.$$

Lemma 6 (See Seto et al. 2018) *Every* $\mathcal{C}$*-convex map is* $\mathcal{C}$*-quasiconvex.*

Next, we formulate sufficient conditions for the continuity of solution map $\mathrm{Sol}_+(\mathrm{SOP})(\cdot, \cdot)$.

Corollary 1 *Assume that*

(i) *K is continuous with convex and compact values on Λ;*
(ii) *Q is $(\pm\mathcal{C})$-continuous with $(\pm\mathcal{C})$-sequentially compact values on $K(\Lambda) \times \Lambda$;*

(iii) *for* $\lambda \in \Lambda$, $Q(\cdot, \lambda)$ *is* $\mathcal{C}$*-quasiconvex on* $K(\lambda)$.

Then, $\mathrm{Sol}_+(\mathrm{SOP})(\cdot, \cdot)$ *is continuous on* $\mathbb{R}_{++} \times \Lambda$.

Proof For each $(x, y, \lambda) \in \Omega \times \Omega \times \Lambda$, we set

$$F(x, y, \lambda) = Q(y, \lambda) - Q(x, \lambda).$$

Thus, $\mathrm{Sol}_+(\mathrm{SOP})(\varepsilon, \lambda) = \mathrm{Sol}_+(\mathrm{SEP})(\varepsilon, \lambda)$, and so we only check the validity of Assumption (ii) and (iii) of Theorem 3.

$\diamond$ *F is $(\mathcal{C}, \mathbb{R}_{++})$-quasiconvex in the first argument on $K(\lambda)$*: For any $y \in K(\lambda)$, $(x_1, r_1), (x_2, r_2) \in K(\lambda) \times \mathbb{R}_{++}$ and $t \in]0, 1[$ with

$$0 \in (F(x_1, y, \lambda) + r_1\mathbb{B}_+ - \mathcal{C}) \cap (F(x_2, y, \lambda) + r_2(\operatorname{int}\mathbb{B}_+) - \mathcal{C}),$$

we have

$$0 \in (Q(y, \lambda) - Q(x_1, \lambda) + r_1\mathbb{B}_+ - \mathcal{C}) \cap (Q(y, \lambda) - Q(x_2, \lambda) + r_2(\operatorname{int}\mathbb{B}_+) - \mathcal{C}).$$

Consequently,

$$0 \in Q(y, \lambda) - (1-t)Q(x_1, \lambda) - tQ(x_2, \lambda) + (1-t)r_1\mathbb{B}_+ + tr_2(\operatorname{int}\mathbb{B}_+) - \mathcal{C},$$

that is,

$$0 \in (1-t)Q(x_1, \lambda) + tQ(x_2, \lambda) - Q(y, \lambda) - (1-t)r_1\mathbb{B}_+ - tr_2(\operatorname{int}\mathbb{B}_+) + \mathcal{C}.$$

Combining this with $(1-t)r_1\mathbb{B}_+ + tr_2(\operatorname{int}\mathbb{B}_+) \subset (tr_2 + (1-t)r_1)(\operatorname{int}\mathbb{B}_+)$, we have

$$0 \in (1-t)Q(x_1, \lambda) + tQ(x_2, \lambda) - Q(y, \lambda) - (tr_2 + (1-t)r_1)(\operatorname{int}\mathbb{B}_+) + \mathcal{C}.$$

Since $Q(\cdot, \lambda)$ is $\mathcal{C}$-quasiconvex on $K(\lambda)$,

$$0 \in Q((1-t)x_1 + tx_2, \lambda) - Q(y, \lambda) - (tr_2 + (1-t)r_1)(\operatorname{int}\mathbb{B}_+) + \mathcal{C}.$$

Equivalently,

$$0 \in F(tx_2 + (1-t)x_1, y, \lambda) + (tr_2 + (1-t)r_1)(\operatorname{int}\mathbb{B}_+) - \mathcal{C}.$$

$\diamond$ *F is $(-\mathcal{C})$-continuous with $(-\mathcal{C})$-sequentially compact values on $K(\Lambda) \times K(\Lambda) \times \Lambda$*: It is clear that if Q is $(\pm\mathcal{C})$-continuous on $K(\Lambda) \times \Lambda$, then F is $(-\mathcal{C})$-continuous on $K(\Lambda) \times K(\Lambda) \times \Lambda$.

