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Right-preordered groups from a categorical perspective

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Abstract. We study categorical properties of right-preordered groups, giving an explicit description of limits and colimits in this category, studying some exactness properties, and showing that it is a quasivariety. We show that, from an algebraic point of view, the category of right-preordered groups shares several properties with the one of monoids. Moreover, we describe split extensions of right-preordered groups, showing in particular that semidirect products of ordered groups always have a natural right-preorder.

Mathematics Subject Classification. 06F15, 18E08, 18E13, 06F05.

Keywords. Right-preordered group, Positive cone, Protomodular object, Split extension.

1. Introduction

In [12] preordered groups have been studied from a categorical point of view. A preordered group is a group equipped with a preorder (i.e. a reflexive and transitive relation) which is compatible with the group operation both on the left and on the right, in the sense that the group operation is monotone with respect to such preorder. In particular, limits and colimits in the category **OrdGrp** of preordered groups and monotone group homomorphisms have been explicitly described, and some exactness properties of **OrdGrp** have been explored. From an algebraic point of view, **OrdGrp** does not share with the category **Grp** of groups some strong properties, like being a protomodular [5] or a Mal'tsev [9] category; such properties are important for several reasons, one of which is that in such contexts several results are valid, like the classical homological lemmas, or the description of cohomology groups in terms of extensions. From this perspective, **OrdGrp** turns out to be more similar

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to the category **Mon** of monoids, where such properties hold only "locally", i.e. for some "good" objects. In the case of **OrdGrp**, such good objects are those preordered groups whose preorder is symmetric (and hence an equivalence relation). Another interesting aspect of this relatively weak algebraic context is that, on a split extension of groups whose kernel and codomain are preordered groups, there may be many compatible preorders, turning it to a split extension in **OrdGrp**, or none.

The wider class of the so-called right- (or left-)preordered groups is also interesting for applications in various mathematical fields. These are groups equipped with a preorder which is compatible with the group operation only on the right (or on the left): if $a \leq b$ then, for all c, $a + c \leq b + c$ (we are using the additive notation although our groups need not be abelian). In fact, right-orders on a group are related to actions of the group on the real line (see, e.g., [18, 10]). Moreover, right-ordered groups are used to give a description of the free lattice-ordered groups on given groups [16]. Following a similar spirit, in [15] spaces of right-orders on partially ordered groups have been related to spectral spaces of lattice-ordered groups. In [19] several important examples of preorders on groups that are compatible with the group operation only on one side are described.

The aim of this paper is to extend the study made in [12] to the category **ROrdGrp** whose objects are the right-preordered groups and whose arrows are the monotone group homomorphisms. In particular, we observe that **ROrdGrp** is isomorphic to the category whose objects are pairs (G, M), where G is a group and M is a submonoid of G, and whose arrows are group homomorphisms that (co)restrict to the submonoids. Using the good properties of the forgetful functors from **ROrdGrp** to, on the one hand, **Grp** and, on the other hand, the category **Ord** of preordered sets and monotone maps, as well as the ones of the functor associating to every right-preordered group its positive cone, we give a description of limits and colimits in **ROrdGrp**, and we show that this category has the same exactness properties as **OrdGrp**. Actually, **ROrdGrp** is shown to be a quasivariety. Moreover, we characterize the "good" objects, from an algebraic point of view, proving that they are still the groups equipped with a (right-compatible) equivalence relation, and observing that they also form a quasivariety. Finally, we explore the possible compatible right-preorders on split extensions, showing that, as for **OrdGrp**, all such preorders are bounded by the product preorder (a pair is positive if and only if both components are) and the so-called lexicographic preorder. We prove that the existence of a compatible right preorder is equivalent to the fact that the lexicographic one is compatible (extending a result of [14]). Using the semidirect product construction, we exhibit examples of split extensions which admit compatible right-preorders without admitting preorders that are compatible on both sides.

2. The category ROrdGrp of right-preordered groups

A *right-preordered group* is a group $(X,+)$ together with a preorder (i.e. a reflexive and transitive relation) $\leq$ such that

$$\forall\, x,y,z \in X, \qquad x \leq y \quad \Rightarrow \quad x+z \leq y+z.$$

A morphism of right-preordered groups is a monotone group homomorphism. We denote the category of right-preordered groups and their morphisms by **ROrdGrp**. We point out that, when the group is abelian, the seemingly weaker condition of being right-preordered coincides with being a preordered group.

A right-preorder on a group X determines a submonoid of X, namely $P_X = \{x \in X \mid x \geq 0\}$, also known as the *positive cone* of X.

Proposition 2.1. *For a group X, right-preorders on X are in bijective correspondence with submonoids of X.*

Proof. If $\leq$ is a right-preorder on X, then, if $x \geq 0$ and $y \geq 0$, $x+y \geq 0+y \geq 0$, hence $P_X = \{x \in X \mid x \geq 0\}$ is a submonoid of X.

Conversely, given a submonoid M of X, define $\leq$ on X by $x \leq y$ if $y - x \in M$. Then $\leq$ is clearly reflexive and transitive: $x \leq y$ and $y \leq z$ implies $y - x \in M$ and $z - y \in M$, hence $(z-y)+(y-x) = z - x \in M$, that is, $x \leq z$. Moreover, for every $x, y, z \in X$, if $y - x \in M$ then $y + z - z - x = (y+z) - (x+z) \in M$; that is, $x \leq y$ implies $x+z \leq y+z$ as claimed. □

Remark 2.2. Given two right-preordered groups X and Y, a group homomorphism $f\colon X \to Y$ is monotone exactly when $f(P_X) \subseteq P_Y$. Hence the category **ROrdGrp** is isomorphic to the category having as objects pairs (X,M), where X is a group and M is a submonoid of X, and as morphisms $f\colon (X,M) \to (Y,N)$ group homomorphisms $f\colon X \to Y$ that (co)restrict to $M \to N$. For simplicity, herein we will refer to both the right-preorder on X and its positive cone as a right-preorder on X.

In order to study the behaviour of the category **ROrdGrp**, we compare it with its full subcategory **OrdGrp** of preordered groups, whose objects are the groups $(X,+)$ equipped with a preorder which is compatible with the group operation both on the left and on the right, namely

$$\forall\, x,y,z \in X, \qquad x \leq y \quad \Rightarrow \quad x+z \leq y+z \text{ and } z+x \leq z+y.$$

For a preordered group $(X,+,\leq)$ the positive cone P_X is not just a submonoid, but it must also be closed under conjugation. We refer to [12] for a detailed study of the category **OrdGrp**.

We start by observing the following.

