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On concentration of positive solutions for a coupled elliptic systems with critical exponential growth in $\mathbb{R}^2$

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Abstract
In this study we prove the existence and concentration of positive solutions for a coupled elliptic system in two-dimensional space specifically $\mathbb{R}^2$. This system features nonlinearities that grow exponentially making the problem more complex. To deal with this challenge we employ Variational Methods which help us find critical points of a functional. Additionally we utilize the Trudinger–Moser inequality a powerful tool that provides essential estimates for our analysis. Through these methods we aim to achieve precise estimations related to the minimax levels which are crucial for understanding the behavior of the solutions.

Keywords Variational methods · Exponential critical growth · Trudinger–Moser inequality

Mathematics Subject Classification 35H30 · 35A15 · 35B30

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1 Introduction

In (1992), Rabinowitz [20] demonstrated the existence of a solution to the problem

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$ and $f(u)$ exhibited a behavior of the form $|u|^{p-2}u$ with $2 < p < 2^* = \frac{2N}{N-2}$. In [30], Wang proved that the solution found by Rabinowitz concentrated at global minimum points of the potential V. In [8], Cingolani and Lazzo showed multiplicity results for the problem (1.1).

Previous and subsequent work on the concentration of solutions around critical points of V can be found in Ambrosetti, Badiale and Cingolani [4], Bartsch and Z. Q. Wang [5], Cao and Noussair [7], Dancer and Wei [9], D'Aprile and Wei [10], Del Pino and Felmer[11], Floer and Weinstein [16] and references therein.

Results concerning the existence, concentration, and multiplicity of solutions to problem (1.1) with critical exponential-type nonlinearity have been less explored. In [1], Alves and Monari established the existence of sign-changing solutions and concentration results. The version for positive solutions and for the N-Laplacian operator was studied by Alves and Figueiredo in [3].

The version for gradient systems of the results by Rabinowitz [20] and Wang [30] has already been studied for $N \geq 3$ and for nonlinearities of polynomial type. For example, Alves and Monari in [2], as well as Figueiredo in [12], demonstrated the existence, concentration, and multiplicity of solutions where the nonlinearity behaved as $F(u, v) = |u|^p|v|^q$ with $2 < p+q < 2^*$. The critical polynomial case was studied by Figueiredo and Furtado in [13].

When studying systems we notice that certain interesting features arise because of the connections in the equations. These connections or couplings can greatly influence the behavior of the system. For example they can lead to new patterns or unexpected results that would not occur if the equations were independent. Understanding these features helps us gain deeper insights into how systems operate and interact. By examining the effects of coupling we can better predict and control the outcomes of various processes. The findings from the studies by [2, 12–14, 21–28] raise an important question about the existence and concentration of solutions in systems featuring exponential growth nonlinearities. These types of nonlinearities can significantly impact the behavior of solutions making it essential to explore new methods to understand their properties. Researchers are now encouraged to investigate whether similar results can be achieved for systems governed by these complex growth patterns. By addressing this question we can expand our understanding of nonlinear systems and potentially discover new mathematical tools for analyzing them. Overall this inquiry could lead to significant advancements in the field.

In this paper, we study the existence and concentration of positive solutions for a coupled elliptic systems in $\mathbb{R}^2$ given by

$$\begin{cases} -\varepsilon^2 \text{div}\,(V_1(x)\nabla u) + u = F_u(u,v) \text{ in } \mathbb{R}^2, \\ -\varepsilon^2 \text{div}\,(V_2(x)\nabla v) + v = F_v(u,v) \text{ in } \mathbb{R}^2, \\ u(x), v(x) > 0 \qquad\qquad\qquad \text{in } \mathbb{R}^2, \end{cases} \tag{S_ε}$$

where $\varepsilon > 0$, $V_i, i = 1, 2$ are continuous functions and F is a function of class C^1 for which the hypotheses will be given later.

(V_1) V_1 and V_2 are continuous functions satisfying:

$$V_{1\infty} = \liminf_{|x|\to\infty} V_1(x) > V_{10} = \inf_{\mathbb{R}^2} V_1(x) > 0,$$
$$V_{2\infty} = \liminf_{|x|\to\infty} V_2(x) > V_{20} = \inf_{\mathbb{R}^2} V_2(x) > 0,$$

where, in this work, we will consider the cases $V_{1\infty} < \infty$ and $V_{2\infty} < \infty$ or $V_{1\infty} = \infty$ and $V_{2\infty} = \infty$.

(V_2) $M = \{x \in \mathbb{R}^2 : V_1(x) = V_{10} \text{ and } V_2(x) = V_{20}\}$ is not empty.

The functions F_u, F_v are partial derivatives of a function F of C^1 class that satisfy

(F_1) $F(u, v) = 0$ for all $u \le 0$ and $v \in \mathbb{R}$ or for all $u \in \mathbb{R}$ and $v \le 0$.

(F_2) The following limits are held:

$$\lim_{(u,v)\to(0,0)} \frac{F_u(u,v)}{|u|+|v|} = 0 \text{ and } \lim_{(u,v)\to(0,0)} \frac{F_v(u,v)}{|u|+|v|} = 0.$$

(F_3) There exists $\alpha_0 > 0$ such that, for all $\alpha > \alpha_0$ and for all $u, v \ge 0$, we have

$$F_u(u,v) \le \exp(\alpha(|u|^2+|v|^2)) - 1, \quad F_v(u,v) \le \exp(\alpha(|u|^2+|v|^2)) - 1.$$

(F_4) There exists $\theta > 2$ such that $0 < \theta F(u,v) \le uF_u(u,v) + vF_v(u,v)$ for all $u, v > 0$.

(F_5) $\Upsilon \to \dfrac{uF_u(\Upsilon u, \Upsilon v) + vF_v(\Upsilon u, \Upsilon v)}{\Upsilon}$ is an increasing function with $u, v, \Upsilon > 0$.

(F_6) There exists $\sigma^* > 1$ such that

$$F(u,v) \ge \sigma u^\beta v^\nu,$$

for all $u, v > 0$, $\beta, \nu > 1$, for all $\sigma > \sigma^*$ and σ^* to be fixed later.

As a prototype of nonlinearity F satisfying conditions $(F_1) - (F_6)$, we can consider $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(u,v) = \begin{cases} \sigma u^\beta v^\nu e^{u^2+v^2}, & u, v \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.1 We say that $(u, v) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ is a positive weak solution for the system (S_ε) if $u, v > 0$ in $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2} \Big(V_1(\varepsilon x)\nabla u \nabla \phi + V_2(\varepsilon x)\nabla v \nabla \psi + u\phi + v\psi \Big) dx$$
$$- \int_{\mathbb{R}^2} F_u(u, v)\phi dx - \int_{\mathbb{R}^2} F_v(u, v)\psi dx = 0,$$

for all $(\phi, \psi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. In addition, (u, v) has minimal energy if $I_\varepsilon(u, v) \leq I_\varepsilon(w, z)$, for all weak solutions (w, z) of (S_ε), where I_ε will be defined in (2.4).

Our main theorem is as follows.

Theorem 1.2 *Assume that hypotheses* $(V_1) - (V_2)$ *and* $(F_1) - (F_6)$ *are held. There exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$*, system* (S_ε) *has a positive weak solution with minimal energy* $(u_\varepsilon, v_\varepsilon) \in L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2)$*, for some* $0 < \iota < 1$*. Moreover, if* $\eta_\varepsilon \in \mathbb{R}^2$ *is the global maximum of* $(u_\varepsilon, v_\varepsilon)$*, then*

$$\lim_{\varepsilon \to 0} V_1(\eta_\varepsilon) = V_{10} \ \text{ and } \ \lim_{\varepsilon \to 0} V_2(\eta_\varepsilon) = V_{20}.$$

Our results complement most of the aforementioned works, since we consider the existence and concentration of solutions and the exponential growth on the nonlinearity. Finally, we would like to emphasize two points: first that, we adapted in Lemma 2.2 aTrudinger–Moser type result for the system we are studying; secondly that, to the best of our knowledge, there is no results concerning existence and concentration of solutions for systems with exponential critical growth.

This paper is organized as follows. In Section 2 we study the variational framework on the system. In Section 3 we study a periodic system, which is essential to prove the main results. In Section 4, we prove the existence of a positive solution with minimal energy for system (S_ε), in other works, we prove Theorem 1.2. Finally, there is a Appendix with a technical result.

2 Variational framework

Changing variables by $x \mapsto \varepsilon x$, we can rewrite system (S_ε) into the following equivalent form

$$\begin{cases} -\text{div}(V_1(\varepsilon x)\nabla u) + u = F_u(u, v) \text{ in } \mathbb{R}^2, \\ -\text{div}(V_2(\varepsilon x)\nabla v) + v = F_v(u, v) \text{ in } \mathbb{R}^2, \\ u, v \in H^1(\mathbb{R}^2), u(x), v(x) \geq 0 \ \text{ for each } x \in \mathbb{R}^2. \end{cases} \tag{$\widehat{S}_\varepsilon$}$$

If (u, v) is a solution of system $(\widehat{S}_\varepsilon)$, then $(\widehat{u}(x), \widehat{v}(x)) := (u(x/\varepsilon), v(x/\varepsilon))$ is a solution of system (S_ε). Thus, for studying system (S_ε), it suffices to study system $(\widehat{S}_\varepsilon)$.

For each $\varepsilon > 0$, we denote by X_ε the Hilbert space

$$X_\varepsilon := \left\{(u, v) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \left(V_1(\varepsilon x)|\nabla u|^2 + V_2(\varepsilon x)|\nabla v|^2\right) dx < \infty\right\}$$

endowed with the norm

$$\|(u, v)\|_\varepsilon^2 := \int_{\mathbb{R}^2} \left(V_1(\varepsilon x)|\nabla u|^2 + V_2(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2\right) dx.$$

We also define on $L^s(\mathbb{R}^2) \times L^s(\mathbb{R}^2)$ the norm

$$|(u, v)|_s^s = \int_{\mathbb{R}^2} |u|^s dx + \int_{\mathbb{R}^2} |v|^s dx.$$

By using condition (V_1) we can verify that

$$X_\varepsilon \hookrightarrow H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2) \times L^s(\mathbb{R}^2) \text{ for all } 2 \le s.$$

We recall [6, Proposition 2.1], which is necessary for the semilinear case with exponential growth.

Proposition 2.1 *If $a > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} [\exp(a|u|^2) - 1] dx < \infty.$$

Moreover, if $\|\nabla u\|_{L^2}^2 \le 1$, $\|u\|_{L^2} \le K < \infty$ and $a < 4\pi$, then there exists a constant $C = C(K, a)$, which depends only on K and a, such that

$$\int_{\mathbb{R}^2} [\exp(a|u|^2) - 1] dx \le C.$$

Inspired by [19, Lemma 1.1], we prove the system version of Proposition 2.2.

Lemma 2.2 *If $a > 0$ and $(u, v) \in X_\varepsilon$, then*

$$\int_{\mathbb{R}^2} \left[\exp\left(a\left(|u|^2 + |v|^2\right)\right) - 1\right] dx < \infty.$$

Moreover, if $\|(u, v)\|_\varepsilon^2 \le 1$ and $a < 2\pi$, then there exists a constant $C = C(a)$, which depends only on a, such that

$$\int_{\mathbb{R}^2} \left[\exp\left(a\left(|u|^2 + |v|^2\right)\right) - 1\right] dx \le C.$$

Proof By using the Yang's inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left[\exp\left(a\left(|u|^2+|v|^2\right)\right)-1\right]dx \\
&\leq \int_{\mathbb{R}^2} \left[\frac{1}{2}\exp(2a|u|^2)-\frac{1}{2}\right]dx + \int_{\mathbb{R}^2} \left[\frac{1}{2}\exp(2a|v|^2)-\frac{1}{2}\right]dx \\
&\leq \int_{\mathbb{R}^2} \left[\exp(2a|u|^2)-1\right]dx + \int_{\mathbb{R}^2} \left[\exp(2a|v|^2)-1\right]dx
\end{aligned}$$

and the result follows by Proposition 2.1. □

Lemma 2.3 *Let $a > 0$ and $r > 1$. There exists a constant $C = C(r)$ such that*

$$\left[\exp\left(a(t^2+s^2)\right)-1\right]^r \leq C\left[\exp\left(ra(t^2+s^2)\right)-1\right] \text{ for } t, s \in \mathbb{R}.$$

Proof It suffices to prove that the limits

$$\lim_{s,t\to 0} \frac{\left[\exp\left(a(t^2+s^2)\right)-1\right]^r}{\left[\exp\left(ra(t^2+s^2)\right)-1\right]} \quad \text{and} \quad \lim_{s,t\to +\infty} \frac{\left[\exp\left(a(t^2+s^2)\right)-1\right]^r}{\left[\exp\left(ra(t^2+s^2)\right)-1\right]}$$

are finite, which is a direct consequence of the L'Hospital's rule. □

Note that the assumption (F_5) implies

$$t \mapsto tuF_u(tu,tv) + tvF_v(tu,tv) - 2F(tu,tv) \tag{2.1}$$

is increasing for $t > 0$ and $u, v > 0$.
Indeed, let $0 < s < t$. By assumption (F_5), we obtain

$$\begin{aligned}
F(tu,tv) - F(su,sv) &= \int_s^t \frac{d}{d\tau} F(\tau u, \tau v) d\tau \\
&= \int_s^t \left[\frac{F_u(\tau u,\tau v)u + F_v(\tau u,\tau v)v}{\tau}\right]\tau d\tau \\
&\leq \int_s^t \left[\frac{F_u(tu,tv)u + F_v(tu,tv)v}{t}\right]\tau d\tau \\
&= \frac{F_u(tu,tv)u + F_v(tu,tv)v}{t}\left(\frac{t^2-s^2}{2}\right).
\end{aligned}$$

Therefore, using the preceding inequality together with assumption (F_5), we deduce that

$$suF_u(su,sv) + svF_v(su,sv) - 2F(su,sv)$$

$$
\begin{aligned}
&= s^2\left(\frac{uF_u(su,sv)+vF_v(su,sv)}{s}\right)-2F(su,sv)\\
&< s^2\left(\frac{uF_u(tu,tv)+vF_v(tu,tv)}{t}\right)-2F(tu,tv)\\
&\quad+\frac{uF_u(tu,tv)+vF_v(tu,tv)}{t}(t^2-s^2)\\
&= tuF_u(tu,tv)+tvF_v(tu,tv)-2F(tu,tv).
\end{aligned}
$$

This establishes the desired monotonicity and completes the proof of assertion (2.1).