For the sequential compactness of F, let $\{z_n\} \subset F(x, y, \lambda) = Q(y, \lambda) - Q(x, \lambda)$. Then, we can find $\{u_n\} \subset Q(y, \lambda)$ and $\{v_n\} \subset Q(x, \lambda)$ such that

$$z_n = u_n - v_n \quad \forall n.$$

Since $Q(y, \lambda)$ is $(-\mathcal{C})$-sequentially compact, there exists $c_n \in (-\mathcal{C})$ such that the sequence $\{u_n - c_n\}$ has a subsequence converging to an element $u_0 \in Q(y, \lambda)$. On the other hand, the set $Q(x, \lambda)$ is $\mathcal{C}$-sequentially compact, there is $d_n \in \mathcal{C}$ such that $\{v_n - d_n\}$ has a subsequence converging to $v_0 \in Q(x, \lambda)$. Thus, the sequence $\{u_n - c_n - v_n + d_n\}$ has a subsequence converging to $u_0 - v_0 \in Q(y, \lambda) - Q(x, \lambda)$, or equivalently $\{z_n - (c_n - d_n)\}$ has a subsequence converging to $u_0 - v_0$ with $c_n - d_n \in (-\mathcal{C})$. It means that $F(x, y, \lambda)$ is $(-\mathcal{C})$-sequentially compact. □

We present the following example to showcase the application of Corollary 1.

Example 3 Let $\mathbb{X} = \Omega = \mathbb{Y} = \mathbb{R}$, $\mathcal{C} = \mathbb{R}_+$ $\Lambda = [0, 1]$, $K(\lambda) = [\lambda, 2]$, $\mathbb{B} = [-1, 1]$ and

$$Q(x, \lambda) = \left[x^2, x^2 + \lambda\right].$$

We only check the $\mathcal{C}$-quasiconvexity of Q because the other are trivial. For each $\lambda \in \mathbb{R}$, $t \in]0, 1]$, let $x_1, x_2 \in \mathbb{R}$ such that for any convex $A \subset \mathbb{R}$,

$$0 \in tQ(x_2) + (1 - t)Q(x_1) + A + \mathcal{C}.$$

We show that

$$0 \in Q(tx_2 + (1 - t)x_1) + A + \mathcal{C}. \tag{9}$$

Indeed, by definition of Q and (9), one has

$$0 \in \left[tx_2^2 + (1 - t)x_1^2, tx_2^2 + t\lambda + (1 - t)x_2^2 + t\lambda\right] + A + \mathcal{C},$$

or equivalently

$$0 \in \left[(tx_2 + (1 - t)x_1)^2 , (tx_1 + (1 - t)x_1)^2 + \lambda\right] + A + \mathcal{C}.$$

This means that

$$0 \in Q(tx_2 + (1 - t)x_1) + A + \mathcal{C}.$$

By some direct computation, we get

$$\text{Sol}_+(\text{SOP})(\varepsilon, \lambda) = \left[\lambda, \frac{2 + \sqrt{\lambda^2 + \lambda + \varepsilon} - |2 - \sqrt{\lambda^2 + \lambda + \varepsilon}|}{2}\right].$$

Remark 5 In (Anh et al. 2020b, Theorem 5.1), the authors investigated the Hausdorff continuity of ideally approximate solution mappings to (SOP) under the framework of upper and lower type set less order relations, relying on the convexity and compactness of the objective maps. In Corollary 1, these assumptions have been further relaxed to quasiconvexity and cone-sequential compactness. Consequently, by employing techniques similar to those used in the proof of Corollary 1, we establish a new result that refines and extends (Anh et al. 2020b, Theorem 5.1).

Acknowledgements The author sincerely thanks the editors and reviewers for their valuable remarks and suggestions, which significantly contributed to improving this paper.

Data availability No data were generated or analyzed in this study.

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