



Non-equilibrium Stationary Solutions for Multicomponent Coagulation Systems with Injection

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Abstract

The existence and non-existence of stationary solutions of multicomponent coagulation equations with a constant flux of mass towards large sizes is investigated. The flux may be induced by a source of small clusters or by a flux boundary condition at the origin of the composition space, and the coagulation kernel can be very general, merely satisfying certain power law asymptotic bounds in terms of the total number of monomers in a cluster. Our set-up, including an appropriate definition of multicomponent flux, allows a sharp classification of the existence of stationary solutions. In particular, this analysis extends previous results for one-component systems to a larger class of kernels.

Keywords Multicomponent Smoluchowski's equation · Non-equilibrium dynamics · Source term · Stationary injection solutions · Constant flux solutions · Mass flux

1 Introduction

Smoluchowski's coagulation equation is a classical model for binary aggregation extensively used in the study of aerosol growth, polymerization, drop formation in rain and several other situations [11, 21, 22]. These systems are made of a large number of particles (clusters) with

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different sizes and compositions. The particle growth is due to the coalescence of particles with smaller sizes and this process determines the size and composition of the clusters. A specific example where particle composition influences the growth of the particles is found in atmospheric science, where multicomponent aerosol particles grow by coagulating with distinct chemical species such as sulfuric acid and either ammonia or dimethylamine molecules (cf. for instance [21]).

The particle clusters are made of aggregates of different types of coagulating molecules, which are called *monomers*. We denote by n_α the concentrations of multicomponent clusters, with composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$ where $\alpha_i \in \mathbb{N}_0$ denotes the number of monomers of type i . Notice that $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and we denote $O = (0, 0, \dots, 0)$. The multicomponent Smoluchowski's coagulation equation, which describes the evolution of the clusters concentrations $\{n_\alpha\}_{\alpha \in \mathbb{N}_0^d \setminus \{O\}}$, is given by

$$\partial_t n_\alpha = \frac{1}{2} \sum_{\beta < \alpha} K_{\alpha-\beta, \beta} n_{\alpha-\beta} n_\beta - n_\alpha \sum_{\beta > 0} K_{\alpha, \beta} n_\beta + s_\alpha, \tag{1.1}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_d)$. The coefficients $K_{\alpha, \beta}$ describe the coagulation rate between clusters with compositions α and β . We use the notation $\beta < \alpha$ to indicate that $\beta_k \leq \alpha_k$ for all $k = 1, 2, \dots, d$, and in addition $\alpha \neq \beta$. We denote as s_α the source of small particles characterized by the composition α . We will allow source terms s_α which are supported on a finite set of values α .

The coefficients $K_{\alpha, \beta}$ yield the coagulation rate between clusters α and β to produce clusters $(\alpha + \beta)$. The form of these coefficients depends on the specific mechanism which is responsible for the aggregation of the clusters. These coefficients have been computed using the method of kinetic theory under different assumptions on the particle sizes and the processes describing the motion of the clusters.

Relevant examples of coagulation kernels have been described in the literature, see e.g. the textbook [11]. Typical ones are given by the *free molecular regime coagulation kernel*

$$K_{\alpha, \beta} = \left(\frac{1}{V(\alpha)} + \frac{1}{V(\beta)} \right)^{\frac{1}{2}} \left((V(\alpha))^{\frac{1}{3}} + (V(\beta))^{\frac{1}{3}} \right)^2. \tag{1.2}$$

and the *diffusive coagulation* or *Brownian kernel*

$$K_{\alpha, \beta} = \left(\frac{1}{(V(\alpha))^{\frac{1}{3}}} + \frac{1}{(V(\beta))^{\frac{1}{3}}} \right) \left((V(\alpha))^{\frac{1}{3}} + (V(\beta))^{\frac{1}{3}} \right). \tag{1.3}$$

Here $V(\alpha)$ is the volume of the cluster characterized by the composition α . More details on the physical properties and the derivation of the kernels can be found in [6, 7]. In these formulas we will assume that the volume scales linearly with the number of monomers in the cluster. More precisely,

$$k_1 |\alpha| \leq V(\alpha) \leq k_2 |\alpha| \quad \text{with } 0 < k_1 \leq k_2 < \infty, \tag{1.4}$$

where $|\cdot|$ denotes the ℓ^1 -norm, i.e.,

$$|\alpha| = \sum_{j=1}^d \alpha_j, \quad \alpha \in \mathbb{N}_0^d \setminus \{O\}. \tag{1.5}$$

The inequalities (1.4) hold, for instance, if we assume $V(\alpha) = \sum_{j=1}^d \alpha_j v_j$ where $v_j > 0$ represents the volume of the monomer of type j for each $j = 1, 2, \dots, d$.

We also consider the continuous version of (1.1) which is given by

$$\begin{aligned} \partial_t f(x) = & \frac{1}{2} \int_{\{0 < \xi < x\}} d\xi K(x - \xi, \xi) f(x - \xi) f(\xi) \\ & - \int_{\mathbb{R}_+^d \setminus \{O\}} d\xi K(x, \xi) f(x) f(\xi) + \eta(x), \quad x \in \mathbb{R}_+^d \setminus \{O\} \end{aligned} \tag{1.6}$$

where f denotes the density of clusters with composition $x \in \mathbb{R}_+^d \setminus \{O\}$. In the same way as in the discrete case, given $x = (x_1, x_2, \dots, x_d)$, $y = (y_1, y_2, \dots, y_d)$ we would say that $x < y$ whenever $x \leq y$ componentwise, and $x \neq y$. In particular,

$$\int_{\{0 < \xi < x\}} d\xi = \int_0^{x_1} d\xi_1 \int_0^{x_2} d\xi_2 \cdots \int_0^{x_d} d\xi_d.$$

We notice that the discrete model (1.1) can be thought as a particular case of the continuous one if we assume that f is the sum of Dirac measures supported at points with integer coordinates.

In this paper we will consider the stationary solutions to the problems (1.1) and (1.6). In order to obtain nontrivial solutions we will require that $\sum_{\beta} s_{\beta} > 0$ in (1.1). In the case of (1.6) we will assume that η is a Radon measure with $0 < \int \eta(dx) < \infty$.

We will restrict our attention to the class of coagulation kernels satisfying

$$c_1 (|\alpha| + |\beta|)^{\gamma} \Phi\left(\frac{|\alpha|}{|\alpha| + |\beta|}\right) \leq K_{\alpha, \beta} \leq c_2 (|\alpha| + |\beta|)^{\gamma} \Phi\left(\frac{|\alpha|}{|\alpha| + |\beta|}\right), \quad \alpha, \beta \in \mathbb{N}_0^d \setminus \{O\} \tag{1.7}$$

$$c_1 (|x| + |y|)^{\gamma} \Phi\left(\frac{|x|}{|x| + |y|}\right) \leq K(x, y) \leq c_2 (|x| + |y|)^{\gamma} \Phi\left(\frac{|x|}{|x| + |y|}\right), \quad x, y \in \mathbb{R}_+^d \setminus \{O\} \tag{1.8}$$

where $|\cdot|$ denotes the ℓ^1 norm as in (1.5) and

$$\Phi(s) = \Phi(1 - s) \text{ for } 0 < s < 1, \quad \Phi(s) = \frac{1}{s^p (1 - s)^p} \text{ for } 0 < s < 1, \quad p \in \mathbb{R} \tag{1.9}$$

where $0 < c_1 \leq c_2 < \infty$. This class of kernels includes the physically relevant kernels (1.2) and (1.3). We stress that even though the estimates (1.7), (1.8) are isotropic, i.e., invariant under permutation of components, the kernels are not necessarily isotropic. For example, this general class includes kernels such as $K(x, y) = x_1 + y_1 + 2(x_2 + y_2)$.

The existence of steady states to the problems (1.1), (1.6) in the case $d = 1$ has been considered in [6] for a less general class of kernels than the one covered by the assumptions (1.7)–(1.9). Indeed, the conditions assumed in [6] (with $d = 1$) are

$$c_1 (\alpha^{\gamma+\lambda} \beta^{-\lambda} + \beta^{\gamma+\lambda} \alpha^{-\lambda}) \leq K_{\alpha, \beta} \leq c_2 (\alpha^{\gamma+\lambda} \beta^{-\lambda} + \beta^{\gamma+\lambda} \alpha^{-\lambda}), \tag{1.10}$$

$$c_1 (x^{\gamma+\lambda} y^{-\lambda} + y^{\gamma+\lambda} x^{-\lambda}) \leq K(x, y) \leq c_2 (x^{\gamma+\lambda} y^{-\lambda} + y^{\gamma+\lambda} x^{-\lambda}) \tag{1.11}$$

for some $\gamma, \lambda \in \mathbb{R}$. It is readily seen that the kernels satisfying (1.10), (1.11) satisfy also (1.7)–(1.9) with $p = \max\{\lambda, -(\gamma + \lambda)\}$ (assuming $d = 1$). On the other hand, for any $p \in \mathbb{R}$ with $p \geq -\frac{\gamma}{2}$ there exists at least one value $\lambda \in \mathbb{R}$ such that $\max\{\lambda, -(\gamma + \lambda)\} = p$. In fact, we can take by definiteness $\lambda = p$, since then $-(\gamma + \lambda) \leq 2p - \lambda = \lambda$, and therefore, $\max\{\lambda, -(\gamma + \lambda)\} = \lambda = p$. If $p < -\frac{\gamma}{2}$, it is not possible to choose λ such that $p = \max\{\lambda, -(\gamma + \lambda)\}$. Therefore, the class of kernels satisfying (1.7)–(1.9) is strictly larger than the class satisfying (1.10), (1.11).

In this paper we will prove that in the multicomponent case and under the assumptions (1.7)–(1.9), there exists a stationary solution to (1.1), (1.6) if and only if

$$\gamma + 2p < 1. \tag{1.12}$$

We note that for $p = \max\{\lambda, -(\gamma + \lambda)\}$, condition (1.12) implies the condition $|\gamma + 2\lambda| < 1$ obtained in [6] for the kernels satisfying (1.10), (1.11). Indeed, if $\lambda \geq -(\gamma + \lambda)$, we have $\gamma + 2\lambda \geq 0$ and, since $p = \lambda$, (1.12) is equivalent to $\gamma + 2\lambda < 1$. Otherwise, if $\lambda < -(\gamma + \lambda)$, we have $\gamma + 2\lambda < 0$, $p = -(\gamma + \lambda)$, and thus (1.12) is equivalent to $\gamma + 2\lambda > -1$. Therefore, (1.12) holds if and only if $|\gamma + 2\lambda| < 1$, whenever the two cases can be compared.

Notice that these steady states yield a transfer of monomers from small clusters to large clusters in the space of clusters sizes. Their existence express the balance between the injection of small clusters (e.g. monomers) and the transport of these monomers towards clusters of infinite size due to the coagulation mechanism. The non-existence of these steady states is due to the fact that the transport of monomers towards large clusters is too fast and cannot be balanced by any monomers injection, and therefore no stationary regime is possible. We emphasize that the steady states of (1.1), (1.6) are stationary non-equilibrium solutions for an open system.

It is worth to mention that in the case of discrete kernels with the form $K_{\alpha,\beta} = \alpha^{\gamma+\lambda}\beta^{-\lambda} + \alpha^{-\lambda}\beta^{\gamma+\lambda}$, and source terms supported at the monomers, the stationary solutions of (1.1) in dimension $d = 1$ have been computed formally in [13] assuming the non gelling condition $\max\{\gamma + \lambda, -\lambda\} \leq 1$. It turns out that these solutions are well defined, non-negative, densities of clusters distributions if and only if $|\gamma + 2\lambda| < 1$ holds.

It is interesting to note that the existence or nonexistence of stationary solutions to (1.1), (1.6) is independent of the number of components d here. Both cases are also already represented by the two example kernels discussed above. In the case of kernels with the form (1.2), we have $\gamma = \frac{1}{6}$ and $p = \frac{1}{2}$. Thus the inequality (1.12) is not satisfied, and there are no stationary solutions. On the other hand, in the case of kernels with the form (1.3) we have $\gamma = 0$ and $p = \frac{1}{3}$. Then the inequality (1.12) holds, and there exists at least one stationary solution.

In this paper we will prove the existence of steady states to (1.1), (1.6) under the assumption (1.12) and nonexistence of steady states if $\gamma + 2p \geq 1$. The key idea of the proofs consists in choosing an appropriate definition of flux in the multidimensional composition space (cf. Sect. 2.1) which allows to make use of the results developed for one-component systems in [6]. More precisely, in the multicomponent setting, the mass flows from small to large sizes through a $(d-1)$ -dimensional surface rather than a point, as in the one-dimensional case. This allows for more possibilities in the choice of the definition of flux and some care is needed in choosing an appropriate definition (cf. Sect. 2.1). Moreover, in the multicomponent setting, the proofs require more refined geometrical arguments than the ones used in the one-component case.

In the physical literature, explicit stationary solutions to the multicomponent equation (1.1) have been obtained in [16] in the case of the constant kernel $K(x, y) = 1$ and additive kernel $K(x, y) = x + y$ and with a source term supported on the monomers.

For non-solving kernels, most of the mathematical analysis of coagulation equations has been made for one-component systems only, i.e., $d = 1$. On the other hand, there are only a few papers addressing the problem of the coagulation equations with injection terms like $\{s_\alpha\}_\alpha$ or η (cf. [3, 4, 6, 17]). This issue has been discussed in [6] and we refer to that paper for additional references.

An interesting property of the steady states to (1.1), (1.6) specific to the multicomponent coagulation system, that does not have a counterpart in the case $d = 1$, is the so-called *localization property*. It consists in the fact that the concentrations n_α localize along a particular line in the space \mathbb{N}_0^d as $|\alpha| \rightarrow \infty$. A similar property holds in the continuous case, namely the density f concentrates along a specific direction of the cone \mathbb{R}_+^d as $|x| \rightarrow \infty$. The precise formulation is the following. If n_α and f are stationary solutions to (1.1), (1.6) respectively then there is a $\zeta > 0$ such that, for any $\varepsilon > 0$,

$$\begin{aligned} \lim_{R \rightarrow \infty} \frac{\sum_{\{R \leq |\alpha| \leq \zeta R\} \cap \left\{ \left| \frac{\alpha}{|\alpha|} - \theta \right| < \varepsilon \right\}} n_\alpha}{\sum_{\{R \leq |\alpha| \leq \zeta R\}} n_\alpha} &= 1 \quad \text{or} \\ \lim_{R \rightarrow \infty} \frac{\int_{\{R \leq |x| \leq \zeta R\} \cap \left\{ \left| \frac{x}{|x|} - \theta \right| < \varepsilon \right\}} f(dx)}{\int_{\{R \leq |x| \leq \zeta R\}} f(dx)} &= 1. \end{aligned} \tag{1.13}$$

where the direction θ is defined by the normalized mass vector of the source s_α or η such that $|\theta| = 1$. The proof of this result is given in [7] for the class of kernels satisfying (1.7)–(1.9). Moreover, we emphasize that asymptotic localization appears to be a very generic feature of multicomponent coagulation, including time-dependent problems. Indeed, we have preliminary evidence that a similar localization property holds for mass conserving solutions of the coagulation equation (i.e. with $\eta = 0$ or $s_\beta = 0$), asymptotically for long times (see the forthcoming paper [8]).

Structure of the Paper

The plan of the paper is the following. In Sect. 2.1 we informally discuss the different types of stationary solutions considered in this paper (constant injection solutions, constant flux solutions, ...). In Sect. 2.2 we introduce rigorously the definitions of solutions studied in this paper. In Sect. 3 we formulate the main results that we prove in this paper, namely, existence or nonexistence of stationary injection solutions or constant flux solutions for several classes of kernels. Section 4 contains two technical results which are repeatedly used in the rest of the paper. The proof of the existence of steady states for some classes of kernels is the content of Sect. 5. The non-existence results for a different class of kernels are given in Sect. 6. Section 7 provides some estimates for the stationary solutions, whenever they exist.