Also by assumptions (F_2) and (F_3), one can show that for a given $\delta > 0$, there exists a positive constant $C_\delta > 0$, such that for all $u, v \geq 0$, $\alpha > \alpha_0$ and $q > 2$, we have

$$
\begin{aligned}
&|F_u(u,v)| + |F_v(u,v)|\\
&\leq \delta(|u|+|v|) + C_\delta(|u|^{q-1}+|v|^{q-1})\left[\exp\left(\alpha\left(|u|^2+|v|^2\right)\right)-1\right]
\end{aligned}
\tag{2.2}
$$

By assumption (F_4), for all $u, v \geq 0$, there exists a positive constant $\widetilde{C}_\delta > 0$ such that

$$
|F(u,v)| \leq \delta(|u|^2+|v|^2) + \widetilde{C}_\delta(|u|^q+|v|^q)\left[\exp\left(\alpha\left(|u|^2+|v|^2\right)\right)-1\right]. \tag{2.3}
$$

Using (2.3) and Lemma 2.2, we have $F(u,v) \in L^1(\mathbb{R}^2)$ for all $(u,v) \in X_\varepsilon$. Therefore, the functional $I_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$
\begin{aligned}
I_\varepsilon(u,v) := &\frac{1}{2}\int_{\mathbb{R}^2}\left(V_1(\varepsilon x)|\nabla u|^2 + V_2(\varepsilon x)|\nabla v|^2 + |u|^2+|v|^2\right)dx\\
&-\int_{\mathbb{R}^2} F(u,v)dx
\end{aligned}
\tag{2.4}
$$

is well defined. Furthermore, using standard arguments as well as the fact that for any given strong convergent sequence (u_n, v_n) in X_ε there is a subsequence (u_{n_k}, v_{n_k}) and there exist $(g,h) \in X_\varepsilon$ such that $|u_{n_k}(x)| \leq g(x)$ and $|v_{n_k}(x)| \leq h(x)$ almost everywhere in $\mathbb{R}^2$, we can see that I_ε is a C^1 functional on X_ε with

$$
\begin{aligned}
I'_\varepsilon(u,v)(\phi,\psi) := &\int_{\mathbb{R}^2}\left(V_1(\varepsilon x)\nabla u\nabla\phi + V_2(\varepsilon x)\nabla v\nabla\psi + u\phi + v\psi\right)dx\\
&-\int_{\mathbb{R}^2}\left(F_u(u,v)\phi + F_v(u,v)\psi\right)dx
\end{aligned}
$$

for all $(\phi,\psi) \in X_\varepsilon$. Consequently, critical points of the functional I_ε are precisely the weak solutions of system $(\widehat{S}_\varepsilon)$.

3 Periodic problem

We define the Sobolev space

$$X_W := H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$$

which is endowed with the norm

$$\begin{aligned}\|(u,v)\|_W^2 := &\int_{\mathbb{R}^2} W_1(x)|\nabla u|^2 dx + \int_{\mathbb{R}^2} W_2(x)|\nabla v|^2 dx \\ &+ \int_{\mathbb{R}^2} |u|^2 dx + \int_{\mathbb{R}^2} |v|^2 dx,\end{aligned} \tag{3.1}$$

where $W_1, W_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous 1-periodic functions, that is,

$$W_1(x+y) = W_1(x) \quad \text{and} \quad W_2(x+y) = W_2(x)$$

for all $y \in \mathbb{Z}^2$ and there are W_{10} and W_{20} positive constants such that

$$W_1(x) \geq W_{10} \quad \text{and} \quad W_2(x) \geq W_{20},$$

for all $x \in \mathbb{R}^2$.

Since W_1, W_2 are positive and 1-periodic, such potential are bounded, which imply that the norm defined by (3.1) is equivalent to usual norm

$$\|(u,v)\|^2 = \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |\nabla v|^2 dx + \int_{\mathbb{R}^2} |u|^2 dx + \int_{\mathbb{R}^2} |v|^2 dx$$

in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.

The main objective of this section is to study the existence of solutions for the problem

$$\begin{cases} -\text{div}\,(W_1(x)\nabla u) + u = F_u(u,v) & \text{in } \mathbb{R}^2, \\ -\text{div}\,(W_2(x)\nabla v) + v = F_v(u,v) & \text{in } \mathbb{R}^2, \\ u(x), v(x) > 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{P_W}$$

where $(u,v) \in X_W$.

From (2.3) and Lemma 2.2, the functional

$$\begin{aligned}J_W(u,v) = &\frac{1}{2}\int_{\mathbb{R}^2} W_1(x)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |u|^2 dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^2} W_2(x)|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |v|^2 dx - \int_{\mathbb{R}^2} F(u,v) dx\end{aligned}$$

is well defined. Furthermore, using (2.2) and standard arguments, J_W is C^1 on X_W with

$$J'_W(u,v)(\phi,\psi) = \int_{\mathbb{R}^2} W_1(x)\nabla u\nabla\phi dx + \int_{\mathbb{R}^2} W_2(x)\nabla v\nabla\psi dx + \int_{\mathbb{R}^2} u\phi dx + \int_{\mathbb{R}^2} v\psi dx - \int_{\mathbb{R}^2} (F_u(u,v)\phi + F_v(u,v)\psi)\, dx,$$

for all $(\phi,\psi) \in X_W$. Consequently, critical points of J_W are precisely the weak solutions of system (P_W). In the next, $\mathcal{M}_W$ denotes the Nehari manifold related to J_W, given by

$$\mathcal{M}_W = \left\{(u,v) \in X_W \setminus \{(0,0)\} : J'_W(u,v)(u,v) = 0\right\}$$

and define c^*_W by setting

$$c^*_W = \inf_{(u,v)\in\mathcal{M}_W} J_W(u,v).$$

Definition 3.1 We say that $(u,v) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ is a positive weak solution for the system (P_W) if $u, v > 0$ in $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2} (W_1(x)\nabla u\nabla\phi + W_2(x)\nabla v\nabla\psi + u\phi + v\psi)\, dx - \int_{\mathbb{R}^2} F_u(u,v)\phi dx - \int_{\mathbb{R}^2} F_v(u,v)\psi dx = 0,$$

for all $(\phi,\psi) \in H^1(\mathbb{R}^2 \times H^1(\mathbb{R}^2)$. In addition (u,v) has minimal energy if $J_W(u,v) \leq J_W(w,z)$, for all (w,z) positive weak solution to (P_W).

The main result in this section is:

Theorem 3.2 *Assume that conditions $(F_1)-(F_6)$ are held. Then, problem (P_W) has a positive weak solution $(u,v) \in L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2)$, for some $0 < \iota < 1$ with minimal energy.*

The next result concerns with the Mountain Pass Geometry of J_W.

Lemma 3.3 *The functional J_W satisfies the following conditions:*
(i) There exists $\rho, \eta > 0$, such that $J_W(u,v) \geq \eta$, if $\|(u,v)\|_W = \rho$.
(ii) For any $(u,v) \in X_W\setminus(0,0)$ with $u \geq 0, v \geq 0$, $J_W(tu,tv) \to -\infty$ as $t \to +\infty$.

Proof For $\alpha > \alpha_0 > 0$, with α_0 given in (F_3) and for $\delta > 0$ sufficient small and (2.3), we get

$$J_W(u,v) \geq \left(\frac{1}{2} - \delta\right)\|(u,v)\|^2_W - \int_{\mathbb{R}^2} \widetilde{C}_\delta(|u|^q + |v|^q)\left[\exp\left(\alpha\left(|u|^2 + |v|^2\right)\right) - 1\right] dx,$$

which implies

$$J_W(u,v) \geq \left(\frac{1}{2}-\delta\right)\|(u,v)\|_W^2 - \int_{\mathbb{R}^2} \widetilde{C}_\delta(|u|^q+|v|^q)\left[\exp\left(\alpha\|(u,v)\|_W^2\frac{|u|^2+|v|^2}{\|(u,v)\|_W^2}\right)-1\right]dx.$$

By considering $\|(u,v)\|_W = \rho$ sufficient small such that $\alpha\rho^2 < 2\pi$, using Lemma 2.2, Sobolev embedding and assumption $2 < q$, we obtain $C > 0$ such that

$$J_W(u,v) \geq \left(\frac{1}{2}-\delta\right)\|(u,v)\|_W^2 - C\|(u,v)\|_W^q.$$

The proof of item (i) is complete.

In order to prove the item (ii), from (F_6), for $t > 0$ and all $\sigma > 0$ and considering that $\beta + \nu > 2$, we get

$$J_W(tu,tv) \leq \frac{t^2}{2}\|(u,v)\|_W^2 - \sigma t^{\beta+\nu}\int_{\mathbb{R}^2}|u|^\beta|v|^\nu,$$

with implies that $\lim_{t\to+\infty} J_W(tu,tv) = -\infty$ and the proof is finished. □

By Lemma 3.3 and [31, Theorem 1.15], there exists a sequence $(u_n,v_n) \subset X_W$ verifying

$$J_W(u_n,v_n) \to c_W \ \text{ and } \ J_W'(u_n,v_n) \to 0 \ \text{ as } \ n\to\infty,$$

where the level c_W is characterized by

$$c_W = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_W(\gamma(t))$$

and

$$\Gamma := \{\gamma \in C([0,1],X_W) : \gamma(0) = (0,0) \ \text{ and } \ J_W(\gamma(1)) < 0\}.$$

In view of assumption (F_5), we have the following characterization (see [31, Chapter 4])

$$c_W = c_W^* = \inf_{(u,v)\in X_W\setminus\{(0,0)\}}\max_{t\geq 0} J_W(tu,tv).$$

In this section, for proving Theorem 3.2, we consider the auxiliary problem

$$\begin{cases} -\text{div}\,(W_1(x)\nabla u) + u = \beta|u|^{\beta-2}u|v|^\nu \ \text{ in } \ \mathbb{R}^2, \\ -\text{div}\,(W_2(x)\nabla v) + v = \nu|u|^\beta|v|^{\nu-2}v \ \text{ in } \ \mathbb{R}^2, \end{cases} \tag{Aux}$$

where β and ν are the constants that appear in the assumption (F_6). We have associated to problem (Aux) the functional

$$T(u,v)=\frac{1}{2}\int_{\mathbb{R}^2}\Big[W_1(x)|\nabla u|^2+W_2(x)|\nabla v|^2+|u|^2+|v|^2\Big]dx-\int_{\mathbb{R}^2}|u|^{\beta}|v|^{\nu}dx.$$

Arguing as [13, Proposition 2.2], we can prove that there exists $(u_0,v_0)\in X_W$ such that

$$T(u_0,v_0)=c_{\beta,\nu},\quad T'(u_0,v_0)=0 \tag{3.2}$$

and

$$c_{\beta,\nu}=\frac{(\beta+\nu)-2}{2}\int_{\mathbb{R}^2}|u_0|^{\beta}|v_0|^{\nu}dx. \tag{3.3}$$

The next result is an estimate to c_W.

