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ZETA ZEROS ON THE CRITICAL LINE

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Dedicated to János Pintz on the occasion of his 75th birthday

Abstract. Montgomery in 1973 introduced the pair correlation method to study the vertical distribution of zeros of the Riemann zeta-function. This work assumed the Riemann Hypothesis (RH). One striking application was a short proof that at least 2/3 of zeta-zeros are simple zeros, the first result of its type. Over the last 50 years, most work on pair correlation of zeta-zeros has continued to assume RH. Here we show that if RH could be removed from Montgomery's simple zero proof, then this would also give a proof that 2/3 of the zeros are simple and on the critical line. This idea has been applied in several recent papers to obtain other results on the zeros.

1. Introduction

In 1973 Montgomery [9] proved, assuming the Riemann Hypothesis (RH), that at least 2/3 of the zeros of the Riemann zeta-function are simple. Prior to this, it was not known either unconditionally or on RH that there were even infinitely many simple zeros. Currently, by a refined version of Levinson's method, Pratt, Robles, Zaharescu, and Zeindler [11] in 2020 proved that at least 41.7% of the zeros of $\zeta(s)$ are on the critical line, and at least 40.7% of the zeros are on the critical line and simple. By large scale computation with careful error analysis, Platt and Trudgian [12] in 2021

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showed that in the critical strip up to height 3,000,175,332,800, there are exactly 12,363,153,437,138 zeros and all of them are on the critical line and are simple. Conditionally, under the RH assumption one can obtain that at least 67.92% of the zeros of $\zeta(s)$ are simple using Montgomery's pair correlation method [4], and with a different method [3,5] one obtains at least 70.37% of the zeros are simple.

2. Zeros of the Riemann zeta-function

Let $s=\sigma+it$ denote any point in the complex plane. The meromorphic function $\zeta(s)$ has a simple pole at $s=1$ with residue 1, and is analytic elsewhere. For $\sigma>1$, we have the Euler product formula $\zeta(s):=\sum_{n=1}^{\infty} n^{-s}=\prod_p (1-p^s)^{-1}$, where the product runs over the primes p. Thus we see that $\zeta(s)\neq 0$ in the half-plane $\sigma>1$ since the individual terms in the Euler product are not zero and converge to 1 as $p\to\infty$. By the functional equation $\zeta(s)=\chi(s)\zeta(1-s)$, where $\chi(s)=2^s\pi^{s-1}\sin\big(\frac{\pi s}{2}\big)\Gamma(1-s)$, we now obtain the analytic continuation of $\zeta(s)$ for $\sigma<0$ where we see the only zeros in this half-plane are the *trivial zeros* $s=-2,-4,-6,-8,\dots$ coming from the factor $\sin(\pi s/2)$ in $\chi(s)$. Thus all the complex zeros of $\zeta(s)$ lie within the strip $0\le\sigma\le 1$. Each of the two proofs of the prime number theorem in 1896 by Hadamard and by de la Vallée Poussin gave different proofs that there are no zeros on the line $\sigma=1$, and therefore all the zeros must lie in the *critical strip* $0<\sigma<1$.

There is one further property of the Riemann zeta-function that is so easy to prove that its importance can often be overlooked; namely that there are no zeros on the real axis $s=\sigma$ for $0\le\sigma<1$. This follows from the formula

$$(1-2^{1-s})\zeta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s},\quad \text{for } 0<\sigma<1,$$

since this shows that $(1-2^{1-\sigma})\zeta(\sigma)>0$ so that $\zeta(\sigma)<0$ on the real line for $0<\sigma<1$. We also see immediately from the functional equation that $\zeta(0)=-1/2$.

If an analytic function $F(s)$ is real on a segment of the real axis it will satisfy the reflection principle that $F(\bar{s})=\overline{F(s)}$ in the symmetric region on either side of that segment where F is analytic. Denoting the complex zeros of the Riemann zeta-function by $\rho=\beta+i\gamma$, we see that $\zeta(\rho)=0=\zeta(\bar{\rho})$ and the functional equation gives as well $\zeta(\rho)=0=\zeta(1-\rho)$. Thus, all nontrivial zeros come in pairs symmetric with the real axis, therefore we have the zeros $\rho=\beta+i\gamma$ and $\bar{\rho}=\beta-i\gamma$, and zeros with $\beta\neq 1/2$ come in symmetric quadruples with respect to both the real axis and the *critical line* (namely, the vertical line $\sigma=1/2$): $\rho=\beta+i\gamma$, $\bar{\rho}=\beta-i\gamma$, $1-\rho=1-\beta-i\gamma$,

$1-\bar{\rho}=1-\beta+i\gamma$. This elementary fact turns out to be fundamental later in this paper.