Notations

We will denote by $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ the non-negative real numbers and integers respectively. We also use a subindex “*” to denote restriction of real-component vectors x to those which satisfy $x > 0$, or equivalently $\max_i(x_i) > 0$. More precisely, we denote $\mathbb{R}_* := \mathbb{R}_+ \setminus \{0\}$, $\mathbb{R}_*^d := \mathbb{R}_+^d \setminus \{O\}$ and $\mathbb{N}_*^d := \mathbb{N}_0^d \setminus \{O\}$. Given a locally compact Hausdorff space X (for instance $X = \mathbb{R}_*^d$) we denote by $C_c(X)$ the space of compactly supported continuous functions from X to \mathbb{C} , and by $C_0(X)$ its completion in the standard supremum norm. The collection of non-negative Radon measures on X , not necessarily bounded, will be denoted by $\mathcal{M}_+(X)$ and its subspace consisting of bounded measures by $\mathcal{M}_{+,b}(X)$. Due to the Riesz–Markov–Kakutani theorem, we can identify $\mathcal{M}_+(X)$ with the space of positive linear functionals on $C_c(X)$.

Both the notation $\eta(x)dx$ and $\eta(dx)$ will be used to denote elements of the above measure spaces. We will use the symbol $\eta(dx)$ when performing integrations or when we want to

emphasize that the measure might not be absolutely continuous with respect to the Lebesgue measure. We will often drop the differential “ dx ” from the first notation, typically when the measure eventually turns out to be absolutely continuous. We will use the symbol $\mathbb{1}P$ to denote the characteristic function of a condition P : $\mathbb{1}P = 1$ if the condition P is true, and $\mathbb{1}P = 0$ if P is false.

2 Different Types of Stationary Solutions for Multicomponent Coagulation Equations

We now introduce different types of stationary solutions of (1.1), (1.6) which will be considered in this paper. These classes of solutions have been discussed in [6] in the case $d = 1$. We will examine here what the convenient definitions in the multicomponent case are. We recall that in all the cases discussed in this Section, the solutions are stationary, nonequilibrium solutions yielding a constant flux of monomers towards large clusters. We discuss shortly these classes of solutions as well as their physical meaning.

2.1 Heuristic Description of Flux and Constant Flux Solutions

In this section, we first introduce different concepts of stationary solutions used in this paper. The rigorous, more detailed, definitions are collected in Sect. 2.2.

Stationary Injection Solutions

The stationary solutions of (1.1), (1.6) satisfy respectively the equations

$$0 = \frac{1}{2} \sum_{\beta < \alpha} K_{\alpha-\beta, \beta} n_{\alpha-\beta} n_{\beta} - n_{\alpha} \sum_{\beta > 0} K_{\alpha, \beta} n_{\beta} + s_{\alpha} \quad , \quad \alpha \in \mathbb{N}_*^d, \tag{2.1}$$

$$0 = \frac{1}{2} \int_{\{0 < y < x\}} K(x-y, y) f(x-y, t) f(y, t) dy - \int_{\mathbb{R}_*^d} K(x, y) f(x, t) f(y, t) dy + \eta(x) \quad , \quad x \in \mathbb{R}_*^d. \tag{2.2}$$

We will assume that the sequence s_{α} is supported in a finite set of values of α . On the other hand, we will assume that $\eta \in \mathcal{M}_+(\mathbb{R}_*^d)$ is a Radon measure compactly supported in the set $x \geq \mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}_*^d$ (for examples of how to relax the assumptions about the source, we refer to a recent preprint [17] where compact support is not required assuming that the solution f is absolutely continuous with respect to the Lebesgue measure). In this paper we are mostly interested in the solutions of the equations (2.1), (2.2) which we call *stationary injection solutions*. Their detailed definition will be given in Sect. 2.2.

Constant Flux Solutions

In addition to the above injection solutions, in the one-component case ($d = 1$) we have considered in [6] a family of solutions of (2.2) with $\eta = 0$ that we have termed as *constant flux solutions*. The terminology and motivation arise from the fact that the coagulation equation without a source, at least formally, conserves “total mass”, the function $\int xf(x, t)dx$. This conservation law leads to a continuity equation, which may be written as

$$\partial_t(xf(x, t)) + \partial_x J(x; f) = 0,$$

where the *flux* can be defined by

$$J(x; f) = \int_0^x dy \int_{x-y}^\infty dz K(y, z) yf(y) f(z).$$

In this case, we find that f is a stationary solution if and only if for all $x > 0$

$$\partial_x J(x; f) = 0, \tag{2.3}$$

i.e., if and only if the flux is constant in x . Therefore, if there is $J_0 \geq 0$ and a measure $f \in \mathcal{M}_+(\mathbb{R}_*)$ such that

$$J(x; f) = J_0, \quad \text{for all } x > 0, \tag{2.4}$$

we say that f is a constant flux solution. We say that the solution has a *non-trivial flux* if $J_0 > 0$. In this case, clearly also $f \neq 0$.

In the above one-dimensional case, any sufficiently regular constant flux solution f also has the property that

$$\begin{aligned} 0 &= -\frac{1}{x} \partial_x J(x; f) = \frac{1}{2} \int_0^x K(x-y, y) f(x-y, t) f(y, t) dy \\ &\quad - \int_0^\infty K(x, y) f(x, t) f(y, t) dy. \end{aligned} \tag{2.5}$$

Comparing the result with (2.2) shows that these are stationary solutions to the original evolution equation without source, albeit with a slightly non-standard physical interpretation as solutions with non-trivial source of “particles” located at $x = 0$. Indeed, one practical use for the constant flux solutions comes from the observation that they can provide the asymptotics of stationary injection solutions. It has been proven in [6] that the stationary injection solutions both of the discrete and the continuous model (cf. (2.1), (2.2)) behave for large values of α or x as a constant flux solution. More precisely, rescaling n_α or f in a suitable manner we obtain some measures that converge for large values to a measure which satisfies (2.4). We refer to [6] for the detailed results.

In the multicomponent case without source, the total mass of each of the particle species is conserved, so there are now d mass continuity equations, as derived below. In addition, the analogue of (2.3) is a vectorial divergence equation. Therefore, it is not possible to characterize the fluxes at a given point just by one number. In order to define a suitable concept of constant flux solutions in the multicomponent case we must take into account that, if $d > 1$, we cannot expect the solutions of (2.5) to be uniquely characterized by the flux of particles across all the surfaces $\{|x| = R\}$ for arbitrary values of $R > 0$, where the norm $|\cdot|$ is as in (1.5). Let us introduce the change of variables $x \rightarrow (|x|, \theta)$ where $\theta = \frac{x}{|x|} \in \Delta^{d-1}$ where we denote by Δ^{d-1} the simplex

$$\Delta^{d-1} = \left\{ \theta \in \mathbb{R}_*^d : |\theta| = 1 \right\}. \tag{2.6}$$

Then, the detailed distribution of the measure f in the variable θ in each surface $\{|x| = R\}$ must be obtained from the generalization of equation (2.3) to the multicomponent case and it cannot be determined just from the values of the fluxes across these surfaces.

We now rewrite equation (2.2) in the form of divergences of fluxes. To this end, we choose a component $j \in \{1, 2, \dots, d\}$ and multiply (2.2) by x_j . Expanding in the first term $x_j = (x - \xi)_j + \xi_j$ and using the symmetry $\xi \leftrightarrow (x - \xi)$, we obtain

$$0 = \int_{\{0 < \xi < x\}} d\xi K(x - \xi, \xi) (x - \xi)_j f(x - \xi) f(\xi) - \int_{\mathbb{R}_*^d} d\xi K(x, \xi) x_j f(x) f(\xi) + x_j \eta(x).$$

We then multiply this equation by a test function $\varphi \in C_c(\mathbb{R}_*^d)$. The support of φ is a compact subset of $\mathbb{R}_+^d \setminus \{O\}$ and thus it is bounded and separated by a finite distance from the origin. Thus we can find $a, b > 0$ with $a < b$ such that the support of φ is contained in the set $\{x : |x| \in [a, b]\}$. Then, using Fubini’s Theorem and assuming that all the integrals appearing in the computations are finite, we obtain

$$0 = \int_{\mathbb{R}_*^d} d\xi \int_{\mathbb{R}_*^d} dx [\varphi(x + \xi) - \varphi(x)] K(x, \xi) x_j f(x) f(\xi) + \int_{\mathbb{R}_*^d} x_j \eta(x) \varphi(x) dx \tag{2.7}$$

Here

$$\varphi(x + \xi) - \varphi(x) = \int_0^1 \xi \cdot \nabla_x \varphi(x + t\xi) dt$$

and thus

$$0 = \int_0^1 dt \int_{\mathbb{R}_*^d} d\xi \int_{\mathbb{R}_*^d} dx [\xi \cdot \nabla_x \varphi(x + t\xi)] K(x, \xi) x_j f(x) f(\xi) + \int_{\mathbb{R}_*^d} x_j \eta(x) \varphi(x) dx$$

Using the change of variables $y = x + t\xi$ in the first integral we obtain

$$0 = \int_0^1 dt \int_{\mathbb{R}_*^d} d\xi \int_{\{t\xi < y\}} dy [\xi \cdot \nabla_y \varphi(y)] K(y - t\xi, \xi) (y - t\xi)_j f(y - t\xi) f(\xi) + \int_{\mathbb{R}_*^d} x_j \eta(x) \varphi(x) dx.$$

Applying Fubini’s Theorem we obtain

$$0 = \int_{\mathbb{R}_*^d} dy \nabla_y \varphi(y) \cdot \left[\int_0^1 dt \int_{\{0 < \xi < \frac{y}{t}\}} d\xi \xi (y - t\xi)_j K(y - t\xi, \xi) f(y - t\xi) f(\xi) \right] + \int_{\mathbb{R}_*^d} x_j \eta(x) \varphi(x) dx.$$

The final result can be interpreted in the sense of distributions as a vector equation

$$\operatorname{div}_x \left(\int_0^1 dt \int_{\{0 < \xi < \frac{x}{t}\}} d\xi \xi \otimes (x - t\xi) K(x - t\xi, \xi) f(x - t\xi) f(\xi) \right) = x \eta(x)$$

or in a more detailed manner for each of the coordinates

$$\operatorname{div} (J_j(x)) = x_j \eta(x), \quad j = 1, 2, \dots, d, \tag{2.8}$$

where each J_j itself is a vector-valued distribution with

$$(J_j)_i(x) = \int_0^1 dt \int_{\{0 < \xi < \frac{x}{t}\}} d\xi \xi_i (x_j - t\xi_j) K(x - t\xi, \xi) f(x - t\xi) f(\xi), \quad j = 1, 2, \dots, d. \tag{2.9}$$

In the case of constant flux solutions, i.e., in the absence of the source term η , equation (2.8) becomes

$$\operatorname{div} (J_j(x)) = 0, \quad j = 1, 2, \dots, d. \tag{2.10}$$

Note that the equations (2.8), (2.10) indeed correspond to the conservation laws associated with the transport of each of the components of the clusters of the system.

In order to quantify the fluxes of different monomer types which characterize the solutions of (2.10) we introduce the following notation. We will write

$$\Sigma_R = \left\{ x \in \mathbb{R}_*^d : |x| = R \right\} \text{ for each } R > 0.$$

Note that then $\Sigma_R = R\Delta^{d-1}$. The outward-pointing unit vector n , with respect to the simplex $\{x \in \mathbb{R}_+^d : 0 \leq |x| \leq R\}$, is given at any point of Σ_R by

$$n = \frac{1}{\sqrt{d}} (1, 1, \dots, 1).$$

Let us for simplicity assume that each $J_j(x)$ is a regular function which satisfies (2.10) and is zero if $x_i \leq 0$ for any component i . We integrate (2.10) over the set $\{x \in \mathbb{R}_+^d : R_1 \leq |x| \leq R_2\}$, for arbitrary $0 < R_1 < R_2$ and use Stokes' theorem. This shows that there is $A \in \mathbb{R}^d$ such that

$$\int_{\Sigma_R} [J_j(x) \cdot n] dS_x = A_j, \text{ for all } j = 1, 2, \dots, d, R > 0 \tag{2.11}$$

where dS_x is the surface area element. It readily follows from (2.9) that $A_j \geq 0$ for each $j \in \{1, 2, \dots, d\}$, i.e., $A \in \mathbb{R}_+^d$. In particular, we find that the flux of monomers of type j is constant across all the surfaces Σ_R .

In contrast to the case $d = 1$, finding f for which equations (2.11) hold does not imply that f satisfies (2.10). This is due to the fact that in the case $d = 1$ the set Σ_R is just a point for each $R > 0$. If $d > 1$ the relation (2.11) does not specify the distribution of the fluxes $J_j(x)$ in each surface Σ_R and this distribution must be obtained from the equations (2.10).

We prove in [7] that the solutions of (2.9), (2.10) are Dirac-like measures f supported along a line $\{x = \lambda b : \lambda > 0\}$ for some vector $b \in \mathbb{R}_+^d$ with $|b| = 1$. Let us point out that indeed there exist solutions with that form. To this end it is convenient to reformulate (2.10) in weak form and to change to the coordinate system $(|x|, \theta)$ indicated above.

Taking into account (2.7), it is natural to define a weak solution of (2.10) as a measure $f \in \mathcal{M}(\mathbb{R}_*^d)$ satisfying

$$0 = \int_{\mathbb{R}_*^d} d\xi \int_{\mathbb{R}_*^d} dx [\varphi(x + \xi) - \varphi(x)] K(x, \xi) x_j f(x) f(\xi) \tag{2.12}$$

for each $j = 1, 2, \dots, d$ and every test function $\varphi \in C_c^1(\mathbb{R}_*^d)$ (see Section 2.2 for a precise definition of weak solutions).

Reformulation of the Problem (2.12) Using a Suitable Change of Variables

It is convenient to rewrite (2.12) using the new coordinates (r, θ) with $r = |x| > 0$ and $\theta = \frac{1}{|x|}x \in \Delta^{d-1}$ for $x \in \mathbb{R}_*^d$. The inverse map $\mathbb{R}_* \times \Delta^{d-1} \rightarrow \mathbb{R}_*^d$ is given by

$$x = r\theta, \quad r > 0, \quad \theta \in \Delta^{d-1}. \tag{2.13}$$

We compute the Jacobian of the mapping $x \rightarrow (r, \theta)$. We use the variables $\theta_1, \theta_2, \dots, \theta_{d-1}$ to parametrize the simplex and set then

$$\theta_d = 1 - \sum_{j=1}^{d-1} \theta_j.$$

Thus, the change of variables is $x \rightarrow (r, \theta_1, \theta_2, \dots, \theta_{d-1})$. Therefore, with the above implicit definition of θ_d ,

$$\frac{\partial (x_1, x_2, \dots, x_d)}{\partial (r, \theta_1, \theta_2, \dots, \theta_{d-1})} = \begin{vmatrix} \theta_1 & \theta_2 & \theta_3 & \dots & \theta_{d-1} & \theta_d \\ r & 0 & 0 & \dots & 0 & -r \\ 0 & r & 0 & \dots & 0 & -r \\ \dots & \dots & r & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -r \\ 0 & 0 & 0 & 0 & r & -r \end{vmatrix} = D_d (r; \theta_1, \theta_2, \dots, \theta_d) .$$

We can iterate, developing the determinant by columns. Then

$$D_d (r; \theta_1, \theta_2, \dots, \theta_d) = (-1)^{d-1} r^{d-1} \theta_1 - r D_{d-1} (r; \theta_2, \theta_3, \dots, \theta_d)$$

Iterating, we arrive to

$$D_d (r; \theta_1, \theta_2, \dots, \theta_d) = (-1)^{d-1} r^{d-1} (\theta_1 + \theta_2 + \dots + \theta_d) = (-r)^{d-1}$$

Then

$$dx = \left| \frac{\partial (x_1, x_2, \dots, x_d)}{\partial (r, \theta_1, \theta_2, \dots, \theta_{d-1})} \right| dr d\theta_1 d\theta_2 \dots d\theta_{d-1} = r^{d-1} dr d\theta_1 d\theta_2 \dots d\theta_{d-1} .$$

We can write $d\theta_1 d\theta_2 \dots d\theta_{d-1}$ in terms of the area element of the simplex. We just use

$$dS (\theta) = \sqrt{1 + (\nabla_\theta h)^2} d\theta_1 d\theta_2 \dots d\theta_{d-1} .$$

with $\theta_d = h (\theta_1, \theta_2, \dots, \theta_{d-1})$ where $h (\theta_1, \theta_2, \dots, \theta_{d-1}) = 1 - \sum_{j=1}^{d-1} \theta_j$. We will denote the element of area of the simplex as $d\tau (\theta)$. Explicitly,

$$d\tau (\theta) = \sqrt{d} d\theta_1 d\theta_2 \dots d\theta_{d-1}$$

Thus,

$$dx = \frac{r^{d-1}}{\sqrt{d}} dr d\tau (\theta) . \tag{2.14}$$

We can now rewrite (2.12) using the above results. Suppose that $x = r\theta$ and $\xi = \rho\sigma$. We then have $|x + \xi| = r + \rho$. On the other hand,

$$\frac{x + \xi}{|x + \xi|} = \frac{r}{r + \rho} \theta + \frac{\rho}{r + \rho} \sigma .$$

We now rewrite the coagulation kernel in this set of variables as

$$G (r, \rho; \theta, \sigma) = K (r\theta, \rho\sigma) . \tag{2.15}$$

We also rewrite the measure f in terms of the measure $F \in \mathcal{M}_+(\mathbb{R}_* \times \Delta^{d-1})$ which is defined as

$$\int \psi (r, \theta) \frac{r^{d-1}}{\sqrt{d}} F (r, \theta) dr d\tau (\theta) = \int \psi (|x|, x/|x|) f (x) dx , \tag{2.16}$$

for a given test function $\psi \in C_c (\mathbb{R}_* \times \Delta^{d-1})$. Notice that if f is absolutely continuous with a smooth density, both sides of (2.16) are the same as it can be seen using an elementary change of variables. Then (2.12) can be equivalently written as

$$\int_0^\infty r^d dr \int_0^\infty \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau (\theta) \int_{\Delta^{d-1}} d\tau (\sigma) G (r, \rho; \theta, \sigma)$$

$$\times \left[\psi \left(r + \rho, \frac{r}{r + \rho} \theta + \frac{\rho}{r + \rho} \sigma \right) - \psi (r, \theta) \right] \theta_j F (r, \theta) F (\rho, \sigma) = 0, \quad (2.17)$$

for all $j = 1, 2, \dots, d$ and $\psi \in C_c^1(\mathbb{R}_* \times \Delta^{d-1})$. Notice that in any open bounded set of \mathbb{R}_*^d the change of variables (2.13) defines a diffeomorphism.