Lemma 3.4 *The value c_W satisfies*

$$c_W\leq\frac{1}{\sigma^{\frac{2}{\beta+\nu-2}}}c_{\beta,\nu}.$$

Proof Notice that the assumption (F_6) implies

$$c_W\leq\max_{t\geq0}J_W(tu_0,tv_0)\leq\max_{t\geq0}\left\{\frac{t^2}{2}\|(u_0,v_0)\|_W^2-\sigma t^{\beta+\nu}\int_{\mathbb{R}^2}|u_0|^{\beta}|v_0|^{\nu}dx\right\}$$

which from (3.2) and (3.3) implies

$$\begin{aligned}c_W&\leq\max_{t\geq0}\left\{\frac{t^2}{2}(\beta+\nu)-\sigma t^{\beta+\nu}\right\}\int_{\mathbb{R}^2}|u_0|^{\beta}|v_0|^{\nu}dx\\&=\max_{t\geq0}\left\{\frac{t^2}{2}(\beta+\nu)-\sigma t^{\beta+\nu}\right\}\frac{2c_{\beta,\nu}}{\beta+\nu-2}.\end{aligned}$$

Some straightforward algebraic manipulations show that

$$c_W\leq\frac{1}{\sigma^{\frac{2}{\beta+\nu-2}}}c_{\beta,\nu}$$

and the result is followed. □

Lemma 3.5 *Consider $\alpha_0>0$ is given in (F_3). If $(u_n,v_n)\subset X_W$ is a $(PS)_{c_W}$ sequence for J_W, then*

$$\limsup_{n\to\infty}\|(u_n,v_n)\|_W^2<\frac{\pi}{2\alpha_0}.$$

Proof First, notice that the assumption (F_4) shows

$$c_W + o_n(1) = J_W(u_n, v_n) - \frac{1}{\theta} J'_W(u_n, v_n)(u_n, v_n) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|(u_n, v_n)\|^2_W,$$

which implies

$$\limsup_{n\to\infty} \|(u_n, v_n)\|^2_W \leq \frac{2\theta c_W}{\theta - 2} \leq \frac{2\theta}{\theta - 2} \frac{1}{\sigma^{\frac{2}{\beta+\nu-2}}} c_{\beta,\nu}.$$

By choosing

$$\sigma* = \left[\frac{4\theta\alpha_0\ c_{\beta,\nu}}{\pi(\theta - 2)}\right]^{(\beta+\nu-2)/2},$$

we have, for $\sigma > \sigma^*$,

$$\limsup_{n\to\infty} \|(u_n, v_n)\|^2_W < \frac{\pi}{2\alpha_0}$$

and the result follows. □

Proposition 3.6 *Let $(u_n, v_n) \subset X_W$ be a $(PS)_{c_W}$ sequence for J_W. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^2$ such that, up to a subsequence, $(w_n, z_n) := (u_n(\cdot + \widetilde{y}_n), v_n(\cdot + \widetilde{y}_n)) \to (w, z) \in X_W$ with $J_W(w, z) = c_W$.*

Proof Note that $(u_n, v_n) \subset X_W$ verifies the following limits

$$J_W(u_n, v_n) \to c_W \quad \text{and} \quad J'_W(u_n, v_n) \to 0.$$

From Lemma 3.5, the sequence (u_n, v_n) is bounded in X_W and, up to a subsequence, we have $(u_n, v_n) \rightharpoonup (u, v)$ converges weakly in X_W. Therefore, for all $(\phi, \psi) \in X_W$,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^2} W_1(x)\nabla u_n \nabla\phi dx \to \int_{\mathbb{R}^2} W_1(x)\nabla u \nabla\phi dx, \\ \displaystyle\int_{\mathbb{R}^2} W_2(x)\nabla v_n \nabla\psi dx \to \int_{\mathbb{R}^2} W_2(x)\nabla v \nabla\psi dx, \\ \displaystyle\int_{\mathbb{R}^2} u_n\phi dx \to \int_{\mathbb{R}^2} u\phi dx, \\ \displaystyle\int_{\mathbb{R}^2} v_n\psi dx \to \int_{\mathbb{R}^2} v\psi dx. \end{cases} \tag{3.4}$$

Now we prove that

$$\begin{aligned} &\int_{\mathbb{R}^2} [F_u(u_n, v_n)\phi dx + F_v(u_n, v_n)\psi] dx \\ &\to \int_{\mathbb{R}^2} [F_u(u, v)\phi dx + F_v(u, v)\psi] dx, \end{aligned} \tag{3.5}$$

for all $(\phi, \psi) \in X_W$. Note that, up to a subsequence,

$$u_n(x) \to u(x), \ v_n(x) \to v(x) \quad \text{a.e. in} \quad \mathbb{R}^2$$

and

$$\begin{cases} F_u(u_n(x), v_n(x))\phi(x) \to F_u(u(x), v(x))\phi(x) & \text{a.e. in} \quad \mathbb{R}^2, \\ F_v(u_n(x), v_n(x))\psi(x) \to F_v(u(x), v(x))\psi(x) & \text{a.e. in} \quad \mathbb{R}^2. \end{cases}$$

By (3.4), it is sufficient to prove that there are $g_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, such that

$$|F_u(u_n, v_n)\phi + F_v(u_n, v_n)\psi| \le u_n\phi + g_1(u_n, v_n)\phi + v_n\psi + g_2(u_n, v_n)\psi$$

where $(g_1(u_n, v_n)\phi + g_2(u_n, v_n)\psi)$ is convergent in $L^1(\mathbb{R}^2)$. Note that by $(F_2)-(F_3)$, for $\alpha > \alpha_0$, α close to α_0, it is possible to prove that

$$\begin{aligned} &F_u(u_n, v_n)\phi + F_v(u_n, v_n)\phi \\ &\le \delta(u_n\phi + v_n\psi) + C_\delta(|u_n|^{q-2}u_n\phi + |v_n|^{q-2}v_n\psi)\left[\exp\left(\alpha(|u_n|^2 + |v_n|^2)\right) - 1\right] \\ &:= \delta(u_n\phi + v_n\psi) + g_1(u_n, v_n)\phi + g_2(u_n, v_n)\psi. \end{aligned}$$

We now show that

$$g_1(u_n, v_n)\phi = C_\delta|u_n|^{q-2}u_n\phi\left[\exp\left(\alpha\left(|u_n|^2 + |v_n|^2\right)\right) - 1\right]$$

is convergent in $L^1(\mathbb{R}^2)$, since the convergence of $(g_2(u_n, v_n)\psi)$ in $L^1(\mathbb{R}^2)$ follows similarly.

Notice that, the sequence $(g_1(u_n, v_n))$ is bounded in $L^2(\mathbb{R}^2)$. In fact, by Lemma 2.3, there exists $C > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^2}\left\{|u_n|^{(q-2)}u_n\left[\exp\left(\alpha\left(|u_n|^2 + |v_n|^2\right)\right) - 1\right]\right\}^2 dx \\ &\le C\int_{\mathbb{R}^2}|u_n|^{2(q-1)}\left[\exp\left(2\alpha\left(|u_n|^2 + |v_n|^2\right)\right) - 1\right]dx. \end{aligned}$$

Considering $s, s' > 1$ such that $\frac{1}{s} + \frac{1}{s'} = 1$ and s close to 1, we obtain, by Hölder's inequality, Lemma 2.3 and Sobolev embedding, that there exists $C_1 > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^2}\left\{|u_n|^{(q-2)}u_n\left[\exp\left(\alpha\left(|u_n|^2 + |v_n|^2\right)\right) - 1\right]\right\}^2 dx \\ &\le C\left(\int_{\mathbb{R}^2}|u_n|^{2s'(q-1)}dx\right)^{1/s'}\left(\int_{\mathbb{R}^2}\left[\exp\left(2s\alpha\left(|u_n|^2 + |v_n|^2\right)\right) - 1\right]dx\right)^{1/s} \\ &\le C_1\left(\int_{\mathbb{R}^2}\left[\exp\left(2s\alpha\|(u_n, v_n)\|_W^2\frac{|u_n|^2 + |v_n|^2}{\|(u_n, v_n)\|_W^2}\right) - 1\right]dx\right)^{1/s}. \end{aligned}$$

From Lemma 3.5, Lemma 2.2 and since $1 < \frac{s\alpha}{\alpha_0} < 2$, we obtain a constant $C_2 > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left\{ |u_n|^{(q-2)} u_n \left[\exp\left(\alpha \left(|u_n|^2 + |v_n|^2 \right) \right) - 1 \right] \right\}^2 dx \\
&\leq C_1 \left(\int_{\mathbb{R}^2} \left[\exp\left(s \frac{\alpha}{\alpha_0} \pi \frac{|u_n|^2 + |v_n|^2}{\|(u_n, v_n)\|^2} \right) - 1 \right] dx \right)^{1/s} \\
&\leq C_1 C_2^{1/s}.
\end{aligned}$$

Then, from [17, Lemma 4.8], we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u_n|^{q-2} u_n \phi \left[\exp\left(\alpha \left(|u_n|^2 + |v_n|^2 \right) \right) - 1 \right] dx \\
&\to \int_{\mathbb{R}^2} |u|^{q-2} u \phi \left[\exp\left(\alpha \left(|u|^2 + |v|^2 \right) \right) - 1 \right] dx
\end{aligned} \tag{3.6}$$

for all $\phi \in H^1(\mathbb{R}^2)$, which implies

$$\int_{\mathbb{R}^2} [F_u(u_n, v_n)\phi + F_v(u_n, v_n)\psi]\, dx \to \int_{\mathbb{R}^2} [F_u(u, v)\phi + F_v(u, v)\psi]\, dx.$$

We can conclude that (u, v) is a critical point of J_W. Now, consider the following two cases.

Case 1. $(u, v) \not\equiv (0, 0)$.

In this case $J'_W(u, v)(u, v) = 0$ and therefore $(u, v) \in \mathcal{M}_W$. Then, by using (F_4) and Fatou's Lemma we obtain

$$\begin{aligned}
c_W &\leq J_W(u, v) - \frac{1}{\theta} J'_W(u, v)(u, v) \\
&= \left(\frac{1}{2} - \frac{1}{\theta} \right) \|(u, v)\|_W^2 + \int_{\mathbb{R}^2} \left[\frac{1}{\theta} \big(u F_u(u, v) + v F_v(u, v) \big) - F(u, v) \right] dx \\
&\leq \liminf_{n\to\infty} \left(\frac{1}{2} - \frac{1}{\theta} \right) \|(u_n, v_n)\|_W^2 \\
&\quad + \liminf_{n\to\infty} \int_{\mathbb{R}^2} \left[\frac{1}{\theta} \big(u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n) \big) - F(u_n, v_n) \right] dx \\
&\leq \liminf_{n\to\infty} \left(J_W(u_n, v_n) - \frac{1}{\theta} J'_W(u_n, v_n)(u_n, v_n) \right) = c_W
\end{aligned}$$

and thus

$$\lim_{n\to\infty} \|(u_n, v_n)\|_W^2 = \|(u, v)\|_W^2.$$

Hence, $(u_n, v_n) \to (u, v)$ in X_W. Consequently, $J_W(u, v) = c_W$ and $\widetilde{y}_n = 0$ for all $n \in \mathbb{N}$.

Case 2. $(u, v) = (0, 0)$.
In this case, there exists $R, \eta > 0$ and $(\widetilde{y}_n) \subset \mathbb{R}^2$ such that

$$\liminf_{n\to\infty} \int_{B_R(\widetilde{y}_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geq \eta.$$

Because if this claim does not hold, we have by [18, Lemma I.1] that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n|^s dx = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |v_n|^s dx = 0$$

for all $s > 2$. Thus, from (2.2) and Lemma 2.2, we obtain

$$\int_{\mathbb{R}^2} (u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n))\, dx \to 0 \quad \text{as} \quad n \to \infty.$$

Since $J'_W(u_n, v_n)(u_n, v_n) = 0$, we can use the above limit to get

$$\|(u_n, v_n)\|^2_W \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, $J_W(u_n, v_n) \to 0$, which contradicts $c_W > 0$.

Now, we define

$$\widetilde{u}_n(x) := u_n(x + \widetilde{y}_n)$$

and

$$\widetilde{v}_n(x) := v_n(x + \widetilde{y}_n).$$

Then, we can verify that

$$J_W(\widetilde{u}_n, \widetilde{v}_n) \to c_W \quad \text{and} \quad J'_W(\widetilde{u}_n, \widetilde{v}_n) \to 0.$$

It is clear that $\|(\widetilde{u}_n, \widetilde{v}_n)\|^2_W < \frac{\pi}{2\alpha_0}$ for n large and there exists $(\widetilde{u}, \widetilde{v}) \in X_W$ with $(\widetilde{u}, \widetilde{v}) \neq (0, 0)$ such that

$$(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\widetilde{u}, \widetilde{v}).$$

Repeating the same arguments used in the Case 1, it follows that $(\widetilde{u}_n, \widetilde{v}_n) \to (\widetilde{u}, \widetilde{v})$ in X_W. □

Now, we prove the existence of a positive weak solution for (P_W), i.e., we present the proof of Theorem 3.2.

Proof of Theorem 3.2 Denote by $u^- = \max\{-u, 0\}$ and $v^- = \max\{-v, 0\}$ the negative part of u and v, respectively. Then, by (F_1) we have

$$0 = J'_W(u, v)(u^-, v^-) = -\|(u^-, v^-)\|^2_W,$$

which implies that $u \geq 0$ and $v \geq 0$. Moreover, by adapting the arguments from Lemma 4.9 in the Appendix, we obtain that $(u, v) \in L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2)$, for some $0 < \iota < 1$. Now we can use Harnack's inequality [29, Theorem 1.1] to conclude that u and v are positive functions. □

4 The nonperiodic problem

In view of assumptions $(F_2) - (F_4)$, we can argue similar to the proof of Lemma 3.3 and check that I_ε satisfies the Mountain Pass Geometry. Moreover, using condition (F_5), we have the following characterization

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)) = c^*_\varepsilon = \inf_{(u,v) \in X_\varepsilon \setminus \{(0,0)\}} \max_{t \geq 0} I_\varepsilon(tu, tv) > 0,$$

where $\Gamma_\varepsilon := \{\gamma \in C([0,1], X_\varepsilon) : \gamma(0) = (0,0), I_\varepsilon(\gamma(1)) < 0\}$ and $c^*_\varepsilon = \inf_{(u,v) \in \mathcal{N}_\varepsilon} I_\varepsilon(u, v)$ with $\mathcal{N}_\varepsilon = \{(u, v) \in X_\varepsilon \setminus \{(0,0)\} : I'_\varepsilon(u, v)(u, v) = 0\}$ denoting the Nehari manifold associated with I_ε.