Proposition 2.3. *The* (*full*) *inclusion of the category* **OrdGrp** *of preordered groups into* **ROrdGrp** *has a left adjoint.*

Proof. Given a right-preordered group X, with positive cone P_X, form the least submonoid $\widetilde{P}_X$ of X closed under conjugation and containing P_X. Define $F\colon$ **ROrdGrp** $\to$ **OrdGrp** by $F(X,P_X) = (X,\widetilde{P}_X)$ and $F(f) = f$; it is easy to check that from $f(P_X) \subseteq P_Y$ it follows that $f(\widetilde{P}_X) \subseteq \widetilde{P}_Y$. The identity maps $(X, P_X) \to (X, \widetilde{P}_X)$ are, by construction of $\widetilde{P}_X$, morphisms in **ROrdGrp** which define pointwise the unit of the adjunction. As we will see below, the unit morphisms are epimorphisms but they are not regular epimorphisms. □

The functor $\mathsf{P}\colon \mathbf{ROrdGrp} \to \mathbf{Mon}$, which sends a right-preordered group to its positive cone, factors through the category $\mathbf{Mon}_{\mathrm{can}}$ of monoids with cancellation, since every submonoid of a group satisfies both left- and right-cancellation properties. The functor $\mathsf{P}_0\colon \mathbf{ROrdGrp} \to \mathbf{Mon}_{\mathrm{can}}$ has a left adjoint:

$$\mathbf{Mon}_{\mathrm{can}} \underset{\mathsf{P}_0}{\overset{\mathsf{roGp}}{\rightleftarrows}} \mathbf{ROrdGrp}, \quad (\perp)$$

which is constructed by considering the group completion $\mathsf{Gp}(M)$ of any cancellative monoid M, with the preorder determined by the image of M into $\mathsf{Gp}(M)$ via the unit of the group completion adjunction. The verification that this construction gives the left adjoint to P_0 is essentially the same as the one described in [12] for preordered groups. Composing this adjunction with the one described in the previous proposition, we get precisely the adjunction (A) in [12]. Following the same arguments as in [12, Proposition 3.1], we conclude that any monoid M which is embeddable in a group is embeddable in its group completion, and this is enough to conclude that it is the positive cone of $\mathsf{roGp}(M)$.

In order to continue our exploration, we recall some categorical notions. The first one is the notion of monadic functor. Given an adjunction between two functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$, with F left adjoint to G, the triple (T, η, μ) where T is the composite functor GF, η is the unit of the adjunction, and μ is the natural transformation $G\varepsilon F\colon T^2 \to T$, with ε the counit of the adjunction, is a monad over $\mathbf{C}$, namely a monoid in the category of endofunctors of $\mathbf{C}$ (η and μ are called unit and multiplication of the monad, respectively). An Eilenberg-Moore algebra for a monad (T, η, μ) is a pair (X, ξ) where X is an object of $\mathbf{C}$ and $\xi\colon T(X) \to X$ is an arrow in T such that $\xi\eta = 1_X$ and $\xi T(\xi) = \xi\mu_X$. Eilenberg-Moore algebras for a monad (T, η, μ) form a category $\mathbf{Alg}(T)$. A functor $G\colon \mathbf{D} \to \mathbf{C}$ is monadic if it has a left adjoint $F\colon \mathbf{C} \to \mathbf{D}$ and the category $\mathbf{D}$ is equivalent to the category $\mathbf{Alg}(T)$ of the algebras for the corresponding monad. The categories that are domain of a monadic functor to the category **Set** of sets are precisely the (possibly infinitary) varieties of universal algebras. For example, the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ is monadic. A criterion for the monadicity of a functor is the following one, due to Beck:

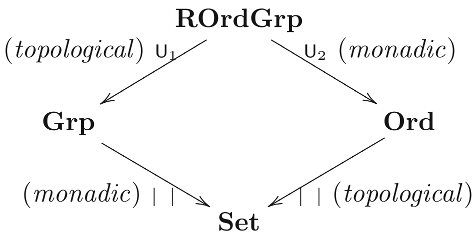
Theorem 2.4. *A functor $G\colon \mathbf{D} \to \mathbf{C}$ is monadic if and only if it has a left adjoint, it reflects isomorphisms, and $\mathbf{C}$ has and G preserves coequalizers of all G-contractible coequalizer pairs.*

To explain the last condition of the criterion above, we recall that a pair of parallel arrows $f, g\colon X \to Y$ is contractible if there exists an arrow $j\colon Y \to X$ such that $fj = 1_Y$ and $gjf = gjg$. A parallel pair of arrows in $\mathbf{C}$ such that its image under G is a contractible pair which has a coequalizer is a G-contractible coequalizer pair (see e.g. Section 2 of [22]).

The second notion we need to recall is the one of topological functor. Given a functor $F\colon \mathbf{C} \to \mathbf{D}$, an F-structured source is an object $D \in \mathbf{D}$ together with a family of arrows $(f_i\colon D \to F(C_i))_{i\in I}$. The functor F is topological if every F-structured source $(D, (f_i)_{i\in I})$ has a unique F-initial lifting, i.e. a unique object $C \in \mathbf{C}$, with arrows $g_i\colon C \to C_i$, such that $F(C) = D$ and $F(g_i) = f_i$. This definition captures the main features of the forgetful functor from the category **Top** of topological spaces to **Set**, which is the prototypical example of a topological functor (and this justifies the name). For more details on this definition, see e.g. [1]. The forgetful functor **Ord** → **Set**, where **Ord** is the category of preorders and monotone maps, is topological, too.

As for the case of **OrdGrp** (see [12, Proposition 2.3]) we have the following results, whose proofs follow arguments similar to those used for **OrdGrp**.

Proposition 2.5. *Consider the forgetful functors* $\mathsf{U}_1\colon \mathbf{ROrdGrp} \to \mathbf{Grp}$ *and* $\mathsf{U}_2\colon \mathbf{ROrdGrp} \to \mathbf{Ord}$*, with* U_1 *forgetting the preorder and* U_2 *the group structure. The functor* U_1 *is topological while* U_2 *is monadic. We have therefore the following commutative diagram*



Proof. U_1 *is a topological functor*: let $(f_i\colon X \to X_i)_{i\in I}$ be a family of group homomorphisms where each X_i, for $i \in I$, is a right-preordered group. Then $P_X = \{x \in X \mid f_i(x) \in P_{X_i} \text{ for every } i \in I\}$ is a submonoid of X such that $f_i(P_X) \subseteq P_{X_i}$, for all $i \in I$. This defines the U_1-initial lifting for (f_i).

U_2 *is a monadic functor*: To prove this we will use Theorem 2.4.

(a) U_2 *has a left adjoint* $\mathsf{L}_2\colon \mathbf{Ord} \to \mathbf{ROrdGrp}$: L_2 assigns to each preorder A the free group $\mathsf{F}(A)$ on the set A equipped with the right-preorder induced by the submonoid of $\mathsf{F}(A)$ generated by the elements of the form $b - a$ for all $a, b \in A$ with $a \leq b$. It is easy to check that L_2 is a functor which is left adjoint to U_2.

(b) U_2 *reflects isomorphisms*: a morphism $f\colon X \to Y$ in **ROrdGrp** with $\mathsf{U}_2(f)$ an isomorphism in **Ord** is a bijective homomorphism whose inverse is both a homomorphism and monotone, hence f is an isomorphism in **ROrdGrp**.

(c) **ROrdGrp** *has and* U_2 *preserves coequalizers of all* U_2*-contractible coequalizer pairs.* First of all **ROrdGrp** is cocomplete, since it is topological over a cocomplete category [1, Theorem 21.16]. Given morphisms $f, g\colon X \to Y$ in **ROrdGrp** with $\mathsf{U}_2(f), \mathsf{U}_2(g)$ a contractible pair in **Ord**, their coequalizer $q\colon Y \to Q$ in **ROrdGrp** is preserved by U_1, and so also by $|\ \ |\cdot \mathsf{U}_1$ because $|\ \ |$ is monadic and $|\mathsf{U}_1(f)|, |\mathsf{U}_1(g)|$ form a contractible pair in **Set**. Therefore $\mathsf{U}_2(q)$

is the coequalizer of $|\mathsf{U}_2(f)|, |\mathsf{U}_2(g)|$ in **Ord**, since it is a split epimorphism and $|\mathsf{U}_2(q)|$ is the coequalizer of $\mathsf{U}_2(f), \mathsf{U}_2(g)$ in **Set**. □

The properties of U_1 and U_2 give important information on the categorical behaviour of **ROrdGrp**.