We observe that the change of variables (2.16) can be understood rigorously via the Riesz–Markov–Kakutani theorem applied to the linear functional

$$\varphi \mapsto \int_{\mathbb{R}_*^d} \frac{\sqrt{d}}{|x|^{d-1}} \varphi(|x|, x/|x|) f(x) dx, \quad \varphi \in C_c(\mathbb{R}_* \times \Delta^{d-1})$$

which allows to define a measure F . Furthermore, if f satisfies the assumptions in Definition 2.1, we have that F is supported in $[1, \infty) \times \Delta^{d-1}$ and

$$\int_{\mathbb{R}_* \times \Delta^{d-1}} r^{d-1+\gamma+p} F(r, \theta) dr d\tau(\theta) < \infty.$$

A Family of Weighted Dirac- δ Solutions

Suppose that K is continuous and homogeneous with homogeneity γ . If the kernel K satisfies (1.8), (1.9) with $\gamma + 2p < 1$, we claim that we then have a family of solutions of (2.17) given by the following weighted Dirac δ -measures

$$F(r, \theta) = \frac{C_0}{r^{\frac{\gamma+1}{2}+d}} \delta(\theta - \theta_0), \quad C_0 > 0, \quad (2.18)$$

where $\theta_0 \in \Delta^{d-1}$ is fixed but arbitrary. To see this, first note that $\frac{r}{r+\rho} \theta_0 + \frac{\rho}{r+\rho} \theta_0 = \theta_0$, and thus then (2.17) is equivalent to

$$(\theta_0)_j \int_0^\infty r^d dr \int_0^\infty \rho^{d-1} d\rho \frac{G(r, \rho; \theta_0, \theta_0)}{r^{\frac{\gamma+1}{2}+d} \rho^{\frac{\gamma+1}{2}+d}} [\psi(r + \rho, \theta_0) - \psi(r, \theta_0)] = 0. \quad (2.19)$$

Notice that the integral in (2.19) is well defined for $\psi \in C_c^1(\mathbb{R}_* \times \Delta^{d-1})$ and $\gamma + 2p < 1$.

We now rewrite (2.19) in a more convenient form. First, since $\theta_0 \in \Delta^{d-1}$, there is at least one j such that $(\theta_0)_j > 0$. Thus the factor $(\theta_0)_j$ may be dropped from (2.19). Then for any $\psi \in C_c^1(\mathbb{R}_* \times \Delta^{d-1})$ we may employ inside the integrand the identity

$$\psi(r + \rho, \theta_0) - \psi(r, \theta_0) = \int_r^{r+\rho} \frac{\partial \psi}{\partial t}(t, \theta_0) dt = \int_a^b \mathbb{1}_{t \geq r} \mathbb{1}_{t < r + \rho} \frac{\partial \psi}{\partial t}(t, \theta_0) dt.$$

where $0 < a < b$ are such that the support of ψ lies in $[a, b] \times \Delta^{d-1}$. Therefore, applying Fubini’s Theorem, (2.19) is seen to be equivalent with

$$\int_a^b dt \frac{\partial \psi}{\partial t}(t, \theta_0) \int_0^t r^d dr \int_{t-r}^\infty \rho^{d-1} d\rho \frac{G(r, \rho; \theta_0, \theta_0)}{r^{\frac{\gamma+1}{2}+d} \rho^{\frac{\gamma+1}{2}+d}} = 0.$$

Integrating by parts we obtain

$$\int_0^\infty dt \psi(t, \theta_0) \frac{\partial}{\partial t} \left(\int_0^t r^d dr \int_{t-r}^\infty \rho^{d-1} d\rho \frac{G(r, \rho; \theta_0, \theta_0)}{r^{\frac{\gamma+1}{2}+d} \rho^{\frac{\gamma+1}{2}+d}} \right) = 0. \quad (2.20)$$

Since this needs to hold for all allowed ψ , we find that it is valid if and only if there is $J_0 \geq 0$ such that

$$J_0 = \int_0^t r^d dr \int_{t-r}^\infty \rho^{d-1} d\rho \frac{G(r, \rho; \theta_0, \theta_0)}{r^{\frac{\gamma+1}{2}+d} \rho^{\frac{\gamma+1}{2}+d}}, \quad \text{for all } t > 0. \tag{2.21}$$

This equation is indeed satisfied if K is a homogeneous kernel, since then $G(\lambda r, \lambda \rho; \theta_0, \theta_0) = \lambda^\gamma G(r, \rho; \theta_0, \theta_0)$ for each $\lambda > 0$, and thus

$$\begin{aligned} & \int_0^t r^d dr \int_{t-r}^\infty \rho^{d-1} d\rho \frac{G(r, \rho; \theta_0, \theta_0)}{r^{\frac{\gamma+1}{2}+d} \rho^{\frac{\gamma+1}{2}+d}} \\ &= \frac{t^{2d+1}}{t^{\gamma+1+2d}} \int_0^1 r^d dr \int_{1-r}^\infty \rho^{d-1} d\rho \frac{G(tr, t\rho; \theta_0, \theta_0)}{r^{\frac{\gamma+1}{2}+d} \rho^{\frac{\gamma+1}{2}+d}} \\ &= \int_0^1 r^d dr \int_{1-r}^\infty \rho^{d-1} d\rho \frac{G(r, \rho; \theta_0, \theta_0)}{r^{\frac{\gamma+1}{2}+d} \rho^{\frac{\gamma+1}{2}+d}} \end{aligned}$$

is constant.

Using the same procedure to simplify the general case (2.17) yields an alternative way of prescribing the fluxes through the surfaces $\{|x| = R\}$. This will be made precise in Section 2.2 (cf. Definitions 2.3).

Stationary Solutions with a Prescribed Concentration of Monomers

As a third possibility used in the literature to obtain stationarity of solutions to coagulation equation, let us briefly mention using, instead of sources, boundary conditions to fix the concentration of monomers to some given value in (2.1). The one-component case has already been considered in [6], but the definition becomes more involved here due to the fact that we have a multicomponent system.

Explicitly, we would then be interested in solutions of

$$0 = \frac{1}{2} \sum_{\beta < \alpha} K_{\alpha-\beta, \beta} n_{\alpha-\beta} n_\beta - n_\alpha \sum_{\beta > \alpha} K_{\alpha, \beta} n_\beta, \quad |\alpha| \notin \{0, 1\} \tag{2.22}$$

We will say that $\{n_\alpha\}_{\alpha \in \mathbb{N}_*^d \setminus \{O\}}$ is a *stationary solution of (2.22) with a prescribed concentration of monomers*, if it solves (2.22) and in addition it satisfies

$$\sum_{\{\alpha: \alpha_j=1\}} n_\alpha = c_j \quad \text{for each } j = 1, 2, 3, \dots, d \tag{2.23}$$

for a given set of concentrations $\{c_j\}_{j=1}^d \in \mathbb{R}_+^d$.

The existence of solutions of the problem (2.22), (2.23) for a given set of concentrations $\{c_j\}_{j=1}^d$ is not evident at all. If we have an injection solution to (2.1) for a set of sources $\{s_\alpha\}_{|\alpha|=1}$, then we have a solution of (2.22), (2.23) for the corresponding values of c_j obtained by means of the sum (2.23). However, there is not any reason to expect that any set of concentrations $\{c_j\}_{j=1}^d$ could be obtained by means of a suitable choice of sources $\{s_\alpha\}_{|\alpha|=1}$. It has been seen in [6] that in the case $d = 1$ the problem (2.22)–(2.23) can be solved if $|\gamma + 2\lambda| < 1$ and kernels with the form (1.10), (1.11).

2.2 Rigorous Definition of the Classes of Steady State Solutions

We define now in a precise mathematical way the solutions that we will consider in this paper.

Definition 2.1 Let $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$ with support contained in $\{x \in \mathbb{R}_*^d \mid 1 \leq |x| \leq L\}$ for some $L > 1$. Suppose that the coagulation kernel K is continuous and satisfies (1.8), (1.9). We say that $f \in \mathcal{M}_+(\mathbb{R}_*^d)$ is a stationary injection solution to (2.2) if the support of f is contained in $\{x \in \mathbb{R}_*^d \mid |x| \geq 1\}$ and f satisfies

$$\int_{\mathbb{R}_*^d} |x|^{\gamma+p} f(dx) < \infty, \tag{2.24}$$

as well as

$$0 = \frac{1}{2} \int_{\mathbb{R}_*^d} \int_{\mathbb{R}_*^d} K(x, y) [\varphi(x+y) - \varphi(x) - \varphi(y)] f(dx) f(dy) + \int_{\mathbb{R}_*^d} \varphi(x) \eta(dx) \tag{2.25}$$

for any test function $\varphi \in C^1(\mathbb{R}_*^d)$.

We recall the notation $\mathbb{R}_*^d := \mathbb{R}_+^d \setminus \{O\}$ and that $|x| = \sum_j |x_j|$ denotes the ℓ_1 -norm in \mathbb{R}_*^d .

Stationary injection solutions for the discrete equation (2.1) can be considered as solutions f of (2.2) with f supported on the elements of $\mathbb{N}_*^d = \mathbb{N}_0^d \setminus \{O\}$ by using Dirac- δ measures as explained next. Let the sequence $\{n_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ be a solution of (2.1), i.e., it satisfies

$$0 = \frac{1}{2} \sum_{\beta < \alpha} K_{\alpha-\beta, \beta} n_{\alpha-\beta} n_\beta - n_\alpha \sum_{\beta > 0} K_{\alpha, \beta} n_\beta + s_\alpha, \quad \alpha \in \mathbb{N}_*^d, \tag{2.26}$$

where the source satisfies $s_\alpha = 0$ whenever $|\alpha| > L$ for some $L > 1$. We then define

$$f(x) = \sum_{\alpha \in \mathbb{N}_*^d} n_\alpha \delta(x - \alpha) \tag{2.27}$$

as well as

$$\eta(x) = \sum_{\alpha \in \mathbb{N}_*^d} s_\alpha \delta(x - \alpha). \tag{2.28}$$

Then η satisfies the assumptions of Definition 2.1 with the same parameter L and we can define a solution of (2.26) as follows.

Definition 2.2 Suppose that $\{s_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ is a non-negative sequence supported in a finite collection of values α . A sequence $\{n_\alpha\}_{\alpha \in \mathbb{N}_*^d}$, with $n_\alpha \geq 0$ for $\alpha \in \mathbb{N}_*^d$, is a stationary injection solution of (2.26) if

$$\sum_{\alpha \in \mathbb{N}_*^d} |\alpha|^{\gamma+p} n_\alpha < \infty,$$

and the measure $f \in \mathcal{M}_+(\mathbb{R}_*^d)$ defined as in (2.27) solves (2.2) in the sense of Definition 2.1.

Thanks to the assumptions on n_α the measure f has its support in $\{x \in \mathbb{R}_*^d \mid |x| \geq 1\}$ and satisfies (2.24).

We now provide a rigorous definition of the constant flux to the equation (2.2) with $\eta = 0$, namely

$$0 = \frac{1}{2} \int_{\{0 < y < x\}} K(x - y, y) f(x - y, t) f(y, t) dy - \int_{\mathbb{R}_*^d} K(x, y) f(x, t) f(y, t) dy. \tag{2.29}$$

Definition 2.3 Suppose that the coagulation kernel K is continuous and satisfies (1.8), (1.9). We say that $f \in \mathcal{M}(\mathbb{R}_*^d)$ is a stationary solution to (2.29) if

$$\int_{\{x \in \mathbb{R}_*^d \mid |x| \geq 1\}} |x|^{\gamma+p} f(dx) + \int_{\{x \in \mathbb{R}_*^d \mid |x| < 1\}} |x|^{1-p} f(dx) < \infty$$

is satisfied and (2.25) holds with $\eta = 0$, for every test function $\varphi \in C_c^1(\mathbb{R}_*^d)$.

We define the total flux across the surface $\{|x| = R\}$ as the vector-valued function $A(R) \in \mathbb{R}_+^d$, $R > 0$, defined by means of

$$A_j(R) = \frac{1}{d} \int_{(0, R] \times \Delta^{d-1}} dr d\tau(\theta) \int_{(R-r, \infty) \times \Delta^{d-1}} d\rho d\tau(\sigma) r^d \rho^{d-1} F(r, \theta) F(\rho, \sigma) \theta_j G(r, \rho; \theta, \sigma), \tag{2.30}$$

where the function G is as in (2.15) and the measure F has been defined using (2.16). We say that f is a non-trivial constant flux solution of (2.29) if it is a stationary solution and there is $J_0 > 0$ such that $A(R) = J_0$ for all $R > 0$.

Remark 2.4 The above definition of flux, (2.30), is obtained by a similar computation as leading to the special case in (2.21) with the additional assumption that the test-function is constant in the simplex-variable θ ; the details of this argument may be found in the proof of Theorem 3.1. Note that in the one-component case to impose that the fluxes are constant, i.e. (2.30), implies that f is a solution to the coagulation equation. This does not automatically happen in the multicomponent case and this explains why we need to further assume (2.25) here.

3 Main Theorems

We state in this Section the main results proven in this paper on the existence and non-existence of stationary injection solutions to (2.2) and (2.26), as well as of the constant flux solutions to (2.29), which have been considered in [6] for the one-component case.

We first describe the existence results for the injection solutions:

Theorem 3.1 *Suppose that the coagulation kernel K is a continuous symmetric function that satisfies (1.8), (1.9) with $\gamma + 2p < 1$. Suppose that $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$ has its support in the set $\{x \in \mathbb{R}_*^d \mid 1 \leq |x| \leq L\}$ for some $L > 1$. Then, there exists a stationary injection solution $f \in \mathcal{M}_+(\mathbb{R}_*^d)$ to (2.2) in the sense of Definition 2.1.*

The following theorem is a corollary of the previous result assuming that f is supported on the set \mathbb{N}_*^d . However, since the discrete coagulation equation has an independent interest and it is relevant for applications, we formulate the result as a separate theorem.