3. Counting zeros with multi-sets

Since the zeros are symmetric about the real axis, we only need to study the zeros above the real axis. Analytic functions have their zeros counted by the argument principle, which counts zeros with their multiplicity. Thus a zero ρ with multiplicity m_ρ is counted as m_ρ zeros. Therefore, rather than using "sets" of zeros we will use in this paper multi-sets of zeros where a zero ρ with multiplicity m_ρ has m_ρ copies of itself in the multiset. Thus we replace the set $Z(T)=\{\rho : 0<\gamma\le T\}$ of complex zeros ρ up to height T by the multiset

$$\mathcal{Z}(T)=\big\{\rho : 0<\gamma\le T,\ \rho \text{ has } m_\rho \text{ copies}\big\}. \tag{3.1}$$

The set $Z(T)$ is called the support of $\mathcal{Z}(T)$, written $\mathrm{Supp}(\mathcal{Z}(T))$, and is the set of *distinct* zeros.

Let $N(T)$ denote the number of zeros in the critical strip with $0<\gamma\le T$. In 1905 von Mangoldt proved that, for $T\ge 3$,

$$N(T)=|\mathcal{Z}(T)|=\sum_{\rho\in\mathcal{Z}(T)}1=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\log T), \tag{3.2}$$

and we have the useful special cases

$$N(T)\sim\frac{T}{2\pi}\log T \quad \text{as } T\to\infty, \quad \text{and} \quad N(T+1)-N(T)=O(\log T). \tag{3.3}$$

4. Montgomery's RH proof of simple zeros

In this section we assume RH so that $\rho=1/2+i\gamma$, and prove the following result.

THEOREM 4.1 (Montgomery). *Assuming RH, at least $2/3$ of the zeros of $\zeta(s)$ are simple.*

The proof is based on the following inequality:

$$\sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}}1\le\sum_{\rho,\rho'\in\mathcal{Z}(T)}\left(\frac{\sin(\frac{1}{2}(\gamma-\gamma')\log T)}{\frac{1}{2}(\gamma-\gamma')\log T}\right)^2=\Big(\frac{4}{3}+o(1)\Big)\frac{T}{2\pi}\log T, \tag{4.1}$$

as $T\to\infty$. Here the first inequality is trivially true. The asymptotic formula for the Fejér kernel sum over differences of zeros was proved by Montgomery assuming RH; the interested reader may see the proof in [9]. The Fejér kernel sum itself is obtained without RH from the formula

$$\sum_{\rho,\rho'\in\mathcal{Z}(T)}\left(\frac{\sin(\frac{1}{2}(\gamma-\gamma')\log T)}{\frac{1}{2}(\gamma-\gamma')\log T}\right)^2=2\int_0^1\Big|\sum_{\rho\in\mathcal{Z}(T)}T^{i\alpha\gamma}\Big|^2(1-\alpha)\,d\alpha, \tag{4.2}$$

which follows immediately from the elementary formula

$$2\,\mathrm{Re}\int_0^1 e^{iw\alpha}(1-\alpha)\,d\alpha=\left(\frac{\sin(\frac{1}{2}w)}{\frac{1}{2}w}\right)^2. \tag{4.3}$$

Thus in (4.1) the double sums are over two copies of the sequence of zeros with $0<\gamma\le T$, where each sequence already has m_γ copies of each zero of multiplicity m_γ.

Now consider the first sum in (4.1) with the condition that $\gamma=\gamma'$. If we think of the terms as ordered pairs (γ,γ') where $\gamma=\gamma'$, then a simple zero gets counted once, but a double zero with $m_\gamma=2$ gets counted 4 times, since $\gamma_1=\gamma_2=\gamma_1'=\gamma_2'$ and we get the 4 terms (γ_1,γ_1'), (γ_1,γ_2'), (γ_2,γ_1'), (γ_2,γ_2'). Thus a single zero of multiplicity m_γ gets counted m_γ^2 times. The proof of Theorem 4.1 is now just a few lines.