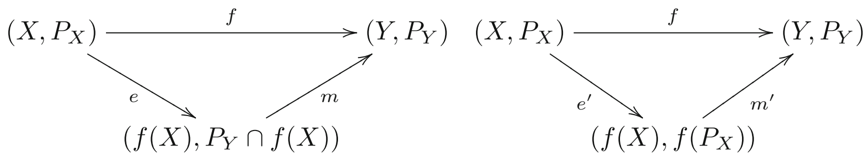
Remark 2.6. (1) Topologicity of $\mathsf{U}_1\colon$ **ROrdGrp** $\to$ **Grp** guarantees that it has both a left and a right adjoint. The former one equips a group X with the discrete order, so that $P_X = \{0\}$, while the latter one equips a group with the total preorder (so that $P_X = X$). Moreover, with **Grp** complete and cocomplete, also **ROrdGrp** is complete and cocomplete.

(2) Both $\mathsf{U}_1\colon$ **ROrdGrp** $\to$ **Grp** and $\mathsf{U}_2\colon$ **ROrdGrp** $\to$ **Ord** preserve limits, which gives us a complete description of limits in **ROrdGrp**: algebraically they are formed like in **Grp**, and then equipped with the limit preorder.

(3) The forgetful functor $\mathsf{P}\colon$ **ROrdGrp** $\to$ **Mon** which assigns to each right-preordered group its positive cone, and to each morphism its (co)restriction to the positive cones, preserves limits and coproducts, but not coequalizers. Indeed, as for **OrdGrp**, the positive cone of a product of right-preordered groups is the product of their positive cones, and analogously for equalizers. For coproducts the situation differs from what happens in **OrdGrp**, because in general the coproduct of positive cones is not closed under conjugation as a submonoid of the coproduct of the preordered groups (see, for instance, [12, Example 2.10]). On the contrary, in **ROrdGrp** the positive cone of a coproduct is the coproduct of the positive cones, just because the coproduct of submonoids is a submonoid of the coproduct.
We point out that the functor P does not preserve coequalizers; for instance, the coequalizer of the pair of morphisms $f, g\colon (\mathbb{Z}, 0) \to (\mathbb{Z}, \mathbb{N})$, with $f(1) = 1$ and $g(1) = 2$, is the constant morphism into $\{0\}$, while the coequalizer of $\mathsf{P}(f), \mathsf{P}(g)\colon \{0\} \to \mathbb{N}$ is the identity on $\mathbb{N}$.

(4) Analogously to the case of **OrdGrp** (see [12, Remark 2.4]), given a morphism $f\colon (X, P_X) \to (Y, P_Y)$ in **ROrdGrp**:
   (a) f is an epimorphism if and only if f is surjective; epimorphisms are stable under pullback;
   (b) f is a regular epimorphism (i.e. a coequalizer of a parallel pair of morphisms) if and only if both f and $\mathsf{P}(f)$ are surjective, and regular epimorphisms are stable under pullback;
   (c) f is a monomorphism if and only if f is injective;
   (d) f is a regular monomorphism (i.e. an equalizer of a parallel pair of morphisms) if and only if f is injective and $P_X = f^{-1}(P_Y)$.

The last part of the remark tells us that in **ROrdGrp** there are two proper and stable factorization systems, (*Epi, Reg Mono*) and (*Reg Epi, Mono*), factoring each morphism as outlined in the following diagram:



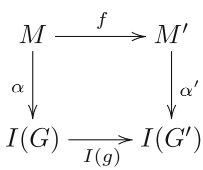
We recall that a factorization system in a category **C** is given by two classes of arrows $\mathcal{E}$ and $\mathcal{M}$ that contain all isomorphisms and are closed under composition, and such that every arrow in **C** factor, uniquely up to isomorphisms, as an arrow in $\mathcal{E}$ followed by one in $\mathcal{M}$. A factorization system $(\mathcal{E}, \mathcal{M})$ is stable when both classes are stable under pullbacks. It is proper when all arrows in $\mathcal{E}$ are epimorphisms and all arrows in $\mathcal{M}$ are monomorphisms. The fact that in a finitely complete category (*Reg Epi, Mono*), where *Reg Epi* is the class of regular epimorphisms and *Mono* is the class of monomorphisms, is a stable factorization system means that the category is *regular*. This is the case, in particular, of every (quasi)variety, where the *Reg Epi-Mono*-factorization is the usual image factorization.

From the observations above, we conclude that **ROrdGrp** is a regular category. As observed in [12], the equivalence relation

$$(\mathbb{Z} \times \mathbb{Z}, \Delta_{\mathbb{N}}) \underset{p_2}{\overset{p_1}{\underset{\longleftarrow\langle 1,1\rangle\longrightarrow}{\longrightarrow}}} (\mathbb{Z}, \mathbb{N}),$$

where $\Delta_{\mathbb{N}} = \{(x, x);\, x \in \mathbb{N}\}$, is not effective in **ROrdGrp** (i.e. it is not the kernel pair of a morphism), since the morphism $\langle p_1, p_2\rangle \colon (\mathbb{Z} \times \mathbb{Z}, \Delta_{\mathbb{N}}) \to (\mathbb{Z} \times \mathbb{Z}, \mathbb{N} \times \mathbb{N})$, which is the identity map, is not a regular monomorphism. We therefore conclude that **ROrdGrp** is not a Barr-exact category [3], hence it is not a variety. Still, we will show next that **ROrdGrp** is a quasivariety. To do it, we will use a sufficient condition described in [11].

Let $I \colon \mathbf{Grp} \to \mathbf{Mon}$ be the inclusion functor, and consider the comma category $\mathbf{Mon} \downarrow I$; that is, an object of $\mathbf{Mon} \downarrow I$ is a homomorphism of monoids $\alpha \colon M \to I(G)$, and a morphism from $\alpha \colon M \to I(G)$ to $\alpha' \colon M' \to I(G')$ is a pair (f, g) where $f \colon M \to M'$ and $g \colon G \to G'$ are homomorphisms making the following diagram



commute. Denote by $\mathbf{Mon} \rightarrowtail\!\!\!\downarrow I$ its full subcategory of monomorphisms (that is, an object is a monomorphism $M \to I(G)$).

Lemma 2.7. **ROrdGrp** *is equivalent to* $\mathbf{Mon} \rightarrowtail\!\!\!\downarrow I$.

Proof. Straightforward. □

The following result will be fundamental in our proof.

Theorem 2.8 [11, Theorem 1.8 and Remark 1.10]. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are varieties admitting at least one constant and* $I\colon \mathbf{B} \to \mathbf{A}$ *is a right adjoint functor, then* $\mathbf{A} \downarrow I$ *is a variety. Moreover, if* $\mathbf{A}$ *and* $\mathbf{B}$ *are finitary varieties, then* $\mathbf{A} \downarrow I$ *is a finitary variety.*

Theorem 2.9. ROrdGrp *is a finitary quasivariety.*

Proof. The functor $I\colon \mathbf{Grp} \to \mathbf{Mon}$ is a right adjoint. Then, by Proposition 1.6 of [11], $\mathbf{Mon} \rceil\!\!\downarrow I$ is a regular epireflective subcategory of $\mathbf{Mon} \downarrow I$. Hence, since **Mon** and **Grp** are finitary varieties with constants, by the Theorem above we may conclude that $\mathbf{Mon} \rceil\!\!\downarrow I$, and hence **ROrdGrp**, is a finitary quasivariety. □

Remark 2.10. Instead of applying directly the results of [11] we could prove directly that **ROrdGrp** is a regular category with a regular projective, regular generator, which, for cocomplete categories, is equivalent to being a quasivariety (see Corollaries 4.4 and 4.6 in [24]). In the Appendix we will use this strategy to show that **OrdGrp** is a quasivariety. It is also finitary, as we show there, but this result needs some extra effort.