Theorem 3.2 *Suppose that the coagulation kernel K is symmetric and satisfies (1.7), (1.9) with $\gamma + 2p < 1$. Let $\{s_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ be a non-negative sequence supported on a finite number of values α . Then, there exists a stationary injection solution $\{n_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ to (2.26) in the sense of Definition 2.2.*

Remark 3.3 Notice that the assumptions on the kernels K in Theorems 3.1, 3.2 are more general than those in [6], since the assumptions on the kernels (1.10), (1.11) were used there. Therefore, the results in this paper provide an improvement of the earlier results even in the case of one component, $d = 1$.

Remark 3.4 Let us point out that no uniqueness of the solutions is claimed in Theorems 3.1, 3.2. The issue of uniqueness of stationary injection solutions is an interesting open problem.

The restrictions for the values of γ an p in Theorems 3.1, 3.2 are not only sufficient to have stationary injection solutions, but they are also necessary. Indeed, we have the following non-existence results, which are analogous to Theorems 2.4, 5.3. in [6].

Theorem 3.5 *The following results hold:*

- (i) *Suppose that K is a continuous symmetric function that satisfies (1.8), (1.9) with $\gamma + 2p \geq 1$. Suppose that $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$ is supported inside the set $\{x \in \mathbb{R}_*^d \mid 1 \leq |x| \leq L\}$ for some $L > 1$. Then there are no solutions to (2.2) in the sense of Definition 2.1.*
- (ii) *Suppose that K is symmetric and satisfies (1.7), (1.9) with $\gamma + 2p \geq 1$. Let $\{s_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ be a nonnegative sequence supported on a finite number of values α . Then there are no solutions to (2.26) in the sense of Definition 2.2.*

Concerning the constant flux solutions to (2.29) we have already seen in Section 2 that (2.18) defines a constant flux solution to (2.29) in the sense of Definition 2.3 if $\gamma + 2p < 1$, at least when K is a homogeneous kernel function. If $\gamma + 2p \geq 1$, such solutions do not exist. This is the content of the following Theorem.

Theorem 3.6 *Suppose that K satisfies (1.8), (1.9) with $\gamma + 2p \geq 1$. There are no non-trivial constant flux solutions to (2.29), in the sense of Definition 2.3.*

Remark 3.7 We observe that we did not require the kernel K to be homogeneous in Theorems 3.1, 3.2, nor in Theorems 3.5, 3.6, for which the upper and lower estimates (1.7), (1.8), (1.9) suffice. Notice that, if the kernel K is homogeneous, there are constant flux solutions to (2.29) with $H(r) = r^{-\frac{\gamma+3}{2}}$ (cf. (2.18)). However, not all constant flux solutions to the one-component equation are necessarily power-laws, even if one assumes homogeneity of the kernel. Examples of non-power law solutions for certain one-dimensional coagulation kernels are given in [9].

4 Some Auxiliary Results

4.1 A Convenient Reformulation of the Problem

The kernels K satisfying (1.7)–(1.9) are characterized by the two parameters γ, p . It turns out that, using a suitable change of variable, we can reformulate the problems described in

Sect. 2 with kernels K into similar problems with new kernels \tilde{K} characterized by parameters $\tilde{\gamma} = \gamma + 2p$ and $\tilde{p} = 0$.

To see this, we will use an idea used in [2] (see also [1] and the recent paper [17]). Before formulating the precise results, we explain the idea in the case of the continuous coagulation equation (2.2). Suppose that $f \in \mathcal{M}_+(\mathbb{R}_*^d)$ solves (2.2). We can rewrite this equation as

$$0 = \frac{1}{2} \int_{\{0 < \xi < x\}} d\xi [K(x - \xi, \xi) |x - \xi|^p |\xi|^p] \frac{f(x - \xi) f(\xi)}{|x - \xi|^p |\xi|^p} - \int_{\mathbb{R}_*^d} d\xi [K(x, \xi) |x|^p |\xi|^p] \frac{f(x) f(\xi)}{|x|^p |\xi|^p} + \eta(x).$$

Therefore, if we multiply the measure f by the strictly positive continuous function $x \mapsto |x|^{-p}$, i.e., if we define $h(x) dx = |x|^{-p} f(x) dx$, we find that it solves the following equation

$$0 = \frac{1}{2} \int_{\{0 < \xi < x\}} d\xi \tilde{K}(x - \xi, \xi) h(x - \xi) h(\xi) - \int_{\mathbb{R}_+^d} d\xi \tilde{K}(x, \xi) h(x) h(\xi) + \eta(x) \tag{4.1}$$

where

$$\tilde{K}(x, y) = K(x, y) |x|^p |y|^p. \tag{4.2}$$

Notice that (4.1) has the same form as (2.2). However, the bounds for the kernel \tilde{K} are simpler than those for K . Namely, we recall that K satisfies (1.8), (1.9), and thus obtain the bounds

$$c_1 (|x| + |y|)^\gamma |x|^p |y|^p \Phi\left(\frac{|x|}{|x| + |y|}\right) \leq \tilde{K}(x, y) \leq c_2 (|x| + |y|)^\gamma |x|^p |y|^p \Phi\left(\frac{|x|}{|x| + |y|}\right).$$

Denoting $s = \frac{|x|}{|x| + |y|}$, we then obtain

$$c_1 (|x| + |y|)^{\gamma+2p} s^p (1 - s)^p \Phi(s) \leq \tilde{K}(x, y) \leq c_2 (|x| + |y|)^{\gamma+2p} s^p (1 - s)^p \Phi(s).$$

By (1.9), this implies

$$c_1 (|x| + |y|)^{\gamma+2p} \leq \tilde{K}(x, y) \leq c_2 (|x| + |y|)^{\gamma+2p}, \quad x, y \in \mathbb{R}_*^d. \tag{4.3}$$

Therefore, h solves (4.1) which is the same equation as (2.2) with a kernel \tilde{K} satisfying (4.3). The new kernel thus satisfies (1.8) after replacing γ by $\tilde{\gamma} = \gamma + 2p$ and p by $\tilde{p} = 0$. In addition, the supports of h and f are the same, and the moment bound (2.24) is true for f , γ , and p , if and only if it is true for h , $\tilde{\gamma}$, and \tilde{p} . Notice that exactly the same argument can be made in the weak formulation of (2.2), as well as in the discrete problem (2.1). Therefore we can reduce the discrete problem considered in this paper to an analogous problem with kernel

$$\tilde{K}(\alpha, \beta) = K(\alpha, \beta) |\alpha|^p |\beta|^p \tag{4.4}$$

which satisfies an estimate

$$c_1 (|\alpha| + |\beta|)^{\gamma+2p} \leq \tilde{K}(\alpha, \beta) \leq c_2 (|\alpha| + |\beta|)^{\gamma+2p}. \tag{4.5}$$

These observations can be summarized as follows.

Lemma 4.1 *The following statements hold:*

- (i) Let $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$ with the support in the set $\{x \in \mathbb{R}_*^d \mid 1 \leq |x| \leq L\}$ for some $L > 1$. Let us assume that K is continuous and satisfies (1.8), (1.9). Then, the Radon measure $f \in \mathcal{M}_+(\mathbb{R}_*^d)$ is a stationary injection solution to (2.2) in the sense of Definition 2.1 if and only if the Radon measure $h(x) = \frac{f(x)}{|x|^p}$ is a stationary injection solution to (2.2) with kernel \tilde{K} defined as in (4.2). Moreover, (4.3) holds.
- (ii) Suppose that $\{s_\alpha\}_{\alpha \in \mathbb{N}_*^d \setminus \{O\}}$ is a sequence supported in a finite number of values α . Let us assume that the kernel K satisfies (1.7), (1.9). Then the sequence $\{n_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ is a stationary injection solution to (2.26) in the sense of Definition 2.2 if and only if the sequence $\{|\alpha|^{-p} n_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ is a stationary injection solution to (2.26) with kernel \tilde{K} defined as in (4.4). Moreover, (4.5) is satisfied.
- (iii) Let us assume that K is continuous and satisfies (1.8), (1.9). The Radon measure $f \in \mathcal{M}_+(\mathbb{R}_+^d)$ is a constant flux solution to (2.29) in the sense of Definition 2.3 if and only if the Radon measure $h(x) = \frac{f(x)}{|x|^p}$ is a constant flux solution to (2.29) with the kernel \tilde{K} defined as in (4.2). Moreover, (4.3) is satisfied.

Therefore, it is sufficient to consider kernels satisfying (1.7)–(1.9) with $p = 0$. Equivalently, we need to examine kernels satisfying (4.3), (4.5).

4.2 An Auxiliary Lemma

The following result which will be extensively used in the rest of the paper has been proven as Lemma 2.10 in [6]. It allows to transform estimates of averaged integrals into estimates on the whole line.

Lemma 4.2 *Suppose $a > 0$ and $b \in (0, 1)$, and assume that $R \in (0, \infty]$ is such that $R \geq a$. Consider some $f \in \mathcal{M}_+(\mathbb{R}_*)$ and $\varphi \in C(\mathbb{R}_*)$, with $\varphi \geq 0$.*

- 1. *Suppose $R < \infty$, and assume that there is $g \in L^1([a, R])$ such that $g \geq 0$ and*

$$\frac{1}{z} \int_{[bz, z]} \varphi(x) f(dx) \leq g(z), \quad \text{for } z \in [a, R]. \tag{4.6}$$

Then

$$\int_{[a, R]} \varphi(x) f(dx) \leq \frac{\int_{[a, R]} g(z) dz}{|\ln b|} + Rg(R). \tag{4.7}$$

- 2. *Consider some $r \in (0, 1)$, and assume that $a/r \leq R < \infty$. Suppose that (4.6) holds for $g(z) = c_0 z^q$, with $q \in \mathbb{R}$ and $c_0 \geq 0$. Then there is a constant $C > 0$, which depends only on r, b and q , such that*

$$\int_{[a, R]} \varphi(x) f(dx) \leq Cc_0 \int_{[a, R]} z^q dz. \tag{4.8}$$

- 3. *If $R = \infty$ and there is $g \in L^1([a, \infty))$ such that $g \geq 0$ and*

$$\frac{1}{z} \int_{[bz, z]} \varphi(x) f(dx) \leq g(z), \quad \text{for } z \geq a, \tag{4.9}$$

then

$$\int_{[a, \infty)} \varphi(x) f(dx) \leq \frac{\int_{[a, \infty)} g(z) dz}{|\ln b|}. \tag{4.10}$$

5 Proof of the Existence of Stationary Injection Solutions

In this Section we prove Theorems 3.1, 3.2.

5.1 Continuous Coagulation Equation

We first prove Theorem 3.1. Notice that due to Lemma 4.1 it is enough to prove the Theorem under the additional assumptions $\gamma < 1, p = 0$. In particular, then the coagulation kernel satisfies (4.3) with $p = 0$.

We will follow a strategy that has been used in the literature to show existence results for some classes of unbounded coagulation kernels (cf. [1]). This consists in proving first the existence of stationary injection solutions for a truncated version of the problem in which the kernel K is replaced by a compactly supported kernel. We will then derive estimates for the solutions of these truncated problems that are uniform in the truncation parameter and we can then take the limit in the truncated problem and derive a solution to (2.2).

We first define the truncated kernel. We will make two truncations, the first one to obtain a bounded kernel and the second one to obtain a kernel with compact support. Before describing these truncations in detail we prove that there exists a stationary injection solution for a large class of coagulation equations with bounded, compactly supported kernels. This result will be used later as an auxiliary tool.

We will use the following Assumptions to characterize a class of solutions in a simplified setup where M is a cutoff parameter, $K = K_M$ is a suitably bounded kernel, and we additionally cut off the “gain term” for large values of $|x|$. This will result in unique solvability of the coagulation evolution equation, and imply existence of stationary solutions.

Assumption 5.1 (*Cutoff model*) We will make the following assumptions on the fixed source term η , on the kernel K_M and on the cutoff function ζ_M .

- (i) Consider a source term $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$. There exists a real number $L \geq 1$ such that $\text{supp}(\eta) \subset \{x : 1 \leq |x| \leq L\}$.
- (ii) The kernel $K_M : \mathbb{R}_*^d \times \mathbb{R}_*^d \rightarrow \mathbb{R}_+$ is a continuous, nonnegative, symmetric function. Suppose that M, a_1, a_2 are constants such that $M > 2L$, with L as in item (i), and $0 < a_1 < a_2$. Assume that the kernel K_M satisfies

$$K_M(x, y) \leq a_2 \quad \text{for all } (x, y) \in (\mathbb{R}_*^d)^2.$$

We assume also

$$K_M(x, y) \in [a_1, a_2] \quad \text{for } (|x|, |y|) \in [1, M]^2$$

and

$$K_M(x, y) = 0 \quad \text{if } (|x|, |y|) \notin (0, 2M)^2.$$

- (iii) We assume that ζ_M is a fixed cutoff function such that $\zeta_M \in C(\mathbb{R}_*^d)$,

$$0 \leq \zeta_M \leq 1, \quad \zeta_M(x) = 1 \quad \text{for } 0 < |x| \leq \frac{1}{2}M, \quad \text{and} \quad \zeta_M(x) = 0 \quad \text{for } |x| \geq M.$$

The cutoff function ζ_M will be used to inactivate the “gain term” for large cluster sizes. Explicitly, in the simplified problem we study solutions to the evolution equations

$$\partial_t f(x, t) = \frac{\zeta_M(x)}{2} \int_{\{0 < y < x\}} K_M(x - y, y) f(x - y, t) f(y, t) dy$$

$$- \int_{\mathbb{R}_*^d} K_M(x, y) f(x, t) f(y, t) dy + \eta(x), \tag{5.1}$$

where the kernel K_M , the source η , and the function ζ_M are as in Assumption 5.1. In particular we are interested in the steady states associated to (5.1). These satisfy

$$0 = \frac{\zeta_M(x)}{2} \int_{\{0 < y < x\}} K_M(x - y, y) f(x - y, t) f(y, t) dy - \int_{\mathbb{R}_*^d} K_M(x, y) f(x, t) f(y, t) dy + \eta(x). \tag{5.2}$$

We will restrict our attention to solutions of (5.2) which vanish for small and very large cluster sizes. More precisely, we will use the following concept of solution to (5.2).

Definition 5.2 Suppose that Assumption 5.1 holds. We will say that $f \in \mathcal{M}_+(\mathbb{R}_*^d)$, satisfying $f(\{x \in \mathbb{R}_*^d : |x| < 1 \text{ or } |x| > M\}) = 0$, is a stationary injection solution to (5.1) if the following identity holds for any test function $\varphi \in C_c(\mathbb{R}_*^d)$:

$$0 = \frac{1}{2} \int_{\mathbb{R}_*^d} \int_{\mathbb{R}_*^d} K_M(x, y) [\varphi(x + y) \zeta_M(x + y) - \varphi(x) - \varphi(y)] f(dx) f(dy) + \int_{\mathbb{R}_*^d} \varphi(x) \eta(dx). \tag{5.3}$$

In order to prove the existence of a stationary injection solution to (5.1) in the sense of Definition 5.2 we will obtain these solutions as a fixed point for the corresponding evolution problem. We first prove the following result.

Proposition 5.3 *Suppose that Assumption 5.1 holds. Then, for any initial condition f_0 satisfying $f_0(\{x \in \mathbb{R}_*^d : |x| < 1 \text{ or } |x| > M\}) = 0$ and for any $T > 0$, there exists a unique time-dependent solution $f \in C^1([0, T], \mathcal{M}_{+,b}(\mathbb{R}_*^d))$ to (5.1) which solves it in the classical sense. Moreover, we have*

$$f\left(\left\{x \in \mathbb{R}_*^d : |x| < 1 \text{ or } |x| > M\right\}, t\right) = 0, \quad \text{for } 0 \leq t \leq T, \tag{5.4}$$

and the following estimate holds

$$\int_{\mathbb{R}_*^d} f(dx, t) \leq \int_{\mathbb{R}_*^d} f_0(dx) + Ct, \quad t \geq 0, \tag{5.5}$$

for $C = \int_{\mathbb{R}_*^d} \eta(dx) \geq 0$ which is independent of f_0, t , and T .