By Lemma 3.3 and [31, Theorem 1.15], there exists a $(PS)_{c_\varepsilon}$ sequence $(u_n, v_n) \subset X_\varepsilon$, that is, a sequence satisfying

$$I_\varepsilon(u_n, v_n) \to c_\varepsilon \ \text{ and } \ I'_\varepsilon(u_n, v_n) \to 0 \ \text{ in } \ X'_\varepsilon.$$

By (F_4), we can assume that (u_n, v_n) is bounded in X_ε and from (F_1), we may assume without loss of generality that $u_n, v_n \geq 0$. Once X_ε is reflexive, there exist a subsequence, still denote by (u_n, v_n), and $(u, v) \in X_\varepsilon$ such that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u, v) \ \text{ weakly in } \ X_\varepsilon, \\ u_n \to u, \ v_n \to v \ \text{ stronly in } \ L^s_{\text{loc}}(\mathbb{R}^2), \ 2 \leq s, \\ u_n(x) \to u(x), \ v_n(x) \to v(x) \ \text{ for a.e. } \ x \in \mathbb{R}^2. \end{cases}$$

4.1 Palais-Smale compactness condition

Lemma 4.1 *Consider $\alpha_0 > 0$ is given in (F_3). Let $\Omega \subset \mathbb{R}^2$ be a compact set and $(u_n, v_n) \subset X_\varepsilon$ be a $(PS)_d$ sequence for I_ε such that $(u_n, v_n) \rightharpoonup (0,0)$ with*

$$\limsup_{n \to \infty} \|(u_n, v_n)\|^2_\varepsilon < \frac{\pi}{2\alpha_0}.$$

Then

$$\lim_{n \to \infty} \int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) dx = 0.$$

Proof Let $\psi \in C^\infty_0(\mathbb{R}^2)$ be such that $\psi \equiv 1$ on Ω and $0 \leq \psi \leq 1$. Then

$$\int_\Omega V_1(\varepsilon x) |\nabla u_n|^2 dx \leq \int_{\mathbb{R}^2} V_1(\varepsilon x) |\nabla u_n|^2 \psi \, dx \tag{4.1}$$

and

$$\int_{\Omega} V_2(\varepsilon x)|\nabla v_n|^2 dx \leq \int_{\mathbb{R}^2} V_2(\varepsilon x)|\nabla v_n|^2 \psi \, dx. \tag{4.2}$$

Since (u_n, v_n) is a $(PS)_d$ sequence for I_ε and $(\psi u_n, \psi v_n)$ is bounded in X_ε, we get

$$I'_\varepsilon(u_n, v_n)(\psi u_n, \psi v_n) = o_n(1).$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(V_1(\varepsilon x)|\nabla u_n|^2 + V_2(\varepsilon x)|\nabla v_n|^2 \right) \psi \, dx \\
&= -\int_{\mathbb{R}^2} (V_1(\varepsilon x) u_n \nabla u_n \nabla \psi + V_2(\varepsilon x) v_n \nabla v_n \nabla \psi) \, dx \\
&\quad -\int_{\mathbb{R}^2} \left(|u_n|^2 + |v_n|^2 \right) \psi \, dx \\
&\quad +\int_{\mathbb{R}^2} (u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n)) \, \psi \, dx + o_n(1).
\end{aligned} \tag{4.3}$$

Let $K := \mathrm{supp}(\psi)$ denote the support of ψ. Since $\psi \in C_0^\infty(\mathbb{R}^2)$, K is compact subset of $\mathbb{R}^2$. Then, by using Hölder's inequality, for some $C_{V_{1,\varepsilon}} > 0$, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} V_1(\varepsilon x) u_n \nabla u_n \nabla \psi \, dx \right| \leq \int_{\mathbb{R}^2} V_1(\varepsilon x)|\nabla u_n||u_n||\nabla \psi| dx \\
&\leq C_{V_{1,\varepsilon}} |\nabla \psi|_\infty \left(\int_K V_1(\varepsilon x)|\nabla u_n|^2 dx \right)^{1/2} \left(\int_K |u_n|^2 dx \right)^{1/2} \\
&\leq C_{V_{1,\varepsilon}} |\nabla \psi|_\infty \|(u_n, v_n)\|_\varepsilon |u_n|_{L^2(K)}
\end{aligned} \tag{4.4}$$

and similarly, for some $C_{V_{2,\varepsilon}} > 0$

$$\left| \int_{\mathbb{R}^2} V_2(\varepsilon x) v_n \nabla v_n \nabla \psi \, dx \right| \leq C_{V_{2,\varepsilon}} |\nabla \psi|_\infty \|(u_n, v_n)\|_\varepsilon |v_n|_{L^2(K)}. \tag{4.5}$$

On the other hand, from (2.2) we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} (u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n)) \, \psi \, dx \right| \\
&\leq \delta \int_{\mathbb{R}^2} \left(|u_n|^2 + |v_n|^2 \right) \psi \, dx \\
&\quad + C_\delta \int_{\mathbb{R}^2} \left(|u_n|^q + |v_n|^q \right) \psi \, \left[\exp\left(\alpha(|u_n|^2 + |v_n|^2) \right) - 1 \right] dx.
\end{aligned}$$

Considering $\alpha > \alpha_0$ with α close to α_0, $r > 1$ with r close to 1, and using Hölder's inequality with exponents r and $r' = \dfrac{r}{r-1}$, along with Lemma 2.3, we obtain a

constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^2} \left(|u_n|^q + |v_n|^q\right) \psi \left[\exp\left(\alpha(|u_n|^2 + |v_n|^2)\right) - 1\right] dx$$
$$\leq C_1 |(u_n, v_n)|^q_{L^{qr'}(K)} \left(\int_{\mathbb{R}^2} \left[\exp\left(r\alpha \|(u_n, v_n)\|^2_\varepsilon \left(\frac{|u_n|^2 + |v_n|^2}{\|(u_n, v_n)\|^2_\varepsilon}\right)\right) - 1\right] dx\right)^{1/r}.$$

Since $\limsup_{n\to\infty} \|(u_n, v_n)\|^2_\varepsilon < \dfrac{\pi}{2\alpha_0}$, we get

$$\int_{\mathbb{R}^2} \left(|u_n|^q + |v_n|^q\right) \psi \left[\exp\left(\alpha(|u_n|^2 + |v_n|^2)\right) - 1\right] dx$$
$$\leq C_1 |(u_n, v_n)|^q_{L^{qr'}(K)} \left(\int_{\mathbb{R}^2} \left[\exp\left(r\frac{\alpha}{2\alpha_0}\pi \left(\frac{|u_n|^2 + |v_n|^2}{\|(u_n, v_n)\|^2_\varepsilon}\right)\right) - 1\right] dx\right)^{1/r}.$$

Since $0 < \frac{r\alpha}{2\alpha_0} < 2$, by Lemma 2.2, there exists $C_2 > 0$ such that

$$\left|\int_{\mathbb{R}^2} (u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n)) \psi \, dx\right|$$
$$\leq \delta |(u_n, v_n)|^2_{L^2(K)} + C_\delta C_1 C_2 |(u_n, v_n)|^q_{L^{qr'}(K)}. \tag{4.6}$$

Then, inserting inequalities (4.4), (4.5) and (4.6) in (4.3) we get

$$\int_{\mathbb{R}^2} \left(V_1(\varepsilon x)|\nabla u_n|^2 + V_2(\varepsilon x)|\nabla v_n|^2\right) \psi \, dx$$
$$\leq C_3 |\nabla\psi|_\infty \|(u_n, v_n)\|_\varepsilon |(u_n, v_n)|_{L^2(K)}$$
$$+\delta |(u_n, v_n)|^2_{L^2(K)} + C_\delta C_1 C_2 |(u_n, v_n)|^q_{L^{qr'}(K)}.$$

Since (u_n, v_n) is bounded in X_ε and $(u_n, v_n) \to (0, 0)$ converges strongly in $L^s_{\text{loc}}(\mathbb{R}^2) \times L^s_{\text{loc}}(\mathbb{R}^2)$ for $s \geq 2$ we deduce that

$$\lim_{n\to\infty} \int_\Omega V_1(\varepsilon x)|\nabla u_n|^2 dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega V_2(\varepsilon x)|\nabla v_n|^2 dx = 0.$$

The proof is complete. □

If $\max\{V_{1\infty}, V_{2\infty}\} < \infty$, we can use Proposition 3.6 for $W_1(x) = V_{1\infty}$ and $W_2(x) = V_{2\infty}$ to guarantee that the minimax level c_{V_∞} of the functional associated to system (P_{V_∞}), given by

$$J_{V_\infty}(u, v) = \frac{1}{2} \int_{\mathbb{R}^2} \left(V_{1\infty}|\nabla u|^2 + V_{2\infty}|\nabla v|^2 + |u|^2 + |v|^2\right) dx - \int_{\mathbb{R}^2} F(u, v) dx$$

is attained for some $(w, z) \in L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2)$, for some $0 < \iota < 1$ and $w, z > 0$.

Proposition 4.2 *Consider $\alpha_0 > 0$ is given in (F_3). Suppose that $\max\{V_{1\infty}, V_{2\infty}\} < \infty$. Let $(u_n, v_n) \subset X_\varepsilon$ be a $(PS)_d$ sequence for I_ε such that $(u_n, v_n) \rightharpoonup (0,0)$ weakly in X_ε satisfying $\limsup_{n\to\infty} \|(u_n, v_n)\|_\varepsilon^2 < \frac{\pi}{2\alpha_0}$. If $(u_n, v_n) \not\to (0,0)$ converges strongly in X_ε. Then $d \geq c_{V_\infty}$.*

Proof We consider $t_n > 0$ such that $(t_n u_n, t_n v_n) \in \mathcal{M}_{V_\infty}$. We start by proving that

$$t_0 = \limsup_{n\to\infty} t_n \leq 1.$$

Arguing by contradiction, we suppose that there exists $\lambda > 0$ and a subsequence still denoted by (t_n) such that

$$t_n \geq 1 + \lambda \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.7}$$

Since (u_n, v_n) is bounded in X_ε, we have that $I'_\varepsilon(u_n, v_n)(u_n, v_n) = o_n(1)$, or equivalently

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(V_1(\varepsilon x)|\nabla u_n|^2 + V_2(\varepsilon x)|\nabla v_n|^2 \right) dx + \int_{\mathbb{R}^2} \left(|u_n|^2 + |v_n|^2 \right) dx \\ &= \int_{\mathbb{R}^2} \left(u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n) \right) dx + o_n(1). \end{aligned} \tag{4.8}$$

Moreover, recalling that $(t_n u_n, t_n v_n) \in \mathcal{M}_{V_\infty}$, we also have

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(V_{1\infty}|\nabla u_n|^2 + V_{2\infty}|\nabla v_n|^2 \right) dx + \int_{\mathbb{R}^2} \left(|u_n|^2 + |v_n|^2 \right) dx \\ &= \int_{\mathbb{R}^2} \frac{u_n F_u(t_n u_n, t_n v_n) + v_n F_v(t_n u_n, t_n v_n)}{t_n} dx. \end{aligned} \tag{4.9}$$

Putting together (4.8) and (4.9), we get

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{u_n F_u(t_n u_n, t_n v_n) + v_n F_v(t_n u_n, t_n v_n)}{t_n} dx \\ &\quad - \int_{\mathbb{R}^2} \left(u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n) \right) dx \\ &= \int_{\mathbb{R}^2} \left[V_{1\infty} - V_1(\varepsilon x) \right] |\nabla u_n|^2 dx \\ &\quad + \int_{\mathbb{R}^2} \left[V_{2\infty} - V_2(\varepsilon x) \right] |\nabla v_n|^2 dx + o_n(1). \end{aligned} \tag{4.10}$$

Now, using assumption (V_1) we can see that, given $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that

$$V_1(\varepsilon x) \geq V_{1\infty} - \zeta, \quad V_2(\varepsilon x) \geq V_{2\infty} - \zeta \quad \text{for all} \quad |\varepsilon x| \geq R. \tag{4.11}$$

From this, using Lemma 4.1 and the boundedness of (u_n, v_n) in X_ε, we can infer that

$$\begin{aligned}
&\int_{\mathbb{R}^2} [V_{1\infty} - V_1(\varepsilon x)] |\nabla u_n|^2 dx \\
&= \int_{B_{R/\varepsilon}(0)} [V_{1\infty} - V_1(\varepsilon x)] |\nabla u_n|^2 dx \\
&\quad + \int_{\mathbb{R}^2 \setminus B_{R/\varepsilon}(0)} [V_{1\infty} - V_1(\varepsilon x)] |\nabla u_n|^2 dx \\
&\leq V_{1\infty} \int_{B_{R/\varepsilon}(0)} |\nabla u_n|^2 dx + \zeta \int_{\mathbb{R}^2 \setminus B_{R/\varepsilon}(0)} |\nabla u_n|^2 dx \\
&\leq o_n(1) + \zeta C_1,
\end{aligned} \tag{4.12}$$

for some positive constant C_1. Similarly, we get

$$\int_{\mathbb{R}^2} [V_{2\infty} - V_2(\varepsilon x)] |\nabla v_n|^2 dx \leq o_n(1) + \zeta C_1. \tag{4.13}$$