Despite not being Barr-exact, **ROrdGrp** shares with **OrdGrp** the property of being *efficiently regular* in the sense of [7]: a regular category is efficiently regular when effective equivalence relations are stable under regular monomorphisms: if R is an effective equivalence relation over an object X and T is another equivalence relation over X which is a *regular subobject* $j\colon T \rightarrowtail R$ of R (i.e. j is a regular monomorphism), then T is itself effective. The proof of this fact is the same as for the case of **OrdGrp** (see [12]): if R is effective and T is a regular subobject of R, then T is a kernel pair of a morphism in **Grp**, since **Grp** is Barr-exact. Moreover, being j a regular monomorphism in **ROrdGrp**, $P_T = T \cap P_R$. The equivalence relation R is effective in **ROrdGrp**, hence $P_R = R \cap P_{X\times X}$, and so

$$P_T = T \cap R \cap P_{X\times X} = T \cap P_{X\times X},$$

which proves that T is effective in **ROrdGrp**. As outlined in [12, Propositions 2.7, 2.8], in every efficiently regular category the change-of-base functor induced by a regular epimorphism is monadic. Therefore:

Corollary 2.11. *In* **ROrdGrp** *a morphism is effective for descent if and only if it is a regular epimorphism.*

Another property that **ROrdGrp** shares with **OrdGrp** is normality in the sense of [20]: a pointed regular category is *normal* if every regular epimorphism is a normal epimorphism, i.e. the cokernel of a morphism. It was observed in [20] that a pointed variety is normal if and only if it is a *variety with ideals* in the sense of Fichtner [17]; such varieties are also called 0-*regular*. A variety with a constant 0 is said to be 0-regular if every congruence in it is determined by its 0-class, meaning that no different congruence has the same 0-class.

Proposition 2.12. ROrdGrp *is a normal category.*

Proof. Following the same argument as in [12, Proposition 2.5] for **OrdGrp**, given any regular epimorphism $f\colon X \to Y$, $\mathsf{U}_1(f)$ is a regular epimorphism in **Grp**, hence a normal epimorphism in **Grp**. Then, since colimits in **ROrdGrp** are built as in **Grp** thanks to the fact that U_1 is topological (see Proposition 2.5), f is a normal epimorphism in **ROrdGrp**. □

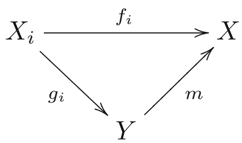
In any regular category, a morphism $f\colon X \to Y$ in **ROrdGrp** is a regular epimorphism if and only if it is an extremal epimorphism: if f factors through a monomorphism $m\colon Z \to Y$ as $f = m\,e$, m is necessarily an isomorphism. In varieties, these morphisms are precisely the surjective morphisms. In the next section we would need to consider the extension of this notion to families of morphisms with the same codomain: a family $(f_i\colon X_i \to X)_{i\in I}$ of morphisms is jointly extremally epimorphic if, whenever there is a monomorphism $m\colon Z \to X$ such that, for all i, $f_i = mg_i$ for arrows $g_i\colon X_i \to Z$, then m is an isomorphism. In algebraic terms, this means that X is generated by the images of the f_i's.

Remark 2.13. As observed in Remark 2.6, exactly as for **OrdGrp**, a morphism $f\colon X \to Y$ in **ROrdGrp** is an extremal epimorphism (or regular epimorphism) if and only if both f and its (co)restriction to the positive cones are surjective.

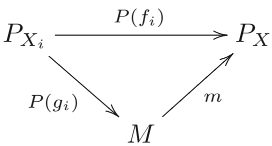
In [12, Remark 2.9] we pointed out that this does not extend to families of morphisms in **OrdGrp**; that is, for a family of morphisms, to be jointly extremally epimorphic in **OrdGrp** does not imply the family of (co)restrictions to the positive cones to be jointly extremally epimorphic. The situation in **ROrdGrp** differs from this one: a family of morphisms $(f_i\colon X_i \to X)_{i\in I}$ is jointly extremally epimorphic in **ROrdGrp** if and only if

(1) $(f_i\colon X_i \to X)_{i\in I}$ is jointly extremally epimorphic in **Grp**;
(2) $(f_i\colon P_{X_i} \to P_X)_{i\in I}$ is jointly extremally epimorphic in **Mon**.

Indeed, if (1) does not hold, given a factorization of (f_i) through a monomorphism in **Grp**



we can equip Y with $P_Y = m^{-1}(P_X)$, making this factorization live in **ROrdGrp**. If (2) does not hold, so that we can obtain a factorization



with m monic (and not an isomorphism), we can factor (f_i) in **ROrdGrp** as follows

$$\begin{array}{ccc} (X_i, P_{X_i}) & \xrightarrow{\quad f_i \quad} & (X, P_X) \\ & {\scriptstyle f_i}\searrow \quad \nearrow{\scriptstyle 1_X} & \\ & (X, M) & \end{array}$$

showing that (f_i) is not jointly extremally epimorphic. The converse implication is obvious.

3. Algebraic properties of ROrdGrp

The notion of jointly extremally epimorphic family of morphisms allows to express in categorical terms the varietal condition of being a *Jónsson-Tarski* variety [21], namely a variety with a unique constant and a binary operation $+$ such that for all x one has $x + 0 = 0 + x = x$. We say that a pointed, finitely complete category is *unital* [6] if, for every pair of objects X, Y, the canonical morphisms $\langle 1, 0\rangle\colon X \to X \times Y$ and $\langle 0, 1\rangle\colon Y \to X \times Y$ are jointly extremally epimorphic. It was observed in [4] that a variety is a unital category if and only if it is a Jónsson-Tarski-variety. In [12] it was shown that the category **OrdGrp** of preordered groups is unital; the same argument works for **ROrdGrp**, showing that the category of right-preordered groups is unital, as well.

In a unital category the notions of commutative and abelian object have a simple expression: an object A in a unital category $\mathbf{C}$ is *commutative* if there exists a morphism $\varphi\colon A \times A \to A$ making the following diagram commutative:

$$\begin{array}{ccccc} A & \xrightarrow{\langle 1,0\rangle} & A \times A & \xleftarrow{\langle 0,1\rangle} & A \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle \varphi} & \swarrow\!\!\swarrow & \\ & & A. & & \end{array}$$

The fact that $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are jointly extremally epimorphic implies that such φ is unique, making commutativity a property of the object A. Moreover, in a Jónsson-Tarski-variety, the morphism φ is necessarily the binary operation $+$: indeed, for any $a, b \in A$

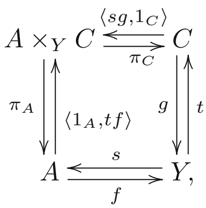
$$\varphi(a, b) = \varphi((a, 0) + (0, b)) = \varphi(a, 0) + \varphi(0, b) = \varphi\langle 1, 0\rangle(a) + \varphi\langle 0, 1\rangle(b) = a + b$$

and so an object is commutative if and only if the operation $+$ is a morphism. This forces $+$ to be commutative and associative, giving a structure of internal commutative monoid (and this explains the name "commutative" for such objects). An object A in $\mathbf{C}$ is *abelian* if it is an internal abelian group, meaning that there is also an inversion $-$ which is a morphism of $\mathbf{C}$. For the same reason as before, also the inversion is uniquely determined, making abelianness a property of an object.