The proof of Proposition 5.3 reduces to the reformulation of (5.1) as an integral equation by means of an application of the Duhamel principle. The well-posedness result then follows by means of a standard fixed point argument. The estimate (5.5) is a consequence of the inequality $\partial_t \left(\int_{\mathbb{R}_*^d} f(dx, t)\right) \leq \int_{\mathbb{R}_*^d} \eta(dx)$ which follows by integration of (5.1). Given that the argument is just a small adaptation of the similar one in [6, Proposition 3.6] we will skip the details of the proof.

The solution f obtained in Proposition 5.3 has a number of useful properties needed later:

Proposition 5.4 *Suppose that Assumption 5.1 holds. Let $f_0 \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$ be as in the statement of Proposition 5.3 and let $f \in C^1([0, T], \mathcal{M}_{+,b}(\mathbb{R}_*^d))$ be the solution to (5.1) which has been obtained in Proposition 5.3. Then, the following properties hold.*

- (i) The function f is a weak solution to (5.1), i.e., for a test function $\varphi \in C^1([0, T], C_c(\mathbb{R}_*^d))$ and any $T > 0$ the following identity holds

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}_*^d} \varphi(x, t) f(dx, t) - \int_{\mathbb{R}_*^d} \partial_t \varphi(x, t) f(dx, t) \\ &= \frac{1}{2} \int_{\mathbb{R}_*^d} \int_{\mathbb{R}_*^d} K_M(x, y) [\varphi(x + y, t) \zeta_M(x + y) - \varphi(x, t) - \varphi(y, t)] f(dx, t) f(dy, t) \\ &+ \int_{\mathbb{R}_*^d} \varphi(x, t) \eta(dx), \quad \text{for every } t \in [0, T]. \end{aligned} \tag{5.6}$$

- (ii) The following inequality is satisfied

$$\partial_t \left(\int_{\mathbb{R}_*^d} f(dx, t) \right) \leq -\frac{a_1}{2} \left(\int_{\mathbb{R}_*^d} f(dx, t) \right)^2 + \int_{\mathbb{R}_*^d} \eta(dx). \tag{5.7}$$

- (iii) For each $M > 1$ we define a topological vector space

$$\mathcal{X}_M = \left\{ f \in \mathcal{M}_{+,b}(\mathbb{R}_*^d) : f \left(\left\{ x \in \mathbb{R}_*^d : |x| < 1 \text{ or } |x| > M \right\} \right) = 0 \right\}$$

endowed with the weak topology of measures, i.e., the $*$ -weak topology inherited as a closed subspace of $C_0(\mathbb{R}_*^d)^*$. If $R > 0$, the subset $\mathcal{U}_R := \left\{ f \in \mathcal{X}_M \mid \int_{\mathbb{R}_*^d} f(dx) \leq R \right\}$ is compact and metrizable in this topology. For each $t > 0$ we define an operator $S(t) : \mathcal{X}_M \rightarrow \mathcal{X}_M$ by means of $S(t)f_0 = f(\cdot, t)$, with $f(\cdot, t)$ as in Proposition 5.3. Then the family of operators $\{S(t)\}_{t \geq 0}$ define a continuous semigroup in \mathcal{X}_M . More precisely, we have

$$S(0) = I, \quad S(t_1 + t_2) = S(t_1)S(t_2) \text{ for each } t_1, t_2 \in \mathbb{R}_+, \tag{5.8}$$

the mapping $f \mapsto S(t)f$ with $f \in \mathcal{X}_M$ is continuous in \mathcal{X}_M for each $t \geq 0$. Moreover, the mapping $t \mapsto S(t)f$ from \mathbb{R}_+ to \mathcal{X}_M is continuous for each $f \in \mathcal{X}_M$.

Proof The proof of (i) is a consequence of the fact that f is a classical solution to (5.1). We then compute $\frac{d}{dt} \left(\int_{\mathbb{R}_*^d} \varphi(x, t) f(dx, t) \right)$ and write it in terms of the coagulation kernel using (5.1). Using the regularity properties of φ and f , and applying Fubini’s theorem to rewrite the term containing convolution in (5.1), we obtain (5.6).

Inequality (5.7) in item (ii) follows using a test function $\varphi(x, t)$ which takes the value 1 for $1 \leq |x| \leq M$, combined with the fact that $\zeta_M \leq 1$ and that $K(x, y) \geq a_1 > 0$ in the support of the measure $f(dx, t)f(dy, t)$.

The statements in item (iii) about the space \mathcal{X}_M and its intersections with norm-topology balls, \mathcal{U}_R , follow from standard results in functional analysis, in particular, by using the Banach–Alaoglu theorem; more details may be found in [6]. The properties of $S(t)$ in (5.8) follow from the definition of $S(t)$ and the uniqueness result in Proposition (5.3). The only nontrivial results to be proven are the continuity properties of $S(t)$. The continuity of $t \mapsto S(t)f$ follows from the differentiability of f in the t variable, yielding even Lipschitz-type estimates for the dependence, cf. Eq. (3.20) in [6].

The continuity of $f \mapsto S(t)f$ in the weak topology of measures is the most involved part of the results but it can also be proven as in [6]. The basic idea is to prove that the mappings $f_0 \mapsto \int_{\mathbb{R}_*^d} \varphi(x) f(dx, t)$ change continuously for every test function $\varphi \in C_c(\mathbb{R}_*^d)$. Considering the evolution of φ in terms of the adjoint equation from the time t to the time 0

we obtain a new function, denoted as $\varphi_0 \in C_c(\mathbb{R}_*^d)$, such that

$$\int_{\mathbb{R}_*^d} \varphi(x) f(dx, t) = \int_{\mathbb{R}_*^d} \varphi_0(x) f_0(dx).$$

Therefore, $\int_{\mathbb{R}_*^d} \varphi(x) f(dx, t)$ changes continuously if $\int_{\mathbb{R}_*^d} \varphi_0(x) f_0(dx)$ does. This gives the desired continuity in the weak topology of measures and the result follows. \square

We can now prove the existence of stationary injection solutions to (5.1) in the sense of Definition 5.2. We observe that, to prove the existence of stationary solutions, finding fixed points for the corresponding evolution semigroup has often been used in the study of coagulation equations. We refer for instance to [5, 10, 18–20]. Similar ideas have been used also to construct stationary solutions of more general classes of kinetic equations. See for instance [12, 14, 15].

Proposition 5.5 *Under the assumptions of Proposition 5.3, there exists a stationary injection solution $\hat{f} \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$ to (5.1) as in Definition 5.2.*

Proof The argument is made along the same lines as the one developed in [6], and we just provide a sketch of the main details. The key idea is to construct an invariant region in the space \mathcal{X}_M under the evolution semigroup $S(t)$. To this end, notice that (5.7) implies that for

$$R \geq \sqrt{\frac{2 \int_{\mathbb{R}_*^d} \eta(dx)}{a_1}}$$

the set

$$\mathcal{U}_R = \left\{ f \in \mathcal{X}_M : \int_{\mathbb{R}_*^d} f(dx) \leq R \right\} \tag{5.9}$$

is invariant under the evolution semigroup $S(t)$, i.e., $S(t)(\mathcal{U}_R) \subset \mathcal{U}_R$. By Proposition 5.4, the set \mathcal{U}_R is convex and compact in the weak topology of measures. Since the operator $S(\delta) : \mathcal{U}_R \rightarrow \mathcal{U}_R$ is continuous in the same topology, it follows from Schauder’s fixed point theorem that there exists a fixed point \hat{f}_δ for each $\delta > 0$. Since \mathcal{U}_R is metrizable and hence sequentially compact, we can use Theorem 1.2 of [5], and conclude that there is \hat{f} such that $S(t)\hat{f} = \hat{f}$ for all t . Thus \hat{f} is a stationary injection solution to (5.1). \square

We can now prove Theorem 3.1.

Proof of Theorem 3.1 Due to Lemma 4.1, item (i), it is enough to prove the result for $p = 0$ and $\gamma < 1$. Therefore, we will restrict our attention to kernels of the form

$$K(x, y) = (|x| + |y|)^\gamma \Phi(x, y) \tag{5.10}$$

where Φ is continuous, symmetric, and

$$0 < C_1 \leq \Phi(x, y) \leq C_2 < \infty, \text{ for } x, y \in \mathbb{R}_*^d. \tag{5.11}$$

We recall that we do not assume that the kernels $K(x, y)$ are homogeneous for this result.

We define a class of truncated kernels by means of

$$K_\varepsilon(x, y) = \min \left\{ (|x| + |y|)^\gamma, \frac{1}{\varepsilon} \right\} \Phi(x, y) + \varepsilon, \quad \varepsilon > 0 \tag{5.12}$$

The kernels K_ε are bounded in $\mathbb{R}_*^d \times \mathbb{R}_*^d$ for each $\varepsilon > 0$. Moreover, they satisfy also $K_\varepsilon(x, y) \geq \varepsilon > 0$ for any $(x, y) \in \mathbb{R}_*^d \times \mathbb{R}_*^d$. We will assume in the following that $\varepsilon \leq 1$, and add further restrictions to its upper bound later on. Note that if $\gamma \leq 0$, the truncation by

the minimum in (5.12) will not have any effect since we are interested in solutions supported in $|x| \geq 1$ and $|y| \geq 1$.

We now introduce another truncation to obtain compactly supported kernels. To this end we choose a symmetric function $o_M \in C_c(\mathbb{R}_* \times \mathbb{R}_*)$ such that for $x, y \in \mathbb{R}_*^d, r, s > 0$,

$$\omega_M(x, y) := o_M(|x|, |y|), \quad o_M(r, s) = \begin{cases} 1, & \text{if } 1 \leq r, s \leq M, \\ 0, & \text{if } r \geq 2M \text{ or } s \geq 2M. \end{cases} \tag{5.13}$$

Note that then $\omega_M \in C_c(\mathbb{R}_*^d \times \mathbb{R}_*^d)$ and it is symmetric. We then define truncated kernels $K_{\varepsilon, M}$ by means of

$$K_{\varepsilon, M}(x, y) = K_\varepsilon(x, y) \omega_M(x, y), \quad x, y \in \mathbb{R}_*^d. \tag{5.14}$$

Each of these kernels satisfies the requirements of K_M in Assumption 5.1.

We finally choose a function $p_M \in C(\mathbb{R}_*)$ satisfying Assumption 5.1 for $d = 1$. Then defining $\zeta_M(x) := p_M(|x|)$ satisfies Assumption 5.1 for general d . The hypothesis of Theorem 3.1 imply that η satisfies the requirements for this function in Assumption 5.1. Therefore, the above choices result in a system which satisfies all conditions required in Assumption 5.1, and thus Proposition 5.5 implies the existence of a stationary injection solution $f_{\varepsilon, M} \in \mathcal{M}_{+, b}(\mathbb{R}_*^d)$ satisfying $f_{\varepsilon, M}(\{x \in \mathbb{R}_*^d : |x| < 1 \text{ or } |x| > M\}) = 0$ and

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}_*^d} \int_{\mathbb{R}_*^d} K_{\varepsilon, M}(x, y) [\varphi(x+y) \zeta_M(x+y) - \varphi(x) - \varphi(y)] f_{\varepsilon, M}(dx) f_{\varepsilon, M}(dy) \\ & + \int_{\mathbb{R}_*^d} \varphi(x) \eta(dx) = 0 \end{aligned} \tag{5.15}$$

for any test function $\varphi \in C_c(\mathbb{R}_*^d)$.

We now derive uniform estimates for the family of solutions $f_{\varepsilon, M}$ in order to take the limits $M \rightarrow \infty$ and then $\varepsilon \rightarrow 0$. To this end, it will be convenient to use the coordinates (r, θ) introduced in (2.13) and defined as $r = |x|, \theta = \frac{x}{|x|}$. We write also $\rho = |y|, \sigma = \frac{y}{|y|}$ as well as $\varphi(x) = \psi(r, \theta), K_{\varepsilon, M}(x, y) = G_{\varepsilon, M}(r, \rho; \theta, \sigma)$, and $f_{\varepsilon, M}(x) = F_{\varepsilon, M}(r, \theta)$, as defined in (2.16). Then (5.15) becomes

$$\begin{aligned} & \frac{1}{2d} \int_{\mathbb{R}_*} r^{d-1} dr \int_{\mathbb{R}_*} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G_{\varepsilon, M}(r, \rho; \theta, \sigma) F_{\varepsilon, M}(r, \theta) F_{\varepsilon, M}(\rho, \sigma) \times \\ & \times \left[p_M(r+\rho) \psi\left(r+\rho, \frac{r}{r+\rho}\theta + \frac{\rho}{r+\rho}\sigma\right) - \psi(r, \theta) - \psi(\rho, \sigma) \right] + \int_{\mathbb{R}_*^d} \varphi(x) \eta(dx) = 0. \end{aligned} \tag{5.16}$$

Given $z, \delta > 0$, we introduce a function $\chi_\delta \in C_c^\infty(\mathbb{R}_*)$ such that $0 \leq \chi_\delta \leq 1, \chi_\delta(s) = 1$ for $1 \leq |s| \leq z$, and $\chi_\delta(s) = 0$ for $|s| \geq z + \delta$. We then take the radial test function $\varphi(x) = |x| \chi_\delta(|x|)$ for $x \in \mathbb{R}_*^d$. Then $\psi(r, \theta) = r \chi_\delta(r)$. Plugging this test function into (5.16), and using $p_M \leq 1$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}_*} r^{d-1} dr \int_{\mathbb{R}_*} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G_{\varepsilon, M}(r, \rho; \theta, \sigma) F_{\varepsilon, M}(r, \theta) F_{\varepsilon, M}(\rho, \sigma) \times \\ & \times [(r+\rho) \chi_\delta(r+\rho) - r \chi_\delta(r) - \rho \chi_\delta(\rho)] + 2d \int_{\mathbb{R}_*^d} |x| \chi_\delta(|x|) \eta(dx) \geq 0. \end{aligned}$$

Taking the limit $\delta \rightarrow 0$ and using arguments analogous to the ones in the proof of Lemma 2.7 in [6], yields

$$\begin{aligned} & \frac{1}{d} \int_{(0,z]} r^d dr \int_{(z-r,\infty)} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau (\theta) \int_{\Delta^{d-1}} d\tau (\sigma) G_{\varepsilon,M}(r, \rho; \theta, \sigma) F_{\varepsilon,M}(r, \theta) F_{\varepsilon,M}(\rho, \sigma) \\ & \leq \int_{\{0 < |x| \leq z\}} |x| \eta(dx), \quad \text{for any } z > 0. \end{aligned} \tag{5.17}$$

We can now derive uniform estimates for the measures $F_{\varepsilon,M}$. We have $d \int_{\{|x| \leq z\}} |x| \eta(dx) \leq d \int_{\{|x| \leq L\}} |x| \eta(dx) =: c$. On the other hand (5.12), (5.14) imply that $K_{\varepsilon,M}(x, y) = G_{\varepsilon,M}(r, \rho; \theta, \sigma) \geq \varepsilon > 0$ for $(r, \rho) \in [1, M]^2$. Thus we may use the information about the support of $f_{\varepsilon,M}$ to conclude that for any $z > 0$

$$\varepsilon \int_{(0,z]} r^d dr \int_{(z-r,\infty)} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau (\theta) \int_{\Delta^{d-1}} d\tau (\sigma) F_{\varepsilon,M}(r, \theta) F_{\varepsilon,M}(\rho, \sigma) \leq c.$$

Assume next $z \in [1, M]$. Then $[2z/3, z]^2 \subset \{(x, y) \in \mathbb{R}_+^2 : 0 < x \leq z, z - x < y \leq M\}$, and we obtain

$$\varepsilon z^{2d-1} \left(\int_{[\frac{2z}{3}, z]} dr \int_{\Delta^{d-1}} d\tau (\theta) F_{\varepsilon,M}(r, \theta) \right)^2 \leq C.$$

Therefore,

$$\frac{1}{z} \int_{[\frac{2z}{3}, z]} dr \int_{\Delta^{d-1}} d\tau (\theta) F_{\varepsilon,M}(r, \theta) \leq \frac{C_\varepsilon}{z^{\frac{2d+1}{2}}} \quad \text{for } 0 < z \leq M$$

where C_ε is a numerical constant depending on ε but independent of M . Then, Lemma 4.2 implies

$$\int_{\mathbb{R}_*^d} f_{\varepsilon,M}(dx) = \int_{\{1 \leq |x| \leq M\}} f_{\varepsilon,M}(dx) = \int_{[1,M]} r^{d-1} dr \int_{\Delta^{d-1}} d\tau (\theta) F_{\varepsilon,M}(r, \theta) \leq \bar{C}_\varepsilon. \tag{5.18}$$

with \bar{C}_ε independent on M .