Proof of Theorem 4.1. Since by RH, $\gamma=\gamma'$ is true if and only if $\rho=\rho'$, we have

$$\sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}}1=\sum_{\rho\in\mathcal{Z}(T)}m_\rho\le\left(\frac{4}{3}+o(1)\right)\frac{T}{2\pi}\log T,\quad\text{as } T\to\infty. \tag{4.4}$$

Since ρ being a simple zero is equivalent to $m_\rho=1$, we obtain the simple inequality

$$\begin{aligned}\sum_{\rho\in\mathcal{Z}(T)}m_\rho&=\sum_{\substack{\rho\in\mathcal{Z}(T)\\ m_\rho=1}}m_\rho+\sum_{\substack{\rho\in\mathcal{Z}(T)\\ m_\rho\ge2}}m_\rho\\ &\ge\sum_{\substack{\rho\in\mathcal{Z}(T)\\ m_\rho=1}}1+2\sum_{\substack{\rho\in\mathcal{Z}(T)\\ m_\rho\ge2}}1=\sum_{\substack{\rho\in\mathcal{Z}(T)\\ \rho\text{ simple}}}1+2\sum_{\substack{\rho\in\mathcal{Z}(T)\\ m_\rho\ge2}}1,\end{aligned} \tag{4.5}$$

and thus

$$\sum_{\substack{\rho\in\mathcal{Z}(T)\\ \rho\text{ simple}}}1+\sum_{\rho\in\mathcal{Z}(T)}m_\rho\ge2\sum_{\rho\in\mathcal{Z}(T)}1=2N(T). \tag{4.6}$$

Then using (3.2) and (3.3), we have as $T \to \infty$,

$$\sum_{\substack{\rho \in \mathcal{Z}(T) \\ \rho \text{ simple}}} 1 \geq 2N(T) - \sum_{\rho \in \mathcal{Z}(T)} m_\rho \tag{4.7}$$

$$\geq \Big(2 - \frac{4}{3} + o(1)\Big)\frac{T}{2\pi}\log T = \Big(\frac{2}{3} + o(1)\Big) \sum_{\rho \in \mathcal{Z}(T)} 1. \quad \square$$

The first inequality in (4.7) can also be justified in a simpler way using Montgomery's argument by noting that $2 - m_\rho = 1$ if $m_\rho = 1$, and $2 - m_\rho \leq 0$ if $m_\rho \geq 2$, and thus we have

$$\sum_{\substack{\rho \in \mathcal{Z}(T) \\ \rho \text{ simple}}} 1 \geq \sum_{\rho \in \mathcal{Z}(T)} (2 - m_\rho) = 2N(T) - \sum_{\rho \in \mathcal{Z}(T)} m_\rho. \tag{4.8}$$

This latter argument is much shorter but may leave the question of how crude the inequality is since we only see contributions from simple zeros. Meanwhile, from (4.5), we see that contributions from double zeros are included and those from higher multiplicities are only slightly degraded.

5. Montgomery's simple zero method if RH is not assumed

For 50 years it never seemed to occur to anyone (including the first-named author of this paper) to seriously think about this proof when RH is not assumed. As we will see in a moment, the key idea here is that, despite the title of this section, not assuming RH is not enough; what you have to do is take seriously the possibility that RH is false. The result of this mental block is we spent almost two years proving (4.1) holds with conditions weaker than RH without seeing the connection to zeros on the critical line, which we call for brevity *critical zeros* in the rest of this exposition.

Recall for the first sum in (4.1) and (4.4) we saw that, assuming RH, the condition $\gamma = \gamma'$ is equivalent to $\rho = \rho'$. But if RH is false then this is also sometimes false, since then there are in $\mathcal{Z}(T)$ at least two symmetric zeros $\rho = \beta + i\gamma$ and $1 - \overline{\rho} = 1 - \beta + i\gamma$ with $\beta \neq 1/2$. Hence, if $\beta \neq 1/2$ then the pairs of zeros $(\beta + i\gamma, 1 - \beta + i\gamma)$ and $(1 - \beta + i\gamma, \beta + i\gamma)$ are also solutions of the equation $\gamma = \gamma'$ on the horizontal line $t = \gamma$. We call these "symmetric diagonal terms". Thus, in addition to the diagonal terms $\rho = \rho'$ in (4.4), which occur for every zero whether RH is assumed or not, we also need to include the symmetric diagonal terms for zeros with $\beta \neq 1/2$. This is not the end. Since we are working with pairs of zeros, if there are three or more zeros on the same horizontal line, we can also have terms which are not diagonal and not symmetric.

To conclude, we have

$$\sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}} 1 = \sum_{\rho\in\mathcal{Z}(T)} m_\rho + \sum_{\substack{\rho\in\mathcal{Z}(T)\\ \beta\neq\frac12}} m_\rho + \sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \beta+\beta'\neq 1,\, \gamma=\gamma'}} 1, \tag{5.1}$$

where the first sum on the right-hand side comes from the diagonal terms, the second is given by the symmetric diagonal terms, and the rest in the third sum are non-symmetric horizontal terms.

We can now immediately prove our RH-free version of Montgomery's simple zero result.