Both in **OrdGrp** and in **ROrdGrp**, given an object $(A, +, \leq)$, the operation $+$ is a morphism if and only if it is commutative and monotone. So, in both cases, the commutative objects are precisely the preordered abelian groups. If, moreover, we require the inversion $-$ to be monotone, the preorder $\leq$ becomes symmetric. Hence, in both categories, the abelian objects are the abelian groups equipped with a congruence. These objects can be identified with pairs (A, N) where A is an abelian group and N is a subgroup of A. So:

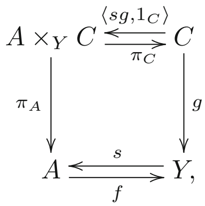
Proposition 3.1. *The (full) subcategory of abelian objects in both* **OrdGrp** *and* **ROrdGrp** *is equivalent to the category of monomorphisms of abelian groups.*

Other important categorical-algebraic properties can be expressed by means of jointly extremally epimorphic pairs of arrows. For example, a finitely complete category $\mathbf{C}$ is a *Mal'tsev* category [9] if every internal reflexive relation in $\mathbf{C}$ is an equivalence relation. For varieties, this corresponds precisely to the classical notion of Mal'tsev variety. It was shown in [6] that a finitely complete category $\mathbf{C}$ is Mal'tsev if and only if, for every pullback of split epimorphisms as in the following diagram



the morphisms $\langle 1_A, tf \rangle$ and $\langle sg, 1_C \rangle$ are jointly extremally epimorphic.

Similarly, a finitely complete category $\mathbf{C}$ is *protomodular* [5] if, for every pullback of a split epimorphism f along any morphism g as in the following diagram



the morphisms π_A and s are jointly extremally epimorphic. For pointed categories, this request is equivalent to the fact that, for every split epimorphism f with section s, the section and the kernel of f are jointly extremally epimorphic. This is moreover equivalent to the validity of the Split Short Five Lemma in $\mathbf{C}$, opening the way to the development of non-abelian homological algebra in categorical terms. Protomodular varieties of universal algebras have been characterized in [8]. It is also known (see, e.g., [4]) that every protomodular category is a Mal'tsev category.

It was observed in [12] that the category **OrdGrp** of preordered groups is not protomodular nor Mal'tsev. The same arguments work for **ROrdGrp**, showing that the category of right-preordered groups is not protomodular nor Mal'tsev, too. In [23], in order to give a categorical characterization of groups among monoids, and more generally to describe the properties of "good" objects in categories with weak algebraic properties, local versions of the notions of Mal'tsevness and protomodularity, relatively to an object, have been considered. Let us recall them.

Definition 3.2. A point $A \underset{f}{\overset{s}{\leftrightarrows}} B$ (i.e. a split epimorphism with a fixed section) with kernel $k\colon X \to A$ in a pointed finitely complete category is *strong* if k and s are jointly extremally epimorphic. It is *stably strong* if every pullback of it along any morphism $g\colon C \to B$ is strong.

Definition 3.3 ([23]). An object Y of a pointed finitely complete category $\mathbf{C}$ is

(1) a *strongly unital object* if the point $Y \times Y \underset{\pi_2}{\overset{\langle 1,1\rangle}{\leftrightarrows}} Y$ is stably strong;

(2) a *Mal'tsev object* if, for every pullback of split epimorphisms over Y as in the following diagram

$$\begin{array}{ccc} A \times_Y C & \underset{\pi_C}{\overset{\langle sg, 1_C\rangle}{\leftrightarrows}} & C \\ {\scriptstyle \pi_A}\downarrow\uparrow{\scriptstyle \langle 1_A, tf\rangle} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} \\ A & \underset{f}{\overset{s}{\leftrightarrows}} & Y, \end{array}$$

the morphisms $\langle 1_A, tf\rangle$ and $\langle sg, 1_C\rangle$ are jointly extremally epimorphic;

(3) a *protomodular* object if every point over Y is stably strong.

As shown in [23], every protomodular object is a Mal'tsev one; in a regular category, every Mal'tsev object is strongly unital. In a unital category, an object is strongly unital if, and only if, it is *gregarious* in the sense of [4, Definition 1.9.1]. Gregarious objects have been the first, unfortunate attempt to characterize groups amongst monoids categorically. Indeed, strongly unital objects in the category **Mon** of monoids are characterized in [4, Proposition 1.9.2]: a monoid M is strongly unital if and only if for any m in M there exist $u, v \in M$ such that $u+m+v=0$. Clearly, every group satisfies this condition, but there are gregarious monoids that are not groups ([4, Counterexample 1.9.3]). An important fact for us is that every cancellative, strongly unital monoid is a group. The proof of this fact that we present here was suggested to us by Alfredo Costa, to whom we are grateful.

Lemma 3.4. *Every cancellative, strongly unital monoid M is a group.*

Proof. Let $m \in M$ and let $u, v \in M$ be such that $u+m+v=0$. We show that $v+u$ is the inverse of m. From $u+m+v=0$ we get

$$u+m+v = 0 = u+m+v+u+m+v;$$

cancelling u on the left and v on the right we get

$$m = 0+m = m+v+u+m.$$

Now, cancelling m on the right we obtain $m + v + u = 0$. Similarly, one gets $v + u + m = 0$. □

We observe that a finitely complete category is a Mal'tsev category if and only if every object in it is a Mal'tsev object, and it is protomodular if and only if every object is protomodular. In the category **Mon** of monoids, the Mal'tsev and the protomodular objects are precisely the groups. The following characterization of these kinds of objects in **ROrdGrp** extends the one we obtained in [12] for **OrdGrp**.

Theorem 3.5. *For a right-preordered group Y, the following conditions are equivalent:*

(i) *Y is a protomodular object in* **ROrdGrp**;
(ii) *Y is a Mal'tsev object in* **ROrdGrp**;
(iii) *Y is a strongly unital object in* **ROrdGrp**;
(iv) *P_Y is a group;*
(v) *the preorder relation on Y is an equivalence relation.*

Proof. The equivalence between (iv) and (v) is obvious.

(iv) $\Rightarrow$ (i): since, in particular, every pair of morphisms in **ROrdGrp** with the same codomain is jointly extremally epimorphic in **ROrdGrp** provided that it is jointly extremally epimorphic in **Grp** and its restriction to the positive cones is jointly extremally epimorphic in **Mon**, we can use the argument of [12, Theorem 4.6].

(i) $\Rightarrow$ (ii) follows from [23, Proposition 7.2].

(ii) $\Rightarrow$ (iii) follows from [23, Proposition 6.3], because **ROrdGrp** is a regular category.