By the earlier mentioned sequential compactness, this bound implies that for each $\varepsilon > 0$ there exists a sequence $\{M_n\}_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} M_n = \infty$ and $f_\varepsilon \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$ such that $f_{\varepsilon,M_n} \rightarrow f_\varepsilon$ in the weak topology of measures. Moreover, we have

$$\int_{\mathbb{R}_*^d} f_\varepsilon(dx) \leq \bar{C}_\varepsilon$$

and also $f_\varepsilon(\{x \in \mathbb{R}_*^d : |x| < 1\}) = 0$.

We now notice that $\zeta_{M_n}(x + y) \rightarrow 1$ as $n \rightarrow \infty$ if $|x + y| > 0$. Using this, as well as (5.18), we can now take the limit $n \rightarrow \infty$ in (5.15) with M replaced by M_n (more details about these estimates can be found from the proof of Theorem 2.3 in [6]). Since $K_\varepsilon(x, y) = \lim_{M \rightarrow \infty} K_{\varepsilon,M}(x, y)$, and using that K_ε is bounded for each $\varepsilon > 0$, we then obtain

$$\frac{1}{2} \int_{\mathbb{R}_*^d} \int_{\mathbb{R}_*^d} K_\varepsilon(x, y) [\varphi(x + y) - \varphi(x) - \varphi(y)] f_\varepsilon(dx) f_\varepsilon(dy) + \int_{\mathbb{R}_*^d} \varphi(x) \eta(dx) = 0 \tag{5.19}$$

for any test function $\varphi \in C_c(\mathbb{R}_*^d)$. We now argue again as in the derivation of (5.17), using the test function $\varphi(x) = |x| \chi_\delta(|x|)$, or equivalently $\psi(r, \theta) = r \chi_\delta(r)$. We define $F_\varepsilon(r, \theta) = f_\varepsilon(x)$ and $K_\varepsilon(x, y) = G_\varepsilon(r, \rho; \theta, \sigma)$. Then, taking the limit $\delta \rightarrow 0$ we arrive at

$$\begin{aligned} & \frac{1}{d} \int_{(0,z]} r^d dr \int_{(z-r,\infty)} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G_\varepsilon(r, \rho; \theta, \sigma) F_\varepsilon(r, \theta) F_\varepsilon(\rho, \sigma) \\ &= \int_{\{0 < |x| \leq z\}} |x| \eta(dx), \quad \text{for any } z > 0. \end{aligned}$$

Using (5.11) and (5.12) we have $K_\varepsilon(x, y) \geq \varepsilon + C_0 \min\{z^\gamma, \frac{1}{\varepsilon}\}$ for $|x|, |y| \in [\frac{z}{2}, z]$ with $C_0 > 0$. Using then that $[2z/3, z]^2 \subset \{(r, \rho) \in \mathbb{R}_+^2 : 0 < r \leq z, z - x < y < \infty\}$ we obtain

$$\left(\varepsilon + \min\{z^\gamma, \frac{1}{\varepsilon}\}\right) z \left(\int_{[2z/3, z]} r^{d-1} dr \int_{\Delta^{d-1}} F_\varepsilon(r, \theta) d\tau(\theta)\right)^2 \leq c \int_{\mathbb{R}_*^d} |x| \eta(dx) \text{ for all } z > 0,$$

where c is independent of ε and η . Thus

$$\begin{aligned} \frac{1}{z} \int_{[2z/3, z]} r^{d-1} dr \int_{\Delta^{d-1}} F_\varepsilon(r, \theta) d\tau(\theta) &\leq \frac{\tilde{C}}{z^{\frac{3}{2}} (\varepsilon + \min\{z^\gamma, \frac{1}{\varepsilon}\})^{\frac{1}{2}}} \left(\int_{\mathbb{R}_*^d} |x| \eta(dx)\right)^{\frac{1}{2}} \\ &\leq \frac{\tilde{C}}{z^{\frac{3}{2}} (\min\{z^\gamma, \frac{1}{\varepsilon}\})^{\frac{1}{2}}} \left(\int_{\mathbb{R}_*^d} |x| \eta(dx)\right)^{\frac{1}{2}} \end{aligned} \tag{5.20}$$

with \tilde{C} independent of ε and η , for $z > 0$. If $z \geq 1$, we have $\min\{z^\gamma, \frac{1}{\varepsilon}\} \geq z^{\gamma_-}$ where $\gamma_- := \min(0, \gamma) \leq 0$. Thus we can conclude that

$$\frac{1}{z} \int_{[2z/3, z]} r^{d-1} dr \int_{\Delta^{d-1}} F_\varepsilon(r, \theta) d\tau(\theta) \leq \frac{\tilde{C}}{z^{\frac{3}{2} + \gamma_-}} \left(\int_{\mathbb{R}_*^d} |x| \eta(dx)\right)^{\frac{1}{2}}, \quad \text{for all } z \geq 1. \tag{5.21}$$

This implies the existence of a subsequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that we have $|x|^{\gamma_-} f_{\varepsilon_n}(dx) \rightarrow |x|^{\gamma_-} f(dx)$ in the weak measure topology of $\mathcal{M}_{+,b}(\mathbb{R}_*^d)$. Clearly, then $f \in \mathcal{M}_+(\mathbb{R}_*^d)$ and also $f(\{x \in \mathbb{R}_*^d : |x| < 1\}) = 0$.

It only remains to check that we can take the limit $n \rightarrow \infty$ in (5.19) with $\varepsilon = \varepsilon_n$ for any test function $\varphi \in C_c(\mathbb{R}_*^d)$. We can readily take the limit $n \rightarrow \infty$, using (5.20), in the term $\int_{\mathbb{R}_*^d} \int_{\mathbb{R}_*^d} K_{\varepsilon_n}(x, y) \varphi(x + y) f_{\varepsilon_n}(dx) f_{\varepsilon_n}(dy)$ because φ is compactly supported and hence the integration may be restricted to a compact subset of $\mathbb{R}_*^d \times \mathbb{R}_*^d$. It only remains to examine the terms of (5.19) which contain the functions $\varphi(x), \varphi(y)$. These contributions are identical, as it can be seen exchanging the variables x and y . Therefore, it is enough to consider only one of them, say $\int_{\mathbb{R}_*^d} \int_{\mathbb{R}_*^d} K_{\varepsilon_n}(x, y) \varphi(x) f_{\varepsilon_n}(dx) f_{\varepsilon_n}(dy)$. Since φ is compactly supported we only need to obtain uniform boundedness of $\int_{\mathbb{R}_*^d} K_{\varepsilon_n}(x, y) f_{\varepsilon_n}(dy)$ for x bounded and for $|y|$ large. Due to (5.12), (5.20), as well as Lemma 4.2, it is enough to estimate

$$\int_1^\infty \frac{\min\{\rho^\gamma, \frac{1}{\varepsilon}\} + \varepsilon}{\rho^{\frac{3}{2}} (\varepsilon + \min\{\rho^\gamma, \frac{1}{\varepsilon}\})^{\frac{1}{2}}} d\rho.$$

It is readily seen that this integral can be bounded (up to a multiplicative constant) by the sum

$$\int_1^\infty \frac{\min\{\rho^\gamma, \frac{1}{\varepsilon}\}}{\rho^{\frac{3}{2}} (\min\{\rho^\gamma, \frac{1}{\varepsilon}\})^{\frac{1}{2}}} d\rho + \int_1^\infty \frac{\varepsilon}{\rho^{\frac{3}{2}} (\varepsilon)^{\frac{1}{2}}} d\rho. \tag{5.22}$$

The second integral in (5.22) is given by $2(\varepsilon)^{\frac{1}{2}}$ and thus it tends to zero as $\varepsilon \rightarrow 0$. On the other hand, the first integral in (5.22) can be bounded as

$$\int_1^\infty \frac{(\min\{\rho^\gamma, \frac{1}{\varepsilon}\})^{\frac{1}{2}} d\rho}{\rho^{\frac{3}{2}}} \leq \int_1^\infty \frac{d\rho}{\rho^{\frac{3-\gamma}{2}}} < \infty$$

since $\gamma < 1$. This gives the desired uniform estimate. We can then take the limit $n \rightarrow \infty$ in (5.19).

It remains to check that also $\int_{\mathbb{R}_*^d} |x|^\gamma f(dx) < \infty$ which is obvious from the construction, if $\gamma \leq 0$. If $0 < \gamma < 1$, this follows by first taking $\varepsilon \rightarrow 0$ in (5.20) and then using Lemma 4.2. Therefore, f is a solution to (2.2) in the sense of Definition 2.1. This concludes the proof of Theorem 3.1. \square

5.2 Discrete Coagulation Equation

We now prove Theorem 3.2. Given that the proof is very similar to the one for the continuous case (cf. Sect. 5.1) we just sketch the main ideas.

Proof of Theorem 3.2 Due to item (ii) of Lemma 4.1 it is enough to prove the result for $p = 0$, $\gamma < 1$. By assumption the kernel K can be written as in (5.10) with Φ satisfying (5.11) for all $x, y \in \mathbb{N}_*^d$. We truncate the kernel K as in the proof of Theorem 3.1 (cf. (5.12), (5.13) and (5.14)) with ζ_M chosen as explained in the paragraph after (5.14). Then ζ_M satisfies Assumption 5.1.

We then consider the following time dependent truncated problem, where $M > 2L$,

$$\partial_t n_\alpha = \frac{\zeta_M(\alpha)}{2} \sum_{\beta < \alpha} K_{\varepsilon, M}(\alpha - \beta, \beta) n_{\alpha - \beta} n_\beta - n_\alpha \sum_{\beta > 0} K_{\varepsilon, M}(\alpha, \beta) n_\beta + s_\alpha, \quad \alpha \in \mathbb{N}_*^d. \tag{5.23}$$

We can construct solutions of (5.23) satisfying $n_\alpha = 0$ for $|\alpha| \geq 2M$ for any nonnegative initial distribution $n_{0, \alpha}$ satisfying the same property. Local existence follows from classical ODE theory. Global existence follows from the estimate

$$\partial_t \left(\sum_\alpha n_\alpha \right) \leq -\frac{\varepsilon}{2} \left(\sum_\alpha n_\alpha \right)^2 + \sum_\alpha s_\alpha \tag{5.24}$$

that implies

$$\sum_\alpha n_\alpha \leq \sum_\alpha n_{0, \alpha} + t \sum_\alpha s_\alpha.$$

Due to (5.24) there exists an invariant convex set defined by means of

$$\left\{ n_\alpha : \sum_\alpha n_\alpha \leq \sqrt{\frac{2 \sum_\alpha s_\alpha}{\varepsilon}}, n_\alpha \geq 0 \right\}.$$

Therefore, the existence of a stationary solution is a consequence of Schauder’s Fixed Point Theorem, arguing as in the proof of Proposition 5.5. This solution will be denoted as $\{n_\alpha^{\varepsilon, M}\}_{\alpha \in \mathbb{N}_0^d \setminus \{0\}}$.

We can now take the limit $M \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ in order to obtain a stationary injection solution to (2.26). To this end, we derive uniform estimates for the sequence $\{n_\alpha^{\varepsilon, M}\}_{\alpha \in \mathbb{N}_*^d}$.

More precisely, we have already proved the estimate $\sum_\alpha n_\alpha^{\varepsilon, M} \leq \sqrt{\frac{2 \sum_\alpha s_\alpha}{\varepsilon}}$. Since the right-hand side is independent of M , there exists a sequence $\{M_n\}_{n \in \mathbb{N}}$ such that $M_n \rightarrow \infty$ as $n \rightarrow \infty$ and $n_\alpha^{\varepsilon, M_n} \rightarrow n_\alpha^\varepsilon$ as $n \rightarrow \infty$, where the sequence $\{n_\alpha^\varepsilon\}_{\alpha \in \mathbb{N}_*^d}$ solves

$$\frac{1}{2} \sum_{\beta < \alpha} K_\varepsilon(\alpha - \beta, \beta) n_{\alpha - \beta}^\varepsilon n_\beta^\varepsilon - n_\alpha^\varepsilon \sum_{\beta > 0} K_\varepsilon(\alpha, \beta) n_\beta^\varepsilon + s_\alpha = 0, \quad \alpha \in \mathbb{N}_*^d. \tag{5.25}$$

In order to estimate the sequence $\{n_\alpha^\varepsilon\}_{\alpha \in \mathbb{N}_*^d}$ we use the fact that the measure $f_\varepsilon = \sum_\alpha n_\alpha^\varepsilon \delta(\cdot - \alpha)$ solves a stationary continuous equation. More precisely, f_ε satisfies (5.19) for any test function $\varphi \in C_c(\mathbb{R}_*^d)$, where K_ε is any continuous extension of the discrete kernel. We can then derive, arguing as in the proof of Theorem 3.1 that f_ε satisfies the estimate (5.20) with $f_\varepsilon(x) = F_\varepsilon(r, \theta)$. We can then show that there exists a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that $n_\alpha^{\varepsilon_n} \rightarrow n_\alpha$ as $n \rightarrow \infty$ for each $\alpha \in \mathbb{N}_*^d$. Moreover, using (5.20) we can also pass to the limit in the weak form of (5.25) to show that $\{n_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ is a stationary injection solution to (2.26). Hence the Theorem follows. \square

6 Nonexistence Results

In this Section we prove the non-existence of stationary injection solutions for the continuous and discrete model as well as the non-existence of constant flux solutions.

We first prove Theorem 3.5. Due to Lemma 4.1, item (i), it is enough to prove the result for $\gamma \geq 1, p = 0$. We first recall the following auxiliary Lemma that is a particular case of Lemma 4.1 in [6] with $a \geq 1, b = 0$.

Lemma 6.1 *Let $a \geq 1$ be a constant. Let $W : \mathbb{R}_* \rightarrow \mathbb{R}$ be a right-continuous non-increasing function satisfying $W(R) \geq 0$, for all $R > 0$. Assume that $h \in \mathcal{M}_+(\mathbb{R}_*)$ satisfies $h([1, \infty)) > 0$ and*

$$\int_{[1, \infty)} x^a h(dx) < \infty. \tag{6.1}$$

Suppose that there exists δ such that $0 < \delta < 1$ and the following inequality holds

$$- \int_{[1, \delta R]} [W(R - y) - W(R)] h(dy) \leq -\frac{C}{R^{a+1}}, \quad \text{for } R \geq R_0, \tag{6.2}$$

for some $R_0 > 1/\delta$ and $C > 0$.

Then there are two constants $R'_0 \geq R_0$ and $B > 0$ which depend only on a, h, δ, R_0 , and C , such that

$$W(R) \geq \frac{B}{R^a}, \quad \text{for } R \geq R'_0, \tag{6.3}$$

The proof of Lemma 6.1 relies on a comparison argument and on the construction of a suitable subsolution for the problem (6.2).

Proof of Theorem 3.5 We start proving item (i) which refers to the continuous case. To this end we follow the same strategy as in the proof of Theorem 2.4 in [6] for the one-component case. To get a contradiction, let us assume f is a stationary injection solution. Let F be the corresponding measure in the simplex coordinates, as explained after Definition 2.1. In particular, then the support of F lies in $[1, \infty) \times \Delta^{d-1}$ and F satisfies $\int_{\mathbb{R}_* \times \Delta^{d-1}} r^{d-1+\gamma} F(r, \theta) dr d\tau(\theta) < \infty$. We define G as before from K . Then, after rewriting (2.25) in the simplex coordinate system, we find that

$$\begin{aligned} & \frac{1}{2d} \int_{\mathbb{R}_*} r^{d-1} dr \int_{\mathbb{R}_*} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G(r, \rho; \theta, \sigma) F(r, \theta) F(\rho, \sigma) \\ & \times \left[\psi \left(r + \rho, \frac{r}{r + \rho} \theta + \frac{\rho}{r + \rho} \sigma \right) - \psi(r, \theta) - \psi(\rho, \sigma) \right] + \int_{\mathbb{R}_*^d} \varphi(x) \eta(dx) = 0 \end{aligned} \tag{6.4}$$

for every $\psi \in C^1_c(\mathbb{R}_* \times \Delta^{d-1})$.