THEOREM 5.1. *Suppose there exists a constant* $\mathbf{C}$ *where* $1 \le \mathbf{C} < 2$ *and that, as* $T \to \infty$,

$$\sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}} 1 \le \big(\mathbf{C} + o(1)\big)\frac{T}{2\pi}\log T. \tag{5.2}$$

Then asymptotically at least the proportion $2-\mathbf{C}$ *of the zeros of* $\zeta(s)$ *are simple, and at least the proportion* $2-\mathbf{C}$ *of the zeros of* $\zeta(s)$ *are on the critical line.*

We first remark on the restrictions on the constant $\mathbf{C}$ in our assumption. Note that

$$\sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}} 1 \ge \sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \rho=\rho'}} 1 = \sum_{\rho\in\mathcal{Z}(T)} m_\rho \ge \sum_{\rho\in\mathcal{Z}(T)} 1 = N(T) \sim \frac{T}{2\pi}\log T \tag{5.3}$$

as $T \to \infty$. Thus $\mathbf{C}$ in our assumption has to satisfy $\mathbf{C} \ge 1$. We also restrict $\mathbf{C} < 2$, since for $\mathbf{C} \ge 2$ the method fails to give a positive proportion.

PROOF OF THEOREM 5.1. Substituting (5.1) into (5.2), we have

$$\sum_{\rho\in\mathcal{Z}(T)} m_\rho + \sum_{\substack{\rho\in\mathcal{Z}(T)\\ \beta\neq\frac12}} m_\rho + \sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \beta+\beta'\neq 1,\, \gamma=\gamma'}} 1 \le \big(\mathbf{C} + o(1)\big)\frac{T}{2\pi}\log T \quad \text{as } T \to \infty. \tag{5.4}$$

In this inequality, removing any sum on the left-hand side results in either no change if the sum is empty, or in a smaller quantity since each sum is non-negative. First, we have

$$\sum_{\rho\in\mathcal{Z}(T)} m_\rho \le \big(\mathbf{C} + o(1)\big)\frac{T}{2\pi}\log T \quad \text{as } T \to \infty,$$

which is (4.4) with $\mathbf{C} = 4/3$, and the identical argument (4.5) and (4.6) (or (4.8)) used there gives

$$\sum_{\substack{\rho\in\mathcal{Z}(T)\\ \rho \text{ simple}}} 1 \ge \Big(2-\mathbf{C}+o(1)\Big)\frac{T}{2\pi}\log T \quad \text{as } T\to\infty, \tag{5.5}$$

which proves that the proportion of simple zeros of $\zeta(s)$ is $\ge 2-\mathbf{C}$.

Next, for the symmetric diagonal terms, we have, as $T\to\infty$,

$$\sum_{\substack{\rho\in\mathcal{Z}(T)\\ \beta\ne 1/2}} m_\rho \le \big(\mathbf{C}+o(1)\big)\frac{T}{2\pi}\log T - \sum_{\rho\in\mathcal{Z}(T)} m_\rho.$$

Recalling the lower bound in (5.3), we obtain

$$\sum_{\substack{\rho\in\mathcal{Z}(T)\\ \beta\ne 1/2}} m_\rho \le \big(\mathbf{C}-1+o(1)\big)\frac{T}{2\pi}\log T, \quad \text{as } T\to\infty. \tag{5.6}$$

Thus

$$\sum_{\substack{\rho\in\mathcal{Z}(T)\\ \beta=1/2}} 1 = \sum_{\rho\in\mathcal{Z}(T)} 1 - \sum_{\substack{\rho\in\mathcal{Z}(T)\\ \beta\ne 1/2}} 1 \ge N(T) - \sum_{\substack{\rho\in\mathcal{Z}(T)\\ \beta\ne 1/2}} m_\rho \ge \big(2-\mathbf{C}+o(1)\big)\frac{T}{2\pi}\log T$$

as $T\to\infty$, and we have proved that the proportion of zeros of $\zeta(s)$ on the critical line is $\ge 2-\mathbf{C}$. $\square$

6. Simple and critical zeros using horizontal multiplicity

We can actually show a stronger statement than Theorem 5.1, by obtaining the same $2-\mathbf{C}$ proportion for zeros that are both simple and critical at the same time, under the same assumption. Note that this does not improve each individual proportion stated in Theorem 5.1. We noticed how to make this improvement using an elementary Venn diagram method. After posting this improvement on the arXiv, citing also our earlier preprints on the subject, Soundararajan pointed out to us an alternative approach which would demonstrate our method better.

In place of (5.2) and (5.4), we now use

$$\sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}} 1 = \sum_{\rho\in\mathcal{Z}(T)} H(\gamma) \le \big(\mathbf{C}+o(1)\big)\frac{T}{2\pi}\log T, \quad \text{