(iii) $\Rightarrow$ (iv): according to Lemma 3.4, we only need to show that P_Y is a gregarious monoid. Suppose there is an element $b \in P_Y$ for which there are no $u, v \in P_Y$ with $u + b + v = 0$. Let $X = \langle b \rangle$ be the subgroup of Y generated by b, with the induced preorder, and $j \colon X \hookrightarrow Y$ the inclusion. As a (right-)preordered group, X is isomorphic to $\mathbb{Z}$ with its usual order, namely $P_X = \{nb \mid n \in \mathbb{N}\}$. Consider then the following right-hand side pullback in **ROrdGrp**:

$$
\begin{array}{ccccc}
Y & \xrightarrow{\langle 1,0\rangle} & Y \times X & \underset{\pi_2}{\overset{\langle j,1\rangle}{\rightleftarrows}} & X \\
\Vert & & \downarrow{\scriptstyle 1\times j} & & \downarrow{\scriptstyle j} \\
Y & \xrightarrow{\langle 1,0\rangle} & Y \times Y & \underset{\pi_2}{\overset{\langle 1,1\rangle}{\rightleftarrows}} & Y.
\end{array}
$$

We show that the positive cone of $Y \times X$ (which is $P_Y \times P_X$) contains strictly the submonoid P of $Y \times X$ generated by $\langle 1, 0\rangle(P_Y)$ and $\langle j, 1\rangle(P_X)$; this would prove that the upper point in the pullback above is not strong, contradicting the assumption. In particular, we show that $(0, b) \notin P$. The elements of P are of the form

$$(y_1, 0) + (n_1 b, n_1 b) + (y_2, 0) + (n_2 b, n_2 b) + \cdots + (y_k, 0) + (n_k b, n_k b)$$

for some $k, n_i \in \mathbb{N}$, $y_i \in Y$. If $(0, b) \in P$, then we should have

$$\begin{cases} y_1 + n_1 b + y_2 + n_2 b + \cdots + y_k + n_k b = 0 \\ n_1 + n_2 + \cdots + n_k = 1 \end{cases}$$

but this is possible only if there is a unique $i \in \{1, \ldots k\}$ such that $n_i = 1$, while all the other n_j's are 0. So we would get

$$y_1 + y_2 + \cdots + y_i + b + y_{i+1} + \cdots + y_k = 0,$$

which is against our assumption on b. □

Therefore the (full) subcategory of protomodular objects of **ROrdGrp** is equivalent to the category **Grp** $\wr$ Id, where Id: **Grp** → **Grp** is the identity functor. Using arguments similar to those used in Section 2 we can conclude that:

Proposition 3.6. *The* (*full*) *subcategories of abelian objects and of protomodular objects of* **ROrdGrp** *are finitary quasivarieties.*

4. Split extensions in ROrdGrp

Any split extension

$$X \xrightarrow{k} A \underset{p}{\overset{s}{\rightleftarrows}} B \tag{4.i}$$

in **ROrdGrp** is in particular a split extension in **Grp**. Hence A, as a group, is isomorphic to the semidirect product $X \rtimes_\varphi B$ w.r.t. the action φ of B on X given by $\varphi_b(x) = k^{-1}(s(b) + k(x) - s(b))$. Furthermore, in **Grp** the split extension (4.i) is isomorphic to $X \xrightarrow{\langle 1,0 \rangle} X \rtimes_\varphi B \underset{\pi_B}{\overset{\langle 0,1 \rangle}{\rightleftarrows}} B$. Hence every split extension (4.i) in **ROrdGrp** is isomorphic to a split extension of the form

$$X \xrightarrow{\langle 1,0 \rangle} X \rtimes_\varphi B \underset{\pi_B}{\overset{\langle 0,1 \rangle}{\rightleftarrows}} B \tag{4.ii}$$

where $X \rtimes_\varphi B$ is equipped with a preorder which makes (4.ii) a split extension in **ROrdGrp**. Given two right preordered groups (X, P_X) and (B, P_B), we call *compatible* right preorders those preorders on $X \rtimes_\varphi B$ that turn $X \rtimes_\varphi B$ into a right preordered group and (4.ii) a split extension in **ROrdGrp**. We already know from [12, Section 5] that, given a split extension (4.ii) in **Grp** with X and B preordered groups, there may be more than one compatible preorder in $X \rtimes_\varphi B$, and there may be none. It follows immediately that in **ROrdGrp** there is no uniqueness of compatible right-preorders. We will see next that, although existence is not guaranteed, the condition that guarantees the existence of compatible right-preorders is much less restrictive than those for preorders.

As in **OrdGrp** (see [12, 14] for details), the positive cone P of a compatible right-preorder must contain

$$P_{\text{prod}} = P_X \times P_B = \{(x, b) \in X \rtimes_\varphi B \mid x \geq 0 \text{ and } b \geq 0\},$$

and must be contained in

$$P_{\text{lex}} = \{(x, b) \in X \rtimes_\varphi B \mid b > 0 \text{ or } (b \sim 0 \text{ and } x \geq 0)\},$$

where $b > 0$ means $b \geq 0$ and $b \neq 0$, while $b \sim 0$ means $b \leq 0$ and $b \geq 0$.

Proposition 4.1. *If P is a compatible cone in $X \rtimes_\varphi B$, then $P_{\text{prod}} \subseteq P \subseteq P_{\text{lex}}$.*

Proof. The morphism $\langle 1, 0\rangle : X \to X \rtimes_\varphi B$ is a kernel in **ROrdGrp** if and only if $(x \in P_X \Leftrightarrow (x, 0) \in P)$, while $\langle 0, 1\rangle : B \to X \rtimes_\varphi B$ is monotone if and only if $(0, b) \in P$ whenever $b \in P_B$. Hence $P_{\text{prod}} \subseteq P$. Moreover, monotonicity of π_B gives that $b \in P_B$ whenever $(x, b) \in P$. If $b \sim 0$ then $-b \in P_B$, and so $(0, -b) \in P$; therefore, if $(x, b) \in P$, then $(x, 0) = (x, b) + (0, -b) \in P$, hence $x \in P_X$. □

Theorem 4.2. *For a split extension (4.ii) in* **Grp** *with (X, P_X) and (B, P_B) right-preordered groups, the following conditions are equivalent:*

(i) *P_{lex} is a compatible right-preorder;*
(ii) *there is a compatible right-preorder P;*
(iii) $(\forall b \in B)\ b \sim 0 \Rightarrow \varphi_b$ *monotone.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii): suppose $b \in P_B$, and let $x \in P_X$; then $(0, b) + (x, 0) = (\varphi_b(x), b)$ must belong to P. If $b \sim 0$ then necessarily $\varphi_b(x) \geq 0$ as claimed.

(iii) $\Rightarrow$ (i): we need to show that P_{lex} is a submonoid of $X \rtimes_\varphi B$; let $(x, b), (y, b') \in P_{\text{lex}}$ (hence $b, b' \in P_B$); then $(x, b) + (y, b') = (x + \varphi_b(y), b + b')$ belongs trivially to P_{lex} when $b+b' \not\sim 0$; if $b+b' \sim 0$ then $b \geq 0$ and $b \leq -b' \leq 0$, hence $b \sim 0$, and so, by (iii), $\varphi_b(y) \geq 0$ and consequently $x + \varphi_b(y) \geq \varphi_b(y) \geq 0$. □

Corollary 4.3. *If B is a right-ordered group (i.e., with an antisymmetric preorder), then the right-preorder P_{lex} is compatible.*

One can also identify the split extensions admitting P_{prod} as a compatible cone.