We now choose test functions of the form $\psi(r, \theta) = r \chi_{R,\delta}(r)$ to derive a formula for the fluxes. Let $R > \delta > 0$. We assume that $\chi_{R,\delta} \in C^\infty_c(\mathbb{R}_*)$ is a ‘‘bump function’’, more precisely, it is monotone increasing on $(0, R]$ monotone decreasing on $[R, \infty)$. We also assume that $\chi_{R,\delta}(s) = 1$ for $\delta \leq s \leq R$, $\chi_{R,\delta}(s) = 0$ for $s \geq R + \delta$. We then have

$$\begin{aligned} & \psi \left(r + \rho, \frac{r}{r + \rho} \theta + \frac{\rho}{r + \rho} \sigma \right) - \psi(r, \theta) - \psi(\rho, \sigma) \\ & = (r + \rho) \chi_{R,\delta}(r + \rho) - r \chi_{R,\delta}(r) - \rho \chi_{R,\delta}(\rho) \\ & = -r [\chi_{R,\delta}(r) - \chi_{R,\delta}(r + \rho)] - \rho [\chi_{R,\delta}(\rho) - \chi_{R,\delta}(r + \rho)] . \end{aligned}$$

Plugging this identity in (6.4) and assuming that $R > L$ we obtain

$$\begin{aligned} |J|d & = \int_{\mathbb{R}_*} r^{d-1} dr \int_{\mathbb{R}_*} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G(r, \rho; \theta, \sigma) F(r, \theta) F(\rho, \sigma) \\ & \times r [\chi_{R,\delta}(r) - \chi_{R,\delta}(r + \rho)] , \end{aligned} \tag{6.5}$$

where $J = \int_{\mathbb{R}_*^d} x \eta(dx) \in \mathbb{R}_*^d$. Taking the limit $\delta \rightarrow 0$ we may then conclude that for all $R > L$

$$\begin{aligned} & \int_{(0,R]} r^{d-1} dr \int_{(R-r,\infty)} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau(\theta) \\ & \int_{\Delta^{d-1}} d\tau(\sigma) G(r, \rho, \theta, \sigma) F(r, \theta) F(\rho, \sigma) r = |J|d . \end{aligned} \tag{6.6}$$

By the earlier mentioned known properties of F ,

$$\int_{[1,\infty)} \rho^{d-1+\gamma} d\rho \int_{\Delta^{d-1}} d\tau(\sigma) F(\rho, \sigma) < \infty . \tag{6.7}$$

We estimate first the contribution to the integral in (6.6) due to the region $\{\rho \geq \delta r\}$ where $0 < \delta < \frac{1}{2}$. Using the upper estimate in (1.8) (see also (4.3)) we obtain

$$\begin{aligned} & \int_{\{(r,\rho):r \in (0,R], r+\rho > R, \rho \geq \delta r\}} r^{d-1} \rho^{d-1} dr d\rho \int_{\Delta^{d-1}} d\tau(\theta) \\ & \int_{\Delta^{d-1}} d\tau(\sigma) G(r, \rho; \theta, \sigma) F(r, \theta) F(\rho, \sigma) r \end{aligned}$$

$$\begin{aligned}
 &\leq C_\delta \int_{[1,R]} r^{d-1+1} dr \int_{\left[\frac{\delta R}{1+\delta}, \infty\right)} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) \rho^\gamma F(r, \theta) F(\rho, \sigma) \\
 &\leq C_\delta \left(\int_{[1,R]} r^{d-1+\gamma} dr \int_{\Delta^{d-1}} F(r, \theta) d\tau(\theta) \right) \int_{\left[\frac{\delta R}{1+\delta}, \infty\right)} \rho^{d-1+\gamma} d\rho \int_{\Delta^{d-1}} F(\rho, \sigma) d\tau(\sigma)
 \end{aligned}
 \tag{6.8}$$

where $\gamma \geq 1, p = 0$. It then follows from (6.7) that the right-hand side of (6.8) tends to zero as $R \rightarrow \infty$.

Therefore, we can now conclude from (6.6) that for every $\delta \in (0, 1)$ there is $R_\delta > L, \delta^{-1}$ such that, if $R \geq R_\delta$, then

$$\begin{aligned}
 &\int_{\{(r,\rho): r \in (0,R], r+\rho > R, \rho < \delta r\}} r^d \rho^{d-1} dr d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G(r, \rho; \theta, \sigma) F(r, \theta) F(\rho, \sigma) \\
 &\geq \frac{|J|d}{2}.
 \end{aligned}$$

Using again the upper estimate in (1.8) (or (4.3)) as well as the fact that in the domain of integration r is close to R we obtain

$$\int_{[1,\delta R]} d\rho \int_{(R-\rho,R]} dr \int_{\Delta^{d-1}} F(\rho, \sigma) \rho^{d-1} d\tau(\sigma) \int_{\Delta^{d-1}} r^{d-1} F(r, \theta) d\tau(\theta) \geq \frac{C_1}{R^{\gamma+1}}$$

where $C_1 > 0$ depends on J but is independent of R and δ . We then consider the measure

$$h(r) = r^{d-1} \int_{\Delta^{d-1}} F(r, \theta) d\tau(\theta),$$

which belongs to $\mathcal{M}_+(\mathbb{R}_*)$, by Fubini’s theorem. We can conclude that for all $R \geq R_\delta$

$$\int_{[1,\delta R]} h(\rho) d\rho \left[\int_{(R-\rho,R]} h(r) dr \right] \geq \frac{C_1}{R^{\gamma+1}}.$$

The support of h lies in $[1, \infty)$ and $h \neq 0$. Since $\int_{\mathbb{R}_* \times \Delta^{d-1}} r^{d-1+\gamma} F(r, \theta) dr d\tau(\theta) < \infty$, we have $\int_{\mathbb{R}_*} r^\gamma h(r) dr < \infty$. Here $\gamma \geq 1$, and we may conclude that also $\int_{\mathbb{R}_*} h(r) dr < \infty$. Therefore, we can define a right-continuous, non-negative and non-increasing function W by

$$W(R) = \int_{(R,\infty)} h(r) dr, \quad R > 0,$$

and rewrite the earlier bound as

$$\int_{[1,\delta R]} h(\rho) [W(R - \rho) - W(R)] d\rho \geq \frac{C_1}{R^{\gamma+1}}$$

for $R \geq R_\delta$. Applying Lemma 6.1, we obtain

$$W(R) \geq \frac{B}{R^\gamma}$$

for all R large enough and with a constant $B > 0$. Then, for any R sufficiently large we have

$$\int_{[R,\infty)} \rho^\gamma h(\rho) d\rho \geq R^\gamma W(R) \geq B > 0,$$

but this contradicts (6.7) and the result follows. This concludes the proof of item (i).

We now prove item (ii) concerning the non-existence of stationary injection solutions for the discrete model. We observe that the result is a direct corollary of item (i). Namely, if $\{n_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ solves (2.26), then $f(\cdot) = \sum_\alpha n_\alpha \delta(\cdot - \alpha)$ and $\eta(\cdot) = \sum_\alpha s_\alpha \delta(\cdot - \alpha)$ solves (2.2) which leads to a contradiction. \square

Finally we show the non-existence of nontrivial constant flux solutions stated in Theorem 3.6.

Proof of Theorem 3.6 The proof is similar to that of Theorem 3.5, except that here we do not even need to assume that f solves (2.25).

We begin with the assumption that f is a nontrivial constant flux solution, and hence A defined by (2.30) satisfies $A_j(R) = (J_0)_j$, for all $R > 0$. Summing over j in (2.30), we find that for all $R > 0$

$$|J_0| = \frac{1}{d} \int_{(0,R] \times \Delta^{d-1}} dr d\tau(\theta) \int_{(R-r,\infty) \times \Delta^{d-1}} d\rho d\tau(\sigma) r^d \rho^{d-1} F(r, \theta) F(\rho, \sigma) G(r, \rho; \theta, \sigma). \tag{6.9}$$

Since we assume that the measure f is nontrivial, here $|J_0| > 0$. Therefore, the previous equality in (6.6) is satisfied (the value of $L > 0$ can now be chosen arbitrarily), and we can follow the remaining steps in the proof of Theorem 3.5 to obtain a contradiction using the moment bound (2.24) and the assumption $\gamma \geq 1$.

7 Upper and Lower Estimates for the Solutions

Until now, we have focused on the question of existence of stationary solutions in the three cases of interest. We now consider those kernels which can have a stationary solution. Although no uniqueness of these solutions is claimed here, we prove in this subsection that all such solutions satisfy certain powerlaw growth bounds. We collect the statement for the continuous stationary injection solutions in Proposition 7.1 and the corresponding results for the discrete coagulation equation in Proposition 7.4. These results generalize similar upper and lower estimates proven in [6] to the multicomponent case and the larger class of kernels.

Proposition 7.1 *Let us assume that K is a continuous symmetric function satisfying (1.8), (1.9) with $\gamma + 2p < 1$. Assume also that $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*^d)$ has its support in the set $\{x \in \mathbb{R}_*^d \mid 1 \leq |x| \leq L\}$ for some $L > 1$. We denote by $|J_0|$ the total mass injection rate, where $J_0 = \int_{\mathbb{R}_*^d} x \eta(dx) \in \mathbb{R}_*^d$. Let $f \in \mathcal{M}_+(\mathbb{R}_*^d)$ be a stationary injection solution to (2.2) in the sense of Definition 2.1. There exist constants $C_1, C_2 > 0$ and $b \in (0, 1)$ depending only on γ, p, d and the constants c_1, c_2 in (1.8) such that the following estimates hold with $\xi = \frac{L}{b}$*

$$\frac{1}{z} \int_{\frac{z}{2} \leq |x| \leq z} f(dx) \leq \frac{C_1 \sqrt{|J_0|}}{z^{\frac{3+\gamma}{2}}} \text{ for all } z > 0, \tag{7.1}$$

$$\frac{1}{z} \int_{bz \leq |x| \leq z} f(dx) \geq \frac{C_2 \sqrt{|J_0|}}{z^{\frac{3+\gamma}{2}}} \text{ for all } z > \xi. \tag{7.2}$$

Alternatively, let us assume that f is a nontrivial constant flux solution (cf. Definition 2.3) with flux J_0 . Then there exist constants $C_1, C_2 > 0$ and $b \in (0, 1)$ depending only on γ, p, d and the constants c_1, c_2 in (1.8) such that (7.1) and (7.2) hold with $\xi = 0$.

Remark 7.2 We observe that $|J_0|$ is the total injection rate which means that it includes all of the possible monomer types.

Proof Due to Lemma 4.1 as well as the fact that the estimates (7.1), (7.2) are invariant under the transformation $(f(x), \gamma, p) \rightarrow \left(\frac{f(x)}{|x|^p}, \gamma + 2p, 0\right)$, it is sufficient to prove the Proposition for $p = 0$ and $\gamma < 1$.

Suppose first that f is an appropriate stationary injection solution, in particular, it satisfies (2.25). We define F by means of (2.16) and G by means of (2.15). Then, arguing as in the derivation of (5.17) in the proof of Theorem 3.1 (i.e. using the test function $\varphi(x) = |x|\chi_\delta(|x|)$ and taking the limit $\delta \rightarrow 0$), we obtain

$$\begin{aligned} & \frac{1}{d} \int_{(0,z]} r^d dr \int_{(z-r,\infty)} \rho^{d-1} d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G(r, \rho; \theta, \sigma) F(r, \theta) F(\rho, \sigma) \\ &= \int_{\{0 < |x| \leq z\}} |x| \eta(dx), \quad \text{for any } z > 0. \end{aligned} \tag{7.3}$$

The (r, ρ) -integration can also be rearranged using Fubini’s theorem to occur over the set

$$\Omega_z := \{(r, \rho) \in \mathbb{R}_+^2 : 0 < r \leq z, \rho > z - r\}, \quad z > 0. \tag{7.4}$$

In particular, we find for z larger than the support of the source that

$$\frac{1}{d} \int_{\Omega_z} dr d\rho r^d \rho^{d-1} \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G(r, \rho; \theta, \sigma) F(r, \theta) F(\rho, \sigma) = |J_0|, \quad z \geq L. \tag{7.5}$$

In the second case, if f is a nontrivial constant flux solution as in Definition 2.3, by summing over j in (2.30) we find that (7.5) holds for any choice of $L > 0$. In particular, we may now conclude that, for all $z > 0$ and for either of two cases of a solution f , we have an upper bound

$$\frac{1}{d} \int_{\Omega_z} dr d\rho r^d \rho^{d-1} \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) G(r, \rho; \theta, \sigma) F(r, \theta) F(\rho, \sigma) \leq |J_0|. \tag{7.6}$$

First, let us recall that, since $p = 0$, we have

$$c_1 (r + \rho)^\gamma \leq G(r, \rho, \theta, \sigma) \leq c_2 (r + \rho)^\gamma. \tag{7.7}$$

The integration goes over Ω_z which contains the set $[2z/3, z]^2$. By positivity of the integrand and using the lower bound in (7.7), we thus obtain an estimate

$$\frac{2}{3d} c_1 z^{\gamma+1} \left(\int_{[2z/3,z]} dr r^{d-1} \int_{\Delta^{d-1}} d\tau(\theta) F(r, \theta) \right)^2 \leq |J_0|, \quad \text{for any } z > 0. \tag{7.8}$$

Since $[z/2, z] = [z/2, 3z/4] \cup [2z/3, z]$, we find that (7.1) holds with a choice of $C_1 > 0$ which depends only on c_1, γ , and p . This concludes the proof of the upper estimate, for both of the types of solutions f .