Combining (4.10), (4.12) and (4.13), we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{u_n F_u(t_n u_n, t_n v_n) + v_n F_v(t_n u_n, t_n v_n)}{t_n} dx \\
&\quad - \int_{\mathbb{R}^2} (u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n))\, dx \\
&\leq o_n(1) + 2\zeta C_1.
\end{aligned} \tag{4.14}$$

On the other hand, we can use the fact that $(u_n, v_n) \not\to (0, 0)$ to obtain $(y_n) \subset \mathbb{R}^2$ and constants $\widetilde{R}, \eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_{\widetilde{R}}(y_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geq \eta. \tag{4.15}$$

Let us consider $(w_n(x), z_n(x)) = (u_n(x + y_n), v_n(x + y_n))$. Then we may assume that, up to a subsequence, $(w_n, z_n) \rightharpoonup (w, z)$ converges weakly in X_ε. By (4.15), there exists $\Lambda \subset \mathbb{R}^2$ with positive measure and such that $w, z > 0$ in Λ. From (4.7), (F_5) and (4.14), we can infer that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{u_n F_u((1+\lambda)u_n, (1+\lambda)v_n) + v_n F_v((1+\lambda)u_n, (1+\lambda)v_n)}{(1+\lambda)} dx \\
&\quad - \int_{\mathbb{R}^2} (u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n))\, dx \\
&\leq o_n(1) + 2\zeta C_1.
\end{aligned}$$

Taking the limit as $n \to \infty$ and applying Fatou's Lemma, we obtain

$$\begin{aligned} 0 &< \int_\Lambda \frac{uF_u((1+\lambda)u,(1+\lambda)v) + vF_v((1+\lambda)u,(1+\lambda)v)}{(1+\lambda)}dx \\ &\quad - \int_\Lambda (uF_u(u,v) + vF_v(u,v))\,dx \\ &\le 2\zeta C_1. \end{aligned}$$

Since $\zeta > 0$ is arbitrary, we obtain a contradiction by taking $\zeta \to 0$. Therefore $t_0 \le 1$. Now we divide the proof in two cases:

Case 1. $t_0 < 1$.

In this case we may suppose that there is a subsequence, still denoted by (t_n), such that $t_n \to t_0$ and $t_n < 1$ for all $n \in \mathbb{N}$.

Recalling that $(t_n u_n, t_n v_n) \in \mathcal{M}_{V_\infty}$ and using (2.1), we obtain

$$\begin{aligned} c_{V_\infty} &\le J_{V_\infty}(t_n u_n, t_n v_n) - \frac{1}{2} J'_{V_\infty}(t_n u_n, t_n v_n)(t_n u_n, t_n v_n) \\ &= \frac{1}{2}\int_{\mathbb{R}^2} [t_n u_n F_u(t_n u_n, t_n v_n) + t_n u_n F_v(t_n u_n, t_n v_n) - 2F(t_n u_n, t_n v_n)]\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^2} [u_n F_u(u_n, v_n) + u_n F_v(u_n, v_n) - 2F(u_n, v_n)]\,dx \\ &= I_\varepsilon(u_n, v_n) - \frac{1}{2} I'_\varepsilon(u_n, v_n)(u_n, v_n) = d + o_n(1). \end{aligned}$$

Taking the limit as $n \to \infty$, we get $d \ge c_{V_\infty}$.

Case 2. $t_0 = 1$.

In this case, up to a subsequence, we may suppose that $t_n \to 1$. Taking into account that $I_\varepsilon(u_n, v_n) \to d$, we have

$$\begin{aligned} d + o_n(1) &= I_\varepsilon(u_n, v_n) - J_{V_\infty}(t_n u_n, t_n v_n) + J_{V_\infty}(t_n u_n, t_n v_n) \\ &\ge I_\varepsilon(u_n, v_n) - J_{V_\infty}(t_n u_n, t_n v_n) + c_{V_\infty}. \end{aligned}$$

Now, let us point out that

$$\begin{aligned} I_\varepsilon(u_n, v_n) - J_{V_\infty}(t_n u_n, t_n v_n) &= \frac{1}{2}\int_{\mathbb{R}^2} \left[V_1(\varepsilon x) - t_n^2 V_{1\infty}\right] |\nabla u_n|^2 dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^2} \left[V_2(\varepsilon x) - t_n^2 V_{2\infty}\right] |\nabla v_n|^2 dx + \frac{1}{2}(1 - t_n^2)\int_{\mathbb{R}^2} \left(|u_n|^2 + |v_n|^2\right) dx \\ &+ \int_{\mathbb{R}^2} [F(t_n u_n, t_n v_n) - F(u_n, v_n)]\,dx. \end{aligned} \tag{4.16}$$

Using condition (V_1), Lemma 4.1, the fact that $t_n \to 1$, inequality (4.11), the boundedness of (u_n, v_n) in X_ε, Sobolev embedding, and the fact that

$$V_1(\varepsilon x) - t_n^2 V_{1\infty} = V_1(\varepsilon x) - V_{1\infty} + (1 - t_n^2)V_{1\infty} \ge -\zeta + (1 - t_n^2)V_{1\infty}$$

for all $|\varepsilon x| \geq R$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left[V_1(\varepsilon x) - t_n^2 V_{1\infty}\right] |\nabla u_n|^2 dx \\
&= \int_{B_{R/\varepsilon}(0)} \left[V_1(\varepsilon x) - t_n^2 V_{1\infty}\right] |\nabla u_n|^2 dx \\
&\quad + \int_{\mathbb{R}^2 \setminus B_{R/\varepsilon}(0)} \left[V_1(\varepsilon x) - t_n^2 V_{1\infty}\right] |\nabla u_n|^2 dx \\
&\geq \left[V_0 - t_n^2 V_{1\infty}\right] \int_{B_{R/\varepsilon}(0)} |\nabla u_n|^2 dx - \zeta \int_{\mathbb{R}^2 \setminus B_{R/\varepsilon}(0)} |\nabla u_n|^2 dx \\
&\quad + (1 - t_n^2) V_{1\infty} \int_{\mathbb{R}^2 \setminus B_{R/\varepsilon}(0)} |\nabla u_n|^2 dx \\
&\geq o_n(1) - \zeta C_2, \qquad (4.17)
\end{aligned}$$

for some constant $C_2 > 0$. Similarly, we have

$$\int_{\mathbb{R}^2} \left[V_2(\varepsilon x) - t_n^2 V_{2\infty}\right] |\nabla v_n|^2 dx \geq o_n(1) - \zeta C_2. \qquad (4.18)$$

Since (u_n, v_n) is bounded in X_ε, we conclude that

$$\frac{1}{2}(1 - t_n^2) \int_{\mathbb{R}^2} \left(|u_n|^2 + |v_n|^2\right) dx = o_n(1). \qquad (4.19)$$

On the other hand, using (2.2), $\limsup_{n\to\infty} \|(u_n, v_n)\|_\varepsilon^2 < \dfrac{\pi}{2\alpha_0}$, Lemma 2.2 and the fact that $t_n \to 1$, we have

$$\int_{\mathbb{R}^2} [F(t_n u_n, t_n v_n) - F(u_n, v_n)]\, dx = o_n(1). \qquad (4.20)$$

Thus, putting together (4.16), (4.17), (4.18), (4.19) and (4.20) we get

$$I_\varepsilon(u_n, v_n) - J_{V_\infty}(t_n u_n, t_n v_n) \geq o_n(1) - 2\zeta C_2.$$

Therefore

$$d + o_n(1) \geq o_n(1) - 2\zeta C_2 + c_{V_\infty}$$

and taking the limit as $n \to \infty$ and the arbitrariness of ζ we have $d \geq c_{V_\infty}$. □

Corollary 4.3 *Consider $\alpha_0 > 0$ is given in (F_3). Suppose that $\max\{V_{1\infty}, V_{2\infty}\} = \infty$. If (u_n, v_n) is a $(PS)_d$ sequence for I_ε satisfying $\limsup_{n\to\infty} \|(u_n, v_n)\|_\varepsilon^2 < \dfrac{\pi}{2\alpha_0}$ and $(u_n, v_n) \rightharpoonup (0, 0)$ converges weakly in X_ε, then $(u_n, v_n) \to (0, 0)$ converges strongly in X_ε.*

Proof For any $a, b > 0$ we have by Proposition 3.6 for $W_1(x) = a$ and $W_2(x) = b$ that the number c_{ab} is achieved by any $(u, v) \in X_W$. If $a > \widetilde{a}$, then we can argue as in [15] to get $c_{ab} > c_{\widetilde{a}b}$. Moreover, $\lim_{a^2+b^2\to\infty} c_{ab} = \infty$.

Since $\max\{V_{1\infty}, V_{2\infty}\} = \infty$ we can take $a, b > 0$ in such way that $c_{ab} > d$ and, for any given $\zeta > 0$, there exists $R > 0$ such that

$$V_1(\varepsilon x) \geq a - \zeta, \quad V_2(\varepsilon x) \geq b - \zeta \quad \text{for all} \quad |\varepsilon x| \geq R. \tag{4.21}$$

For example, if $V_{2\infty} < \infty$, we take $b = V_{2\infty}$ and $a > 0$ large. If $V_{1\infty} = V_{2\infty} = \infty$ we take both a and b large.

Now, we suppose by contradiction that $(u_n, v_n) \not\to (0, 0)$. Then, we can use (4.21) and argue as in the proof of Proposition 4.2 to conclude that $d \geq c_{ab}$, which does not make sense. Hence $(u_n, v_n) \to (0, 0)$ and the corollary is proved. □

Lemma 4.4 *Consider $\alpha_0 > 0$ is given in (F_3). Let (u_n, v_n) be a sequence satisfying $\limsup_{n\to\infty} \|(u_n, v_n)\|_\varepsilon^2 < \frac{\pi}{8\alpha_0}$. Assume that $(u_n, v_n) \rightharpoonup (u, v)$. Then it follows that*

(i) $\int_{\mathbb{R}^2} \big|F(w_n, z_n) - F(u_n, v_n) + F(u, v)\big|dx = o_n(1)$,
where $(w_n, z_n) = (u_n - u, v_n - v)$.

(ii) There exists $r > 1$, r close to 1, such that

$$\int_{\mathbb{R}^2} \big|F_w(w_n, z_n) - F_u(u_n, v_n) + F_u(u, v)\big|^r dx = o_n(1)$$

and

$$\int_{\mathbb{R}^2} \big|F_z(w_n, z_n) - F_v(u_n, v_n) + F_v(u, v)\big|^r dx = o_n(1).$$

Proof We will prove only (i) because the same arguments can be used in the proof of (ii).

We observe that

$$\big|F(w_n + u, z_n + v) - F(w_n, z_n)\big| = \left|\int_0^1 \frac{d}{dt} F(w_n + tu, z_n + tv)dt\right|.$$

Thus, using (2.2), given $\delta > 0$, there exists $C_\delta > 0$ and $q > 2$ such that

$$\begin{aligned}
&\big|F(w_n + u, z_n + v) - F(w_n, z_n)\big| \\
&\leq \delta(|u| + |v|)\,(|w_n| + |z_n| + |u| + |v|) \\
&\quad + 2^{q-2} C_\delta (|u| + |v|)\left(|w_n|^{q-1} + |z_n|^{q-1} + |u|^{q-1} + |v|^{q-1}\right) \\
&\quad \times \left[\exp\left(\alpha\big((|w_n| + |u|)^2 + (|z_n| + |v|)^2\big)\right) - 1\right].
\end{aligned}$$

For simplicity, we use the following notations

$$\widetilde{w}_n = |w_n| + |u|, \ \widetilde{z}_n = |z_n| + |v|,$$

$$H_n = \exp\left(\alpha(|\widetilde{w}_n|^2 + |\widetilde{z}_n|^2)\right) - 1 \quad \text{and} \quad H = \exp\left(\alpha(|u|^2 + |v|^2)\right) - 1.$$

In this way, using (2.3)

$$\begin{aligned}
&\left|F(w_n + u, z_n + v) - F(w_n, z_n) - F(u, v)\right| \\
&\leq \delta(|u| + |v|)\,(|w_n| + |z_n| + |u| + |v|) + \delta(|u|^2 + |v|^2) + \widetilde{C}_\delta(|u|^q + |v|^q)H \\
&\quad + 2^{q-2}C_\delta(|u| + |v|)\left(|w_n|^{q-1} + |z_n|^{q-1} + |u|^{q-1} + |v|^{q-1}\right)H_n.
\end{aligned}$$

On the other hand,

$$\|(w_n, z_n)\|_\varepsilon^2 = \|(u_n, v_n)\|_\varepsilon^2 - \|(u, v)\|_\varepsilon^2 + o_n(1).$$

Then

$$\|(u, v)\|_\varepsilon^2 \leq \limsup_{n\to\infty}\left(\|(u, v)\|_\varepsilon^2 + \|(w_n, z_n)\|_\varepsilon^2\right) < \frac{\pi}{8\alpha_0} \tag{4.22}$$

and

$$\limsup_{n\to\infty} \|(\widetilde{w}_n, \widetilde{z}_n)\|_\varepsilon^2 < \left(\frac{1}{8} + \frac{1}{8}\right)\frac{\pi}{\alpha_0} < \frac{\pi}{2\alpha_0}. \tag{4.23}$$

If we define

$$\begin{aligned}
Q_n := {} & \delta(|u| + |v|)\,(|w_n| + |z_n| + |u| + |v|) \\
& + \delta(|u|^2 + |v|^2) + \widetilde{C}_\delta(|u|^q + |v|^q)H \\
& + 2^{q-2}C_\delta(|u| + |v|)\left(|w_n|^{q-1} + |z_n|^{q-1} + |u|^{q-1} + |v|^{q-1}\right)H_n
\end{aligned}$$

and

$$\begin{aligned}
Q := {} & \delta(|u| + |v|)^2 + \delta\left(|u|^2 + |v|^2\right) + \widetilde{C}_\delta(|u|^q + |v|^q)H \\
& + 2^{q-2}C_\delta(|u|^{q-1} + |v|^{q-1})H
\end{aligned}$$

we have that $Q_n(x) \to Q(x)$ a.e. in $\mathbb{R}^2$.