Proposition 4.4. *For a split extension (4.ii) in* **Grp**, *with (X, P_X) and (B, P_B) right-preordered groups, the following conditions are equivalent:*

(i) *P_{prod} is a compatible right-preorder.*
(ii) $(\forall b \in P_B)\ \varphi_b$ *is monotone.*

Proof. (i) $\Rightarrow$ (ii): If $b \in P_B$ and $x \in P_X$, then $(0, b)+(x, 0) = (\varphi_b(x), b) \in P_{\text{prod}}$, hence $\varphi_b(x) \in P_X$ and therefore φ_b is monotone.

(ii) $\Rightarrow$ (i): Given $x, y \in P_X$ and $a, b \in P_B$, $(x, a) + (y, b) = (x + \varphi_a(y), a + b) \in P_{\text{prod}}$ since $a + b \in P_B$ and $x + \varphi_b(y) \in P_X$ by (ii). □

A comparison of Corollary 4.3 above with Theorem 3.2 of [14] gives us the following

Corollary 4.5. *When B is an ordered group and φ_b is not monotone for some $b \in B$, then P_{lex} – and every possible compatible right-preorder on $X \rtimes_\varphi B$ – makes it a right-preordered group but not a preordered one.*

Examples 4.6(2–3) below are concrete instances of the situation described in the previous corollary.

Examples 4.6. (1) As shown in [12, Example 5.8], the split extension

$$(\mathbb{Z},\mathbb{N}) \xrightarrow{\langle 1,0\rangle} \mathbb{Z}\times\mathbb{Z} \underset{\pi_2}{\overset{\langle 0,1\rangle}{\leftrightarrows}} (\mathbb{Z},\mathbb{N})$$

has an uncountable number of compatible (right-)preorders (here every right-preorder is a preorder on $\mathbb{Z}\times\mathbb{Z}$ since the group is abelian).

(2) Consider the split extension

$$(\mathbb{Z},P) \xrightarrow{\langle 1,0\rangle} \mathbb{Z}\rtimes_{\varphi}\mathbb{Z} \underset{\pi_2}{\overset{\langle 0,1\rangle}{\leftrightarrows}} (\mathbb{Z},\mathbb{Z}) \tag{4.iii}$$

with $\varphi_b(x) = (-1)^b x$.
When $P = \mathbb{N}$ then φ_b is not monotone when b is an odd number (although every $b \sim 0$), hence there is no right-preorder making (4.iii) a split extension in **ROrdGrp**.
When $P = \{0\}$, φ_b is trivially monotone, hence P_{prod}, which coincides with P_{lex}, is a compatible right-preorder. Note that there is no compatible preorder making $\mathbb{Z}\rtimes_{\varphi}\mathbb{Z}$ a preordered group, since the existence of such compatible preorder would imply that $\varphi_b \sim \mathrm{id}$ for every $b\in\mathbb{Z}$ (see [14, Theorem 3.2]).

(3) We can extend the previous situation, providing plenty of examples of right-(pre)ordered groups which are not (pre)ordered. Let us describe explicitly some of them. We recall that an action of a group B on a group X can be expressed as a group homomorphism from B to the automorphism group $\mathrm{Aut}(X)$. When X is the additive group of rationals, $\mathrm{Aut}(X) = (\mathbb{Q}\backslash\{0\},\cdot)$. Hence, an action of $\mathbb{Z}$ on $\mathbb{Q}$, namely a group homomorphism $\mathbb{Z}\to\mathbb{Q}\backslash\{0\}$, is uniquely determined by the image of 1, i.e. by a non-zero rational number q. Let us then consider the split extension

$$(\mathbb{Q},\mathbb{Q}^+) \xrightarrow{\langle 1,0\rangle} \mathbb{Q}\rtimes_{\varphi}\mathbb{Z} \underset{\pi_2}{\overset{\langle 0,1\rangle}{\leftrightarrows}} (\mathbb{Z},\mathbb{N})$$

where φ is the action determined by the non-zero rational number q, i.e. $\varphi_b(x) = q^b x$. The lexicographic preorder (actually, order) on $\mathbb{Q}\rtimes_{\varphi}\mathbb{Z}$ is always right-compatible, since $(\mathbb{Z},\mathbb{N})$ is an ordered group (i.e. the preorder is antisymmetric), but when $q<0$ the map φ_b is not monotone for all positive b, and so, according to [14, Theorem 3.2], there is no compatible preorder on $\mathbb{Q}\rtimes_{\varphi}\mathbb{Z}$.

5. Further comments

The passage from preordered groups to right-preordered groups as outlined here can be carried out to the more general context of V-groups, when V is a commutative and unital quantale, as studied in [13]. We take this opportunity to point out that in the statement (ii) of Proposition 3.1 of our paper [13] a condition is missing. Indeed, a V-group is a group which has a V-category structure both left- and right-invariant under shifting, and this is the notion studied in [13]. Condition (ii) of Proposition 3.1 of [13] gives a notion of right-invariant V-group, or simply right-V-group, that may be interesting to explore in future work.

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Declarations

Conflict of interest Not applicable.

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Appendix A. OrdGrp is a finitary quasivariety

In this Appendix we show that the category **OrdGrp** is a finitary quasivariety. We point out that, although by Proposition 2.3 **OrdGrp** is an epireflective subcategory of **ROrdGrp**, it is not regular epireflective, hence we cannot conclude immediately that **OrdGrp** is a quasivariety.

In the proof the coproduct of $(\mathbb{Z}, \{0\})$ and $(\mathbb{Z}, \mathbb{N})$ will play a key role; for simplicity we denote it by $\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}}$.

Proposition A.1. *The ordered group* $\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}}$ *is a regular projective, regular generator of* **OrdGrp**.

Proof. $\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}}$ is a regular generator: given a preordered group (X, P_X), each $x \in X$ and $a \in P_X$ define morphisms $\varphi_x \colon \mathbb{Z}_0 \to (X, P_X)$, with $\varphi_x(1) = x$, and $\psi_a \colon \mathbb{Z}_{\mathbb{N}} \to (X, P_X)$, defined by $\psi_a(1) = a$, and thus a morphism $\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}} \to (X, P_X)$; and in fact every morphism $\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}} \to (X, P_X)$ is of this form. Hence the morphism $\coprod_f (\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}})_f \to (X, P_X)$, indexed by the set of morphisms $f \colon \mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}} \to (X, P_X)$, is clearly a regular epimorphism, that is, $\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}}$ is a regular generator of **OrdGrp**.

$\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}}$ is regular projective: as observed above, the representable functor

$$\mathbf{OrdGrp}(\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}}, -) \to \mathbf{Set} \tag{A.i}$$

is isomorphic to the functor which assigns to each (X, P_X) the set $P_X \times X$, and to each morphism $f \colon (X, P_X) \to (Y, P_Y)$ the map $\overline{f} \times f \colon P_X \times X \to P_Y \times Y$, where $\overline{f}$ is the (co-)restriction of f to the positive cones. Hence, if $f \colon (X, P_X) \to (Y, P_Y)$ is a regular epimorphism, so that both $\overline{f}$ and f are surjective, then $\overline{f} \times f$ is surjective as claimed. □

It was shown in [12, Proposition 2.5] that **OrdGrp** is a regular category. Hence we may conclude that **OrdGrp** is a quasivariety via [24, Corollary 4.4].