For the lower bound, we recall that (7.5) holds for either of the types of solutions f , using an arbitrary $L > 0$ if f is a flux solution. We now prove that the main contribution to the integral in (7.5) is due to the regions where r and ρ are comparable. To this end, for each $\delta \in (0, 1)$ we partition $(0, \infty)^2 = \Sigma_1(\delta) \cup \Sigma_2(\delta) \cup \Sigma_3(\delta)$ where

$$\Sigma_1(\delta) = \{(r, \rho) \in (0, \infty)^2 : \rho > r/\delta\},$$

$$\begin{aligned} \Sigma_2(\delta) &= \{(r, \rho) \in (0, \infty)^2 : \delta r \leq \rho \leq r/\delta\}, \\ \Sigma_3(\delta) &= \{(r, \rho) \in (0, \infty)^2 : \rho < \delta r\}, \end{aligned} \tag{7.9}$$

We define also the related functions

$$J_j(z, \delta) = \frac{1}{d} \int_{\Omega_z \cap \Sigma_j(\delta)} dr d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) r^d \rho^{d-1} G(r, \rho; \theta, \sigma) F(r, \theta) F(\rho, \sigma) \tag{7.10}$$

for $z > 0$, $\delta \in (0, 1)$ and $j = 1, 2, 3$. We now claim that for each $\varepsilon > 0$ there exists δ_ε depending only on ε , as well as on γ, p, d and the constants c_1, c_2 , such that, if $\delta < \delta_\varepsilon$ we have

$$\sup_{z>0} J_1(z, \delta) \leq \varepsilon |J_0| \tag{7.11}$$

and

$$\frac{1}{R} \int_{[R, 2R]} J_3(z, \delta) dz \leq \varepsilon |J_0| \text{ for each } R > 0. \tag{7.12}$$

We prove first (7.11). By (7.7), if $\delta < 1$ and $(r, \rho) \in \Sigma_1(\delta)$, we have $G(r, \rho; \theta, \sigma) \leq 2^{|\gamma|} c_2 \rho^\gamma$, and, therefore,

$$J_1(z, \delta) \leq \frac{1}{d} 2^{|\gamma|} c_2 \int_{\Omega_z \cap \Sigma_1(\delta)} dr d\rho \int_{\Delta^{d-1}} d\tau(\theta) \int_{\Delta^{d-1}} d\tau(\sigma) r^d \rho^{d-1+\gamma} F(r, \theta) F(\rho, \sigma).$$

We have already proven a powerlaw upper bound in (7.1), and this allows further bounding similar integrals via Lemma 4.2. In particular, we may conclude that for any $q \in \mathbb{R}$ there is a constant C , which depends only on q, γ, d , and C_1 , such that if $a > 0$ and $R \geq 2a$, we have

$$\int_{[a, R]} dr \int_{\Delta^{d-1}} d\tau(\theta) r^{d-1+q} F(r, \theta) \leq C \sqrt{|J_0|} \int_a^R r^{q-\frac{\gamma+3}{2}} dr. \tag{7.13}$$

Now $\Omega_z \cap \Sigma_1(\delta)$ is contained in $\{(r, \rho) : 0 < r \leq z, \rho \geq \frac{r}{\delta}, \frac{z}{2}\}$, and we first use the above powerlaw bound to estimate the ρ -integral. Setting $q = \gamma$ and taking $R \rightarrow \infty$, we find that for any $a > 0$

$$\int_{[a, \infty)} d\rho \int_{\Delta^{d-1}} d\tau(\sigma) \rho^{d-1+\gamma} F(\rho, \sigma) \leq C \sqrt{|J_0|} \int_a^\infty \rho^{\gamma-\frac{\gamma+3}{2}} d\rho = \frac{2C}{1-\gamma} \sqrt{|J_0|} a^{-\frac{1-\gamma}{2}}.$$

We employ the estimate with $a = \max(\frac{r}{\delta}, \frac{z}{2})$, and conclude that

$$J_1(z, \delta) \leq C \sqrt{|J_0|} \int_{(0, z]} dr \int_{\Delta^{d-1}} d\tau(\theta) r^d \max\left(\frac{r}{\delta}, \frac{z}{2}\right)^{-\frac{1-\gamma}{2}} F(r, \theta).$$

where the constant C has been adjusted. We can then apply the first item of Lemma 4.2 to the interval $[a', z]$ with a weight $\varphi(r) = r \min(\frac{\delta}{r}, \frac{z}{2})^{\frac{1-\gamma}{2}}$ and a bound function $g(r) = C_1 \sqrt{d|J_0|} r^{-\frac{1+\gamma}{2}} \min(\frac{2\delta}{r}, \frac{z}{2})^{\frac{1-\gamma}{2}}$. Taking then $a' \rightarrow 0$ shows that there is a constant \bar{C}_2 which depends only on variables allowed in the Proposition and for which

$$J_1(z, \delta) \leq \bar{C}_2 |J_0| \left(\int_0^z \frac{dr}{r^{\frac{1+\gamma}{2}}} \min \left\{ \frac{1}{z^{\frac{1-\gamma}{2}}}, \frac{\delta^{\frac{1-\gamma}{2}}}{r^{\frac{1-\gamma}{2}}} \right\} + \delta^{\frac{1-\gamma}{2}} \right). \tag{7.14}$$

We use the change of variables $r = z\xi$ in the remaining integral which thus becomes equal to $\int_0^1 d\xi \xi^{-\frac{1+\gamma}{2}} \min\left(1, \delta^{\frac{1-\gamma}{2}} \xi^{-\frac{1-\gamma}{2}}\right)$. The integral is independent of z , well defined for any $\delta > 0$, and it converges to zero as $\delta \rightarrow 0$ due to Lebesgue’s dominated convergence theorem. Thus (7.14) implies that (7.11) holds for all small enough δ .

We next prove (7.12) and now assume that $0 < \delta \leq \frac{1}{4}$ and choose an arbitrary $R > 0$. Integrating $J_3(z, \delta)$ over $z \in [R, 2R]$ and using that $\Omega_z \cap \Sigma_3 \subset \{(r, \rho) : 0 < \rho \leq \delta z, z - \rho \leq r \leq z\}$ as well as (7.7), we obtain

$$\begin{aligned} & \int_{[R, 2R]} J_3(z, \delta) dz \\ & \leq C_3 \int_{[R, 2R]} dz \int_{(0, \delta z]} d\rho \int_{[z-\rho, z]} r^{\gamma+1} dr \int_{\Delta^{d-1}} r^{d-1} F(r, \theta) d\tau(\theta) \int_{\Delta^{d-1}} \rho^{d-1} F(\rho, \sigma) d\tau(\sigma) \\ & \leq \tilde{C}_3 R^{\gamma+1} \int_{[R, 2R]} dz \int_{(0, 2\delta R]} d\rho \int_{[z-\rho, z]} dr \int_{\Delta^{d-1}} r^{d-1} F(r, \theta) d\tau(\theta) \int_{\Delta^{d-1}} \rho^{d-1} F(\rho, \sigma) d\tau(\sigma) \end{aligned}$$

where we use that $(0, \delta z] \subset (0, 2\delta R]$. Here $\{(r, z) : z - \rho \leq r \leq z, R \leq z \leq 2R\} \subset \{(r, z) : R/2 \leq r \leq 2R, r \leq z \leq r + \rho\}$ if $0 < \rho \leq R/2$, and thus employing Fubini’s theorem we obtain

$$\begin{aligned} & \int_{[R, 2R]} J_3(z, \delta) dz \\ & \leq \tilde{C}_3 R^{\gamma+1} \int_{(0, 2\delta R]} d\rho \int_{[\frac{R}{2}, R]} dr \int_{[r, r+\rho]} dz \int_{\Delta^{d-1}} r^{d-1} F(r, \theta) d\tau(\theta) \int_{\Delta^{d-1}} \rho^{d-1} F(\rho, \sigma) d\tau(\sigma) \\ & = \tilde{C}_3 R^{\gamma+1} \int_{(0, 2\delta R]} \rho d\rho \int_{[\frac{R}{2}, R]} dr \int_{\Delta^{d-1}} r^{d-1} F(r, \theta) d\tau(\theta) \int_{\Delta^{d-1}} \rho^{d-1} F(\rho, \sigma) d\tau(\sigma). \end{aligned}$$

Using here the estimate (7.13) to bound the remaining r - and ρ -integrals separately, we find that

$$\int_{[R, 2R]} J_3(z, \delta) dz \leq C |J_0| R^{\gamma+1} \int_{(0, 2\delta R]} \frac{\rho d\rho}{\rho^{\frac{3+\gamma}{2}}} \int_{[\frac{R}{2}, R]} \frac{dr}{r^{\frac{3+\gamma}{2}}} \leq C_4 R |J_0| \delta^{\frac{1-\gamma}{2}}.$$

This implies (7.12) for all sufficiently small δ .

In particular, we can now conclude that for any $\varepsilon > 0$ there is a $\delta_\varepsilon \in (0, \frac{1}{4}]$ such that for all $0 < \delta < \delta_\varepsilon$ both (7.11) and (7.12) hold. In order to prove (7.2), we first note that (7.5) and (7.10) imply (using an arbitrary $L > 0$ if f is a flux solution)

$$|J_0| = \sum_{j=1}^3 J_j(z, \delta) \text{ for all } z \geq L. \tag{7.15}$$

We fix ε so that $0 < \varepsilon \leq \frac{1}{4}$ and choose some $\delta \in (0, \delta_\varepsilon)$. We then integrate (7.15) over $z \in [R, 2R]$ with $R \geq L$, and use (7.11) and (7.12) to conclude that

$$\frac{|J_0|R}{2} \leq |J_0|R(1 - 2\varepsilon) \leq \int_{[R, 2R]} J_2(z, \delta) dz.$$

A simple geometrical argument shows that there exists a constant $b, 0 < b < 1$, depending only on the choice of δ such that $\bigcup_{z \in [R, 2R]} (\Omega_z \cap \Sigma_2(\delta)) \subset (\sqrt{b}R, R/\sqrt{b}]^2$. (For a fixed z and $(r, \rho) \in \Omega_z \cap \Sigma_2(\delta)$ one finds $(\delta^{-1} + 1)^{-1}z < r, \rho \leq \delta^{-1}z$; thus for example $b = \frac{\delta^2}{4}$ would suffice.) Moreover, we then have $G(r, \rho; \theta, \sigma) \leq C_5 R^\gamma$ for all $(r, \rho) \in (\sqrt{b}R, R/\sqrt{b}]^2$

where C_5 depends only on b, γ and c_2 . Thus using the definition of $J_2(z, \delta)$ given in (7.10) we obtain

$$\frac{|J_0|R}{2} \leq C'R^{\gamma+2} \left(\int_{\{\sqrt{b}R \leq |x| \leq R/\sqrt{b}\}} f(dx) \right)^2 \text{ for } R \geq L.$$

Therefore,

$$\frac{1}{R} \int_{\{\sqrt{b}R \leq |x| \leq R/\sqrt{b}\}} f(dx) \geq \frac{\tilde{C}_1 \sqrt{|J_0|}}{R^{\frac{3+\gamma}{2}}} \text{ for } R \geq L,$$

and (7.2) follows, after replacing R/\sqrt{b} by z (note that if $z > L/b$, then $\sqrt{bz} > L/\sqrt{b} > L$). If f is a constant flux solution, then the result holds for any $L > 0$, and thus we may also set $\xi = 0$ in (7.2). □

Remark 7.3 Notice that for $\gamma > -1$, combining Proposition 7.1 with Lemma 4.2 we obtain that the number of clusters associated to the stationary injection solutions $\int_{\mathbb{R}_*^d} f(dx)$ is finite. Moreover, the following estimates hold:

$$\frac{C'_1 \sqrt{|J_0|}}{R^{(\gamma+1)/2}} \leq \int_{\{|x| \geq R\}} f(dx) \leq \frac{C'_2 \sqrt{|J_0|}}{R^{(\gamma+1)/2}}, \text{ for } R > L,$$

where $|J_0| = \int_{\mathbb{R}_*^d} |x|\eta(dx)$ and $0 < C'_1 \leq C'_2$. (The lower bound follows by setting $z = R/b > L/b$ in the Proposition.)

We can now conclude a result similar to Proposition 7.1 for the solutions of the discrete problem (2.26). Although a fairly direct corollary of the previous result, it is worth recording separately in detail, due to the relevance of discrete coagulation equation in applications.

Proposition 7.4 *Suppose that K is symmetric and satisfies (1.7), (1.9) with $\gamma + 2p < 1$. Let $\{s_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ be a nonnegative sequence supported on a finite number of values α , and suppose $L_s > 1$ is such that $s_\alpha = 0$ if $|\alpha| > L_s$. Suppose that $\{n_\alpha\}_{\alpha \in \mathbb{N}_*^d}$ is a stationary injection solution to (2.26) in the sense of Definition 2.2. We denote as $|J_0|$ the total injection rate of monomers given by $|J_0| = \sum_\alpha |\alpha| s_\alpha$. Then, there exist positive constants C_1, C_2 and $b \in (0, 1)$ depending only on γ, p, d and the constants c_1, c_2 in (1.7) such that*

$$\frac{1}{z} \sum_{\frac{z}{2} \leq |\alpha| \leq z} n_\alpha \leq \frac{C_1 \sqrt{|J_0|}}{z^{\frac{3+\gamma}{2}}} \text{ for all } z \geq 1, \tag{7.16}$$

$$\frac{1}{z} \sum_{bz \leq |\alpha| \leq z} n_\alpha \geq \frac{C_2 \sqrt{|J_0|}}{z^{\frac{3+\gamma}{2}}} \text{ for all } z \geq \frac{L_s}{b}. \tag{7.17}$$

Proof This result is just a Corollary of Proposition 7.1 since $f(\cdot) = \sum_\alpha n_\alpha \delta(\cdot - \alpha)$ and $\eta(\cdot) = \sum_\alpha s_\alpha \delta(\cdot - \alpha)$ satisfy all the assumptions in Proposition 7.1, and (7.1), (7.2) imply (7.16), (7.17) respectively. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

- Banasiak, J., Lamb, W., Laurençot, P.: Analytic Methods for Coagulation-Fragmentation Models. CRC Press, Boca Raton (2019)
- Degond, P., Liu, J., Pego, R.L.: Coagulation-fragmentation model for animal group-size statistics. *J. Nonlinear Sci.* **27**, 379–424 (2017)
- Dubovski, P.B.: Mathematical Theory of Coagulation. Lecture Notes Series, vol. 23. Seoul National University, Seoul (1994)
- Escobedo, M., Mischler, S.: Dust and self-similarity for the Smoluchowski coagulation equation. *Ann. I. H. Poincaré Anal. Non Linéaire* **23**(3), 331–362 (2006)
- Escobedo, M., Mischler, S., Rodriguez Ricard, M.: On self-similarity and stationary problem for fragmentation and coagulation models. *Ann. Inst. Henri Poincaré (C) Anal. Non Linéaire* **22**(1), 99–125 (2005)
- Ferreira, M.A., Lukkarinen, J., Nota, A., Velázquez, J.J.L.: Stationary non-equilibrium solutions for coagulation systems. *Arch. Ration. Mech. Anal.* **240**, 809–875 (2021)
- Ferreira, M.A., Lukkarinen, J., Nota, A., Velázquez, J.J.L.: Localization in stationary non-equilibrium solutions for multicomponent coagulation systems. *Commun. Math. Phys.* **388**, 479–506 (2021)
- Ferreira, M.A., Lukkarinen, J., Nota, A., Velázquez, J.J.L.: Asymptotic localization in multicomponent time-dependent coagulation equations. (2022) [arXiv:2203.08076](https://arxiv.org/abs/2203.08076)
- Ferreira, M.A., Lukkarinen, J., Nota, A., Velázquez, J.J.L.: Non-power law constant flux solutions for the Smoluchowski coagulation equation. [arXiv:2207.09518](https://arxiv.org/abs/2207.09518) (2022)
- Fournier, N., Laurençot, P.: Existence of self-similar solutions to Smoluchowski's coagulation equation. *Commun. Math. Phys.* **256**(3), 589–609 (2005)
- Friedlander, S.K.: Smoke, Dust, and Haze. Oxford University Press, Oxford (2000)
- Gamba, I.M., Panferov, V., Villani, C.: On the Boltzmann equation for diffusively excited granular media. *Commun. Math. Phys.* **246**(3), 503–541 (2004)
- Hayakawa, H.: Irreversible kinetic coagulations in the presence of a source. *J. Phys. A* **20**(12), L801–L805 (1987)
- James, R.D., Nota, A., Velázquez, J.J.L.: Self-similar profiles for homoenergetic solutions of the Boltzmann equation: particle velocity distribution and entropy. *Arch. Rational Mech. Anal.* **231**(2), 787–843 (2019)
- Kierkels, A., Velázquez, J.J.L.: On the transfer of energy towards infinity in the theory of weak turbulence for the nonlinear Schrödinger equation. *J. Stat. Phys.* **159**, 668–712 (2015)
- Krapivsky, P.I., Ben-Naim, E.: Aggregation with multiple conservation laws. *Phys. Rev. E* **53**(1), 291–298 (1996)
- Laurençot, P.: Stationary solutions to Smoluchowski's coagulation equation with source. *North-W. Eur. J. Math.* **6**, 137–164 (2020)

18. Niethammer, B., Velázquez, J.J.L.: Self-similar solutions with fat tails for Smoluchowski's coagulation equation with locally bounded kernels. *Commun. Math. Phys.* **318**(2), 505–532 (2013)
19. Niethammer, B., Throm, S., Velázquez, J.J.L.: Self-similar solutions with fat tails for Smoluchowski's coagulation equation with singular kernels. *Ann. Inst. Henri Poincaré (C) Nonlinear Analysis* **33**(5), 1223–1257 (2016)
20. Niethammer, B., Nota, A., Throm, S., Velázquez, J.J.L.: Self-similar asymptotic behavior for the solutions of a linear coagulation equation. *J. Differ. Equ.* **266**(1), 653–715 (2019)
21. Olenius, T., Kupiainen-Määttä, O., Ortega, I.K., Kurtén, T., Vehkamäki, H.: Free energy barrier in the growth of sulfuric acid-ammonia and sulfuric acid-dimethylamine clusters. *J. Chem. Phys.* **139**, 084312 (2013)
22. Vehkamäki, H., Riipinen, I.: Thermodynamics and kinetics of atmospheric aerosol particle formation and growth. *Chem. Soc. Rev.* **41**(15), 5160 (2012)

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