For $s > 1$, we can take s close to 1 and α close to α_0 such that $1 < \dfrac{s\alpha}{\alpha_0} < 2$. Thus, using (4.22) and (4.23) we can show that $H_n, H \in L^s(\mathbb{R}^2)$ and there exists $C_1 > 0$ such that $|H_n|_{L^s(\mathbb{R}^2)} \leq C_1$. Therefore, by Brezis-Lieb Lemma [17, Lemma 4.8], we conclude that

$$\int_{\mathbb{R}^2} Q_n dx \to \int_{\mathbb{R}^2} Q dx \ \text{ as } \ n \to \infty.$$

Since

$$\left|F(w_n + u, z_n + v) - F(w_n, z_n) - F(u, v)\right| \to 0 \ \text{ a.e. in } \mathbb{R}^2$$

we can apply the Generalized Dominated Convergence Theorem to get

$$\int_{\mathbb{R}^2} \left|F(w_n + u, z_n + v) - F(w_n, z_n) - F(u, v)\right| dx \to 0 \ \text{ as } \ n \to \infty.$$

The lemma is proved. □

Proposition 4.5 *Consider $\alpha_0 > 0$ is given in (F_3). Let (u_n, v_n) be a $(PS)_c$ sequence for I_ε satisfying $\limsup_{n\to\infty} \|(u_n, v_n)\|_\varepsilon^2 < \frac{\pi}{2\alpha_0}$. Assume that $c < c_{V_\infty}$ when $\max\{V_{1\infty}, V_{2\infty}\} < \infty$, or $c \in \mathbb{R}$ if $\max\{V_{1\infty}, V_{2\infty}\} = \infty$. Then (u_n, v_n) has a convergent subsequence in X_ε.*

Proof Let $(u_n, v_n) \subset X_\varepsilon$ be a sequence verifying

$$I_\varepsilon(u_n, v_n) \to c \quad \text{and} \quad I'_\varepsilon(u_n, v_n) \to 0 \quad \text{in} \quad X'_\varepsilon.$$

Since (u_n, v_n) is bounded in X_ε, then, up to a subsequence we have $(u_n, v_n) \rightharpoonup (u, v)$ converges weakly in X_ε with (u, v) being a critical point of I_ε. Moreover, using Lemma 4.4 we can show that

$$I_\varepsilon(u_n - u, v_n - v) = I_\varepsilon(u_n, v_n) - I_\varepsilon(u, v) + o_n(1)$$

and

$$\|I'_\varepsilon(u_n - u, v_n - v) - I'_\varepsilon(u_n, v_n) + I'_\varepsilon(u, v)\| = o_n(1).$$

Thus, $I'_\varepsilon(u_n - u, v_n - v) \to 0$ and

$$\lim_{n\to\infty} I_\varepsilon(u_n - u, v_n - v) = c - I_\varepsilon(u, v) =: d.$$

Recalling that $I'_\varepsilon(u, v) = 0$ and using (F_4) we get

$$\begin{aligned} I_\varepsilon(u, v) &= I_\varepsilon(u, v) - \frac{1}{\theta} I'_\varepsilon(u, v)(u, v) \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|(u, v)\|_\varepsilon^2 + \frac{1}{\theta} \int_{\mathbb{R}^2} [u F_u(u, v) + v F_v(u, v) - \theta F(u, v)]\, dx \\ &\geq 0 \end{aligned}$$

and therefore $d \leq c < c_{V_\infty}$. If $\max\{V_{1\infty}, V_{2\infty}\} < \infty$, it follows from Proposition 4.2 that $(u_n - u, v_n - v) \to (0, 0)$ converges strongly or equivalently, $(u_n, v_n) \to (u, v)$ converges strongly in X_ε. In view of Corollary 4.3, the same occurs in the complementary case $\max\{V_{1\infty}, V_{2\infty}\} = \infty$. The proposition is proved. □

As before, we denote by c_{V_0} the minimax level of the functional $J_{V_0} : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$J_{V_0}(u, v) = \frac{1}{2} \int_{\mathbb{R}^2} \left(V_{10}|\nabla u|^2 + V_{20}|\nabla v|^2 + |u|^2 + |v|^2\right) dx - \int_{\mathbb{R}^2} F(u, v) dx.$$

Thus, we get the following result:

Lemma 4.6 *We have*

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \leq c_{V_0}.$$

Proof Let $\psi \in C_0^\infty(\mathbb{R}^2, [0,1])$ be such that $\psi(x) = 1$, for $|x| < 1$ and $\psi(x) = 0$, for $|x| \geq 2$. For any $r > 0$ we define

$$(w_{1,r}(x), w_{2,r}(x)) = (\psi(x/r)w_1(x), \psi(x/r)w_2(x)),$$

where (w_1, w_2) is a ground state solution of the system (P_{V_0}) given by Proposition 3.6, by considering $W_1(x) = V_{10}$ and $W_2(x) = V_{20}$ for all $x \in \mathbb{R}^2$.

Let $t_{\varepsilon,r} > 0$ be such that $(t_{\varepsilon,r}w_{1,r}, t_{\varepsilon,r}w_{2,r}) \in \mathcal{N}_\varepsilon$. Then

$$\begin{aligned} c_\varepsilon &\leq I_\varepsilon(t_{\varepsilon,r}w_{1,r}, t_{\varepsilon,r}w_{2,r}) \\ &= \frac{t_{\varepsilon,r}^2}{2}\int_{\mathbb{R}^2}\Big(V_1(\varepsilon x)|\nabla w_{1,r}|^2 + V_2(\varepsilon x)|\nabla w_{2,r}|^2 + |w_{1,r}|^2 + |w_{2,r}|^2\Big)dx \\ &\quad - \int_{\mathbb{R}^2} F(t_{\varepsilon,r}w_{1,r}, t_{\varepsilon,r}w_{2,r})dx. \end{aligned}$$

It is easy to check that, for r fixed, $t_{\varepsilon,r} \to t_r > 0$ as $\varepsilon \to 0^+$. Moreover, without loss of generality, we may suppose that $V_1(0) = V_{10}$ and $V_2(0) = V_{20}$. Hence, since $(w_{1,r}, w_{2,r})$ has compact support, we can use Lebesgue's Theorem to get

$$\begin{aligned} \limsup_{\varepsilon\to 0^+} c_\varepsilon &\leq \frac{t_r^2}{2}\int_{\mathbb{R}^2}\Big(V_{10}|\nabla w_{1,r}|^2 + V_{20}|\nabla w_{2,r}|^2 + |w_{1,r}|^2 + |w_{2,r}|^2\Big)dx \\ &\quad - \int_{\mathbb{R}^2} F(t_r w_{1,r}, t_r w_{2,r})dx = J_{V_0}(t_r w_{1,r}, t_r w_{2,r}). \end{aligned}$$

Since $(w_1, w_2) \in \mathcal{M}_{V_0}$ and $(w_{1,r}, w_{2,r}) \to (w_1, w_2)$ in $H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ as $r \to \infty$, we can check that $t_r \to 1$ as $r \to \infty$. Thus, it follows from the above expression that

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \leq \lim_{r\to\infty} J_{V_0}(t_r w_{1,r}, t_r w_{2,r}) = J_{V_0}(w_1, w_2) = c_{V_0}.$$

The lemma is proved. □

Theorem 4.7 *Assume that $(V_1) - (V_2)$ and $(F_1) - (F_6)$ are held. Then, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, problem $(\widehat{S}_\varepsilon)$ admits a positive ground state solution.*

Proof From Lemma 4.6, we obtain $\varepsilon_0 > 0$ such that $c_\varepsilon < c_{V_0}$ for any $\varepsilon \in (0, \varepsilon_0)$. For these values of ε, since I_ε has the Mountain Pass Geometry, we can take a sequence $(u_n, v_n) \subset X_\varepsilon$ such that

$$I_\varepsilon(u_n, v_n) \to c_\varepsilon \quad \text{and} \quad I'_\varepsilon(u_n, v_n) \to 0 \quad \text{in} \quad X'_\varepsilon.$$

Thus

$$\limsup_{n\to\infty} \|(u_n, v_n)\|_\varepsilon^2 \leq \frac{2\theta}{\theta - 2}c_\varepsilon.$$

Considering $W_1(x) = V_{10}$ and $W_2(x) = V_{20}$ for all $x \in \mathbb{R}^2$, we guarantee by Lemma 3.4 that

$$c_{V_0} \leq \frac{c_{\beta,\nu}}{\sigma^{\frac{2}{\beta+\nu-2}}}.$$

Thus

$$\limsup_{n\to\infty} \|(u_n, v_n)\|_\varepsilon^2 \leq \frac{2\theta c_{\beta,\nu}}{(\theta-2)\sigma^{\frac{2}{\beta+\nu-2}}}.$$

If we take

$$\sigma^* = \left[\frac{16\theta\alpha_0\, c_{\beta,\nu}}{\pi(\theta-2)}\right]^{(\beta+\nu-2)/2},$$

then, for $\sigma > \sigma^*$,

$$\limsup_{n\to\infty} \|(u_n, v_n)\|_\varepsilon^2 < \frac{\pi}{8\alpha_0}.$$

Now, since $c_{V_0} < c_{V_\infty}$ we can use Proposition 4.5 to get, along a subsequence $(u_n, v_n) \to (u_\varepsilon, v_\varepsilon)$ with $(u_\varepsilon, v_\varepsilon)$ being a critical point of I_ε and $I_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon$. Arguing as in the proof Theorem 3.2, we can show that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ in $\mathbb{R}^2$. □

Proposition 4.8 *Let $(\varepsilon_n) \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0^+$ and (u_n, v_n) be a solution of system $(\widehat{S}_{\varepsilon_n})$ satisfying $I_{\varepsilon_n}(u_n, v_n) \to c_{V_0}$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^2$ such that the sequence $(w_n, z_n) = (u_n(\cdot + \widetilde{y}_n), v_n(\cdot + \widetilde{y}_n))$ has a convergent subsequence in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. Moreover, up to a subsequence, $(y_n) = (\varepsilon_n \widetilde{y}_n)$ is such that $y_n \to y \in M$.*

Proof Arguing as in the proof of Proposition 3.6, we obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^2$ and constants $R, \eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\widetilde{y}_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geq \eta. \tag{4.24}$$

Let us define

$$(w_n(x), z_n(x)) = (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n)).$$

In view of the boundedness of (w_n, z_n) in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ and (4.24), we may suppose, up to a subsequence that $(w_n, z_n) \rightharpoonup (w, z)$ converges weakly in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ for some $(w, z) \neq (0, 0)$.

Let $t_n > 0$ be such that $(t_n w_n, t_n z_n) = (\widetilde{w}_n, \widetilde{z}_n) \in \mathcal{M}_{V_0}$. Thus, by using the change of variables $x \mapsto x + \widetilde{y}_n$, $V_1(x) \geq V_{10}$, $V_2(x) \geq V_{20}$, the invariance by translation and the fact that $(u_n, v_n) \in \mathcal{N}_{\varepsilon_n}$, one can have

$$c_{V_0} \leq J_{V_0}(\widetilde{w}_n, \widetilde{z}_n) \leq I_{\varepsilon_n}(t_n u_n, t_n v_n) \leq I_{\varepsilon_n}(u_n, v_n) = c_{V_0} + o_n(1),$$

which implies

$$J_{V_0}(\widetilde{w}_n, \widetilde{z}_n) \to c_{V_0}.$$

Since (w_n, z_n) and $(\widetilde{w}_n, \widetilde{z}_n)$ are bounded sequences and $(w_n, z_n) \not\to (0, 0)$, the sequence (t_n) is bounded. Thus, up to a subsequence, $t_n \to t_0 \geq 0$. Let us show that

$t_0 > 0$. Otherwise, if $t_0 = 0$, by the boundedness of (w_n, z_n), we get $(\widetilde{w}_n, \widetilde{z}_n) \to (0, 0)$ in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, that is $J_{V_0}(\widetilde{w}_n, \widetilde{z}_n) \to 0$ which contradicts with $c_{V_0} > 0$. Thus $t_0 > 0$ and, up to a subsequence, we may assume that $(\widetilde{w}_n, \widetilde{z}_n) \rightharpoonup (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ with $(\widetilde{w}, \widetilde{z}) = (t_0 w, t_0 z) \neq (0, 0)$. From Proposition 3.6, we conclude that $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, that is $(w_n, z_n) \to (w, z)$ in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.

Next, we prove that $(y_n) = (\varepsilon_n \widetilde{y}_n)$ has a subsequence satisfying $y_n \to y \in M$. We first claim that (y_n) is bounded. In fact, assume by contradiction that (y_n) is not bounded. Then, there exists a subsequence, still denoted by (y_n), such that $|y_n| \to \infty$. Now, we consider two different cases.

Case 1. $\max\{V_{1\infty}, V_{2\infty}\} = \infty$.

Since (u_n, v_n) is a solution of system $(\widehat{S}_{\varepsilon_n})$, that is $I_{\varepsilon_n}(u_n, v_n) = c_{\varepsilon_n}$ and $I'_{\varepsilon_n}(u_n, v_n)(u_n, v_n) = 0$, we have that

$$\int_{\mathbb{R}^2} \left(V_1(\varepsilon_n x + y_n)|\nabla w_n|^2 + V_2(\varepsilon_n x + y_n)|\nabla z_n|^2 + |w_n|^2 + |z_n|^2 \right) dx = \int_{\mathbb{R}^2} [u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n)]\, dx.$$

By applying Fatou's Lemma we obtain

$$\infty \leq \liminf_{n\to\infty} \int_{\mathbb{R}^2} [u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n)]\, dx.$$

On the other hand, using (2.2) we get

$$\int_{\mathbb{R}^2} \left| u_n F_u(u_n, v_n) + v_n F_v(u_n, v_n) \right| dx \leq 2\delta \int_{\mathbb{R}^2} \left(|u_n|^2 + |v_n|^2 \right) dx + 2C_\delta \int_{\mathbb{R}^2} \left(|u_n|^q + |v_n|^q \right) H_n \, dx,$$

where $H_n = \left[\exp\left(\alpha(|u_n|^2 + |v_n|^2)\right) - 1\right]$. Since $I_{\varepsilon_n}(u_n, v_n) = c_{\varepsilon_n}$, for $r > 1$ close to 1, we can show that (H_n) is bounded in $L^r(\mathbb{R}^2)$. Then, by using Hölder's inequality, the boundedness of (u_n, v_n) in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ and Sobolev embedding, we get that the right hand side in the above expression is bounded. Thus, we obtain a contradiction.

Case 2. $\max\{V_{1\infty}, V_{2\infty}\} < \infty$.

In this case, since $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ converges strongly in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, $V_{10} < V_{1\infty}$, $V_{20} < V_{2\infty}$ and using the change of variables $x \mapsto x + \widetilde{y}_n$, we have

$$\begin{aligned} c_{V_0} &= J_{V_0}(\widetilde{w}, \widetilde{z}) \\ &< \frac{1}{2} \int_{\mathbb{R}^2} \left(V_{1\infty}|\nabla \widetilde{w}|^2 + V_{2\infty}|\nabla \widetilde{z}|^2 + |\widetilde{w}|^2 + |\widetilde{z}|^2 \right) dx - \int_{\mathbb{R}^2} F(\widetilde{w}, \widetilde{z}) dx \\ &\leq \liminf_{n\to\infty} \left[\frac{1}{2} \int_{\mathbb{R}^2} \left(V_1(\varepsilon_n x + y_n)|\nabla \widetilde{w}_n|^2 + V_2(\varepsilon_n x + y_n)|\nabla \widetilde{z}_n|^2 \right) dx \right. \end{aligned}$$

$$+\frac{1}{2}\int_{\mathbb{R}^2}\left(|\widetilde{w}_n|^2+|\widetilde{z}_n|^2\right)dx-\int_{\mathbb{R}^2}F(\widetilde{w}_n,\widetilde{z}_n)dx\Big]$$
$$=\liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n,t_n v_n)\le\liminf_{n\to\infty} I_{\varepsilon_n}(u_n,v_n)=c_{V_0},$$

which is a contradiction. Hence (y_n) is bounded and, up to a subsequence, we may assume that $y_n\to y$. If $y\notin M$ then $V_{10}<V_1(y)$, $V_{20}<V_2(y)$ and we have that

$$c_{V_0}<\frac{1}{2}\int_{\mathbb{R}^2}\left(V_1(y)|\nabla\widetilde{w}|^2+V_2(y)|\nabla\widetilde{z}|^2+|\widetilde{w}|^2+|\widetilde{z}|^2\right)dx-\int_{\mathbb{R}^2}F(\widetilde{w},\widetilde{z})dx.$$

This inequality and the similar calculations performed in Case 2 provide a contradiction. Therefore, we can conclude that $y\in M$. □

4.2 A positive weak solution with minimal energy

In this section, we prove the existence of a positive solution with minimum energy for system (S_ε), i.e. we present the proof of Theorem 1.2.

Proof of Theorem 1.2 From Theorem 4.7, there exists $\varepsilon_0>0$ such that, for any $\varepsilon\in(0,\varepsilon_0)$, we obtain a positive solution $(u_\varepsilon,v_\varepsilon)$ of system $(\widehat{S}_\varepsilon)$.

We consider $\varepsilon_n\to 0^+$ and take $(u_n,v_n)\in X_{\varepsilon_n}$ solution of system $(\widehat{S}_{\varepsilon_n})$. By applying Lemma 4.9 in Appendix we obtain $R>0$ and $(\widetilde{y}_n)\subset\mathbb{R}^2$ such that

$$|(u_n,v_n)|_{L^\infty(\mathbb{R}^2\setminus B_R(\widetilde{y}_n))}<\zeta.$$

Now, we proceed by contradiction and use (4.24) to obtain

$$|(u_n,v_n)|_{L^\infty(B_R(\widetilde{y}_n))}>\zeta. \tag{4.25}$$

From (4.25) we conclude that the maximum point $\pi_{\varepsilon_n,1}\in\mathbb{R}^2$ of u_n and the maximum point $\pi_{\varepsilon_n,2}\in\mathbb{R}^2$ of v_n belong to $B_R(\widetilde{y}_n)$. Hence $\pi_{\varepsilon_n,1}=\widetilde{y}_n+q_{n,1}$, for some $q_{n,1}\in B_R(0)$ and $\pi_{\varepsilon_n,2}=\widetilde{y}_n+q_{n,2}$, for some $q_{n,2}\in B_R(0)$. Recalling that the associated solution of (S_{ε_n}) is of the form $(\widehat{u}_n(x),\widehat{v}_n(x))=(u_n(x/\varepsilon_n),v_n(x/\varepsilon_n))$, we conclude that the maximum point $\Pi_{\varepsilon_n,1}$ of $\widehat{u}_n$ and the maximum point $\Pi_{\varepsilon_n,2}$ of $\widehat{v}_n$ are

$$\Pi_{\varepsilon_n,1}=\varepsilon_n\widetilde{y}_n+\varepsilon_n q_{n,1}\quad\text{and}\quad\Pi_{\varepsilon_n,2}=\varepsilon_n\widetilde{y}_n+\varepsilon_n q_{n,2}.$$

Since $(q_{n,1}),(q_{n,2})\subset B_R(0)$ are bounded and $\varepsilon_n\widetilde{y}_n\to y\in M$, we obtain

$$\lim_{n\to\infty}V_1(\Pi_{\varepsilon_n,1})=V_1(y)=V_{10}\quad\text{and}\quad\lim_{n\to\infty}V_2(\Pi_{\varepsilon_n,2})=V_2(y)=V_{20}.$$

The last limits imply that the maximum points of $\widehat{u}_n$ and $\widehat{v}_n$ are concentrated around the set M. The positivity and the regularity of the solutions follow the arguments that can be found in the proof of Theorem 3.2. □

Appendix

Lemma 4.9 *Let (ε_n) be a sequence such that $\varepsilon_n \to 0^+$ and for each $n \in \mathbb{N}$ let (u_n, v_n) be a solution of system $(\widehat{S}_{\varepsilon_n})$. Then $(u_n, v_n) \in L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$. Moreover, given $\zeta > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$|(w_n, z_n)|_{L^\infty(\mathbb{R}^2 \setminus B_R(0))} < \zeta, \quad \text{for all} \quad n \geq n_0,$$

where $w_n(x) = u_n(x + \widetilde{y}_n)$, $z_n(x) = v_n(x + \widetilde{y}_n)$ and $(\widetilde{y}_n)$ is the sequence given in Proposition 4.8.

Proof Since (u_n, v_n) is a solution of the system $(\widehat{S}_{\varepsilon_n})$, (w_n, z_n), verifies

$$(S_n) \qquad \begin{cases} -\operatorname{div}(V_{1,n}(x)\nabla w_n) + w_n = F_w(w_n, z_n) & \text{in} \quad \mathbb{R}^2 \\ -\operatorname{div}\left(V_{2,n}(x)\nabla z_n\right) + z_n = F_z(w_n, z_n) & \text{in} \quad \mathbb{R}^2, \\ w_n, z_n \in H^1(\mathbb{R}^2),\ w_n(x), z_n(x) > 0, & \text{in} \quad \mathbb{R}^2, \end{cases}$$

where $V_{1,n}(x) = V_1(\varepsilon_n x + y_n)$, $V_{2,n}(x) = V_2(\varepsilon_n x + y_n)$ and $y_n = \varepsilon_n \widetilde{y}_n$.

Fixed $R > 0$, let $\eta \in C^\infty(\mathbb{R}^2)$, $0 \leq \eta \leq 1$, $\eta(x) = 1$ if $|x| \geq R$, $\eta(x) = 0$ if $|x| \leq R/2$ and $|\nabla \eta|_\infty < 2/R$.

For each $n \in \mathbb{N}$ and $L > 0$, we define

$$w_{L,n}(x) = \min\{w_n(x), L\}, \quad \Upsilon_{w,L,n} = \eta^2 w_n w_{L,n}^{2(\tau-1)},$$

and

$$z_{L,n}(x) = \min\{z_n(x), L\}, \quad \Upsilon_{z,L,n} = \eta^2 z_n z_{L,n}^{2(\tau-1)}$$

with $\tau > 1$ to be determined later.

Using $(\Upsilon_{w,L,n}, \Upsilon_{z,L,n})$ as a test function in (S_n) we get

$$\begin{aligned} &\int_{\mathbb{R}^2} V_{1,n}(x)\eta^2 w_{L,n}^{2(\tau-1)}|\nabla w_n|^2 dx + \int_{\mathbb{R}^2} V_{2,n}(x)\eta^2 z_{L,n}^{2(\tau-1)}|\nabla z_n|^2 dx \\ &+ \int_{\mathbb{R}^2} \left(\eta^2 w_n^2 w_{L,n}^{2(\tau-1)} + \eta^2 z_n^2 z_{L,n}^{2(\tau-1)}\right) dx \\ &= -2\int_{\mathbb{R}^2} V_{1,n}(x)\eta w_n w_{L,n}^{2(\tau-1)}\nabla w_n \nabla \eta dx - 2\int_{\mathbb{R}^2} V_{2,n}(x)\eta z_n z_{L,n}^{2(\tau-1)}\nabla z_n \nabla \eta dx \\ &-2(\tau-1)\int_{\mathbb{R}^2} V_{1,n}(x)\eta^2 w_n w_{L,n}^{2(\tau-1)-1}\nabla w_n \nabla w_{L,n} dx \\ &-2(\tau-1)\int_{\mathbb{R}^2} V_{2,n}(x)\eta^2 z_n z_{L,n}^{2(\tau-1)-1}\nabla z_n \nabla z_{L,n} dx \\ &+ \int_{\mathbb{R}^2} \left(\eta^2 w_n w_{L,n}^{2(\tau-1)} F_w(w_n, z_n) + \eta^2 z_n z_{L,n}^{2(\tau-1)} F_z(w_n, z_n)\right) dx. \end{aligned}$$

From (F_2) and (F_3), given $\delta > 0$, there exists $C_\delta > 0$ and $2^* > 2$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\eta^2 w_n w_{L,n}^{2(\tau-1)} F_w(w_n, z_n) + \eta^2 z_n z_{L,n}^{2(\tau-1)} F_z(w_n, z_n)\right) dx \\
&\leq \delta \int_{\mathbb{R}^2} \left(\eta^2 w_n^2 w_{L,n}^{2(\tau-1)} + \eta^2 z_n^2 z_{L,n}^{2(\tau-1)}\right) dx \\
&\quad + C_\delta \int_{\mathbb{R}^2} \left(\eta^2 w_n^{2^*} w_{L,n}^{2(\tau-1)} + \eta^2 z_n^{2^*} z_{L,n}^{2(\tau-1)}\right) H_n dx,
\end{aligned}$$

where $H_n = \left[\exp\left(\alpha(w_n^2 + z_n^2)\right) - 1\right]$.

Thus, taking $\delta > 0$ sufficiently small and using the assumption (V_1), we get

$$\begin{aligned}
&V_{10} \int_{\mathbb{R}^2} \eta^2 w_{L,n}^{2(\tau-1)} |\nabla w_n|^2 dx + V_{20} \int_{\mathbb{R}^2} \eta^2 z_{L,n}^{2(\tau-1)} |\nabla z_n|^2 dx \\
&\leq C_{1,R} \int_{\mathbb{R}^2} \eta w_n w_{L,n}^{2(\tau-1)} |\nabla w_n| |\nabla \eta| dx + C_{2,R} \int_{\mathbb{R}^2} \eta z_n z_{L,n}^{2(\tau-1)} |\nabla z_n| |\nabla \eta| dx \\
&\quad + C_\delta \int_{\mathbb{R}^2} \left(\eta^2 w_n^{2^*} w_{L,n}^{2(\tau-1)} + \eta^2 z_n^{2^*} z_{L,n}^{2(\tau-1)}\right) H_n dx,
\end{aligned}$$

where we have used $w_n, z_n > 0$ and the inequalities below

$$\begin{aligned}
&\int_{\mathbb{R}^2} V_{1,n}(x) \eta^2 w_n w_{L,n}^{2(\tau-1)-1} \nabla w_n \nabla w_{L,n} dx \\
&= \int_{\{w_n \leq L\}} V_{1,n}(x) \eta^2 w_{L,n}^{2(\tau-1)} |\nabla w_n|^2 dx \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} V_{2,n}(x) \eta^2 z_n z_{L,n}^{2(\tau-1)-1} \nabla z_n \nabla z_{L,n} dx \\
&= \int_{\{z_n \leq L\}} V_{2,n}(x) \eta^2 z_{L,n}^{2(\tau-1)} |\nabla z_n|^2 dx \geq 0.
\end{aligned}$$

For any $\varrho > 0$, we can apply Young's inequality to obtain

$$\begin{aligned}
&V_{10} \int_{\mathbb{R}^2} \eta^2 w_{L,n}^{2(\tau-1)} |\nabla w_n|^2 dx + V_{20} \int_{\mathbb{R}^2} \eta^2 z_{L,n}^{2(\tau-1)} |\nabla z_n|^2 dx \\
&\leq \varrho C_{1,R} \int_{\mathbb{R}^2} \eta^2 w_{L,n}^{2(\tau-1)} |\nabla w_n|^2 dx + C_\varrho C_{1,R} \int_{\mathbb{R}^2} w_n^2 w_{L,n}^{2(\tau-1)} |\nabla \eta|^2 dx \\
&\quad + \varrho C_{2,R} \int_{\mathbb{R}^2} \eta^2 z_{L,n}^{2(\tau-1)} |\nabla z_n|^2 dx + C_\varrho C_{2,R} \int_{\mathbb{R}^2} z_n^2 z_{L,n}^{2(\tau-1)} |\nabla \eta|^2 dx \\
&\quad + C_\delta \int_{\mathbb{R}^2} \left(\eta^2 w_n^{2^*} w_{L,n}^{2(\tau-1)} + \eta^2 z_n^{2^*} z_{L,n}^{2(\tau-1)}\right) H_n dx.
\end{aligned}$$

By choosing $\varrho > 0$ sufficiently small, there exists $C_1 > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \eta^2 w_{L,n}^{2(\tau-1)} |\nabla w_n|^2 dx + \int_{\mathbb{R}^2} \eta^2 z_{L,n}^{2(\tau-1)} |\nabla z_n|^2 dx \\
&\leq C_1 \int_{\mathbb{R}^2} w_n^2 w_{L,n}^{2(\tau-1)} |\nabla \eta|^2 dx + C_1 \int_{\mathbb{R}^2} z_n^2 z_{L,n}^{2(\tau-1)} |\nabla \eta|^2 dx \\
&\quad + C_1 \int_{\mathbb{R}^2} \left(\eta^2 w_n^{2^*} w_{L,n}^{2(\tau-1)} + \eta^2 z_n^{2^*} z_{L,n}^{2(\tau-1)} \right) H_n dx.
\end{aligned}$$

If we define $\widehat{w}_{L,n} = \eta w_n w_{L,n}^{\tau-1}$ and $\widehat{z}_{L,n} = \eta z_n z_{L,n}^{\tau-1}$, we can use Sobolev embedding and the above inequality to get

$$\begin{aligned}
&|\widehat{w}_{L,n}|^2_{L^{2^*}(\mathbb{R}^2)} + |\widehat{z}_{L,n}|^2_{L^{2^*}(\mathbb{R}^2)} \\
&\leq C_2 \int_{\mathbb{R}^2} \left| \nabla \left(\eta w_n w_{L,n}^{\tau-1} \right) \right|^2 dx + C_2 \int_{\mathbb{R}^2} \left| \nabla \left(\eta z_n z_{L,n}^{\tau-1} \right) \right|^2 dx \\
&\leq C_3 \int_{\mathbb{R}^2} w_n^2 w_{L,n}^{2(\tau-1)} |\nabla \eta|^2 dx + C_3 \tau^2 \int_{\mathbb{R}^2} \eta^2 w_{L,n}^{2(\tau-1)} |\nabla w_n|^2 dx \\
&\quad + C_3 \int_{\mathbb{R}^2} z_n^2 z_{L,n}^{2(\tau-1)} |\nabla \eta|^2 dx + C_3 \tau^2 \int_{\mathbb{R}^2} \eta^2 z_{L,n}^{2(\tau-1)} |\nabla z_n|^2 dx \\
&\leq C_4 \tau^2 \left(\int_{\mathbb{R}^2} w_n^2 w_{L,n}^{2(\tau-1)} |\nabla \eta|^2 + \int_{\mathbb{R}^2} \eta^2 w_n^{2^*} w_{L,n}^{2(\tau-1)} H_n dx \right) \\
&\quad + C_4 \tau^2 \left(\int_{\mathbb{R}^2} z_n^2 z_{L,n}^{2(\tau-1)} |\nabla \eta|^2 + \int_{\mathbb{R}^2} \eta^2 z_n^{2^*} z_{L,n}^{2(\tau-1)} H_n dx \right).
\end{aligned}$$

Thus, using the fact that $w_{L,n} \leq w_n$ and $z_{L,n} \leq z_n$, we get

$$\begin{aligned}
&|\widehat{w}_{L,n}|^2_{L^{2^*}(\mathbb{R}^2)} + |\widehat{z}_{L,n}|^2_{L^{2^*}(\mathbb{R}^2)} \\
&\leq C_4 \tau^2 \left(\int_{R \geq |x| \geq R/2} w_n^{2\tau} dx + \int_{|x| \geq R/2} w_n^{2^*-2} w_n^{2\tau} H_n dx \right) \\
&\quad + C_4 \tau^2 \left(\int_{R \geq |x| \geq R/2} z_n^{2\tau} dx + \int_{|x| \geq R/2} z_n^{2^*-2} z_n^{2\tau} H_n dx \right).
\end{aligned}$$

If we set

$$t = \sqrt{r} > 1, \quad 2^* > \frac{2t}{t-1} \quad \text{and} \quad \Theta = \frac{2^*(t-1)}{2t} > 1$$

we can apply Hölder's inequality with exponents $t/(t-1)$ and t in (4.26), to get

$$\begin{aligned}
&|\widehat{w}_{L,n}|^2_{L^{2^*}(\mathbb{R}^2)} + |\widehat{z}_{L,n}|^2_{L^{2^*}(\mathbb{R}^2)} \\
&\leq C_4 \tau^2 \Bigg[\left(\int_{R \geq |x| \geq R/2} w_n^{2\tau t/(t-1)} dx \right)^{\frac{t-1}{t}} \left(\int_{R \geq |x| \geq R/2} 1 dx \right)^{1/t}
\end{aligned}$$

$$+\left(\int_{|x|\geq R/2} w_n^{(2^*-2)t} H_n^t dx\right)^{1/t} \left(\int_{|x|\geq R/2} w_n^{2\tau t/(t-1)} dx\right)^{\frac{t-1}{t}}\Bigg]$$
$$+C_4\tau^2\Bigg[\left(\int_{R\geq|x|\geq R/2} z_n^{2\tau t/(t-1)} dx\right)^{\frac{t-1}{t}} \left(\int_{R\geq|x|\geq R/2} 1dx\right)^{1/t}$$
$$+\left(\int_{|x|\geq R/2} z_n^{(2^*-2)t} H_n^t dx\right)^{1/t} \left(\int_{|x|\geq R/2} z_n^{2\tau t/(t-1)} dx\right)^{\frac{t-1}{t}}\Bigg].$$

On the other hand, for $r > 1$, we can use Lemma 2.3 and the invariance by translation to get

$$\int_{|x|\geq R/2} H_n^{t^2} dx \leq \int_{\mathbb{R}^2} \left[\exp\left(r\alpha\|(u_n, v_n)\|_{\varepsilon_n}^2 \left(\frac{|u_n|^2+|v_n|^2}{\|(u_n, v_n)\|_{\varepsilon_n}^2}\right)\right) - 1\right] dx.$$

Since (u_n, v_n) is a solution of system $(\widehat{S}_{\varepsilon_n})$, $\limsup_{n\to\infty} \|(u_n, v_n)\|_{\varepsilon_n}^2 < \dfrac{\pi}{2\alpha_0}$. Thus, there exists $n_0 \in \mathbb{N}$ and choosing r sufficiently close to 1 satisfying $r\alpha\|(u_n, v_n)\|_{\varepsilon_n}^2 < 2\pi$ for all $n \geq n_0$, we can apply Lemma 2.2 to get $C_5 > 0$ such that

$$\int_{|x|\geq R/2} H_n^{t^2} dx \leq C_5 \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus, using again Hölder's inequality, the boundedness of (u_n, v_n) and Sobolev embedding we get $C_6 > 0$ such that

$$\int_{|x|\geq R/2} w_n^{(2^*-2)t} H_n^t dx \leq \left(\int_{|x|\geq R/2} w_n^{(2^*-2)t^2/(t-1)} dx\right)^{\frac{t-1}{t}} \left(\int_{|x|\geq R/2} H_n^{t^2} dx\right)^{1/t}$$
$$\leq C_6$$

and similarly

$$\int_{|x|\geq R/2} z_n^{(2^*-2)t} H_n^t dx \leq C_6.$$

It follows from (4.1) that

$$|\widehat{w}_{L,n}|_{L^{2^*}(\mathbb{R}^2)}^2 + |\widehat{z}_{L,n}|_{L^{2^*}(\mathbb{R}^2)}^2 \leq C_7\tau^2 \left(\int_{|x|\geq R/2} w_n^{2\tau t/(t-1)} dx\right)^{(t-1)/t}$$
$$+C_7\tau^2 \left(\int_{|x|\geq R/2} z_n^{2\tau t/(t-1)} dx\right)^{(t-1)/t}$$

Since η is constant in $\mathbb{R}^2 \setminus B_R(0)$, $w_{L,n} \leq w_n$ and $z_{L,n} \leq z_n$, we have from (4.1) that

$$|w_{L,n}|_{L^{2^*\tau}(B_R^c(0))}^{2\tau} + |z_{L,n}|_{L^{2^*\tau}(B_R^c(0))}^{2\tau}$$

$$\leq \left(\int_{\mathbb{R}^2} \eta^{2^*} w_n^{2^*} w_{L,n}^{2^*(\tau-1)} dx\right)^{2/2^*} + \left(\int_{\mathbb{R}^2} \eta^{2^*} z_n^{2^*} z_{L,n}^{2^*(\tau-1)} dx\right)^{2/2^*}$$
$$\leq C_7 \tau^2 \left(|w_n|^{2\tau}_{L^{2\tau t/(t-1)}(B^c_{R/2}(0))} + |z_n|^{2\tau}_{L^{2\tau t/(t-1)}(B^c_{R/2}(0))}\right).$$

Now, we can apply Fatou's Lemma in the variable L to obtain

$$|w_n|^{2\tau}_{L^{2^*\tau}(B^c_R(0))} + |z_n|^{2\tau}_{L^{2^*\tau}(B^c_R(0))}$$
$$\leq C_7 \tau^2 \left(|w_n|^{2\tau}_{L^{2\tau t/(t-1)}(B^c_{R/2}(0))} + |z_n|^{2\tau}_{L^{2\tau t/(t-1)}(B^c_{R/2}(0))}\right).$$

Thus,

$$|(w_n, z_n)|^{2\tau}_{L^{2^*\tau}(B^c_R(0))} \leq C_8 \tau^2 |(w_n, z_n)|^{2\tau}_{L^{2\tau t/(t-1)}(B^c_{R/2}(0))}.$$

Combining $\Theta = \dfrac{2^*(t-1)}{2t}$, $s = \dfrac{2t}{t-1}$ and the last inequality we derive that

$$|(w_n, z_n)|_{L^{\Theta^{m+1}s}(B^c_R(0))} \leq C_8^{\sum_{i=1}^m \Theta^{-i}} \Theta^{\sum_{i=1}^m i\Theta^{-i}} |(w_n, z_n)|_{L^{2^*}(B^c_{R/2}(0))}$$

which implies

$$|(w_n, z_n)|_{L^\infty(B^c_R(0))} \leq C_9 |(w_n, z_n)|_{L^{2^*}(B^c_{R/2}(0))}.$$

By Proposition 4.8 we have that $(w_n, z_n) \to (w, z)$ converges strongly in $L^{2^*}(\mathbb{R}^2)$. Thus, for $R > 0$ sufficiently large, there holds

$$|(w_n, z_n)|_{L^\infty(B^c_R(0))} < \zeta$$

for a large n, which proves this lemma. □

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Declarations

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