Theorem A.2. **OrdGrp** *is a finitary quasivariety.*

Proof. In order to show that **OrdGrp** is finitary, we need to show that the representable functor (A.i) preserves filtered colimits. As observed above, this functor is isomorphic to the composite

$$\mathbf{OrdGrp} \xrightarrow{G} \mathbf{Mon} \times \mathbf{Grp} \xrightarrow{U} \mathbf{Set}\ ,$$

where $G(X, P_X) = (P_X, X)$ and $U(M, G) = M \times G$. The functor U preserves filtered colimits because both **Mon** and **Grp** are finitary varieties and (filtered) colimits are formed componentwise in $\mathbf{Mon} \times \mathbf{Grp}$. It remains to be shown that G preserves filtered colimits.

Let $\mathbf{D}$ be a filtered category and $J \colon \mathbf{D} \to \mathbf{OrdGrp}$ a functor. Let us denote $J(D)$ by (X_D, P_D). Consider the colimits

$$(P_D \xrightarrow{\rho_D} P) \ \text{ and } \ (X_D \xrightarrow{\sigma_D} X)$$

of $\Pi_1 \cdot G \cdot J \colon \mathbf{D} \to \mathbf{Mon}$ and $\Pi_2 \cdot G \cdot J \colon \mathbf{D} \to \mathbf{Grp}$, respectively, where Π_1, Π_2 are the projections.

Note that $(X_D \xrightarrow{\sigma_D} X)$ is also a colimit in **Mon**, since the inclusion $\mathbf{Grp} \to \mathbf{Mon}$ preserves filtered colimits. Therefore the inclusions $\iota_D \colon P_D \to X_D$ of the positive cones induce a morphism $\iota \colon P \to X$ making the following diagram commute

$$\begin{array}{ccc} P_D & \xrightarrow{\rho_D} & P \\ {\scriptstyle \iota_D}\downarrow & & \downarrow{\scriptstyle \iota} \\ X_D & \xrightarrow[\sigma_D]{} & X \end{array}$$

which is necessarily a monomorphism, by Corollary 1.60 of [2]. Hence, without loss of generality, we may assume P to be a submonoid of X. In order to check that (X, P) is the colimit of J in **OrdGrp** it remains to be checked that P is closed under conjugation. Let $x \in X$ and $a \in P$; we want to show that $x + a - x \in P$. Since U preserves filtered colimits, there exists $D \in \mathbf{D}$ and $x_D \in X_D$ such that $x = \sigma_D(x_D)$, and there exists $D' \in \mathbf{D}$ and $a_{D'} \in P_{D'}$ such that $a = \rho_{D'}(a_{D'})$. Since the category $\mathbf{D}$ is filtered, there exist $D'' \in \mathbf{D}$ and morphisms $\alpha\colon D \to D''$, $\beta\colon D' \to D''$ in $\mathbf{D}$, and so $x = \sigma_{D''}(J\alpha(x_D))$ and $a = \rho_{D''}(J\beta(a_{D'})) = \sigma_{D''}(J\beta(a_{D'}))$. Therefore

$$x + a - x = \sigma_{D''}(J\alpha(x_D) + J\beta(a_{D'}) - J\alpha(x_D)) \in P,$$

since $J\beta(a_{D'}) \in P_{D''}$ and $P_{D''}$ is closed under conjugation in $X_{D''}$. □

Remark A.3. The techniques used for **OrdGrp** may also be used to show that **ROrdGrp** is a finitary quasivariety. For **ROrdGrp** the regular projective, regular generator will be again $\mathbb{Z}_0 \amalg \mathbb{Z}_{\mathbb{N}}$, but note that the coproduct in **ROrdGrp** does not coincide with the coproduct in **OrdGrp** (the positive cones do not coincide). Still, the same proof works.

References

[1] Adámek, J., Herrlich, H., Strecker, G.E.: Abstract and concrete categories: the joy of cats. Repr. Theory Appl. Categ. no. 17 (2006)

[2] Adámek, J., Rosický, J.: Locally presentable and accessible categories. London Mathematical Society Lecture Note Series, vol. 189. Cambridge University Press, Cambridge (1994)

[3] Barr, M.: Exact categories. In: Lecture Notes in Mathematics, vol. 236. Springer, Berlin, pp. 1–120 (1971)

[4] Borceux, F., Bourn, D.: Mal'cev, Protomodular, Homological and Semi-abelian Categories. Mathematics and its Applications, vol. 566. Kluwer, Amsterdam (2004)

[5] Bourn, D.: Normalization equivalence, kernel equivalence and affine categories. In: Lecture Notes in Mathematics, vol. 1488. Springer, Berlin, pp. 43–62 (1991)

[6] Bourn, D.: Mal'cev categories and fibration of pointed objects. Appl. Categ. Struct. **4**, 307–327 (1996)

[7] Bourn, D.: Baer sums in homological categories. J. Algebra **308**, 414–443 (2007)

[8] Bourn, D., Janelidze, G.: Characterization of protomodular varieties of universal algebras. Theory Appl. Categ. **11**, 143–147 (2003)

[9] Carboni, A., Lambek, J., Pedicchio, M.C.: Diagram chasing in Mal'cev categories. J. Pure Appl. Algebra **69**, 271–284 (1990)

[10] Clay, A., Rolfsen, D.: Ordered groups and topology. In: Graduate Studies in Mathematics, vol. 176. American Mathematical Society (2016)

[11] Clementino, M.M., Fitas, C., Hofmann, D.: A variety of co-quasivarieties. Top. Appl. 109226 (to appear) arXiv:2404.05017

[12] Clementino, M.M., Martins-Ferreira, N., Montoli, A.: On the categorical behaviour of preordered groups. J. Pure Appl. Algebra **223**, 4226–4245 (2019)

[13] Clementino, M.M., Montoli, A.: On the categorical behaviour of V-groups. J. Pure Appl. Algebra **225**, 106550 (2021)

[14] Clementino, M.M., Ruivo, C.: On split extensions of preordered groups. Port. Math. **80**, 327–341 (2023)

[15] Colacito, A., Marra, V.: Orders on groups, and spectral spaces of lattice-groups. Algebra Univ. **81**(6) (2020)

[16] Conrad, P.: Free lattice-ordered groups. J. Algebra **16**, 191–203 (1970)

[17] Fichtner, K.: Varieties of universal algebras with ideals. Mat. Sbornik N. S. **75**, 445–453 (1968). (**English translation: Math. USSR Sbornik 4, 411–418 (1968)**)

[18] Ghys, E.: Groups acting on the circle. Enseign. Math. 2 **47**, 329–407 (2001)

[19] Glass, A.M.W.: Partially Ordered Groups. World Scientific, Singapore (1999)

[20] Janelidze, Z.: The pointed subobject functor, 3×3 lemmas, and subtractivity of spans. Theory Appl. Categ. **23**, 221–242 (2010)

[21] Jónsson, B., Tarski, A.: Direct Decompositions of Finite Algebraic Systems. Notre Dame Mathematical Lectures, Notre Dame (1947)

[22] MacDonald, J., Sobral, M.: Aspects of monads. In: Pedicchio, M.C., Tholen, W. (eds.) Categorical Foundations, Encyclopedia of Mathematics and its Applications, vol. 97, pp. 213–268. Cambridge University Press, Cambridge (2004)

[23] Montoli, A., Rodelo, D., Van der Linden, T.: Two characterisations of groups amongst monoids. J. Pure Appl. Algebra **222**, 747–777 (2018)

[24] Pedicchio, M.C., Vitale, E.M.: On the abstract characterization of quasi-varieties. Algebra Univ. **43**, 269–278 (2000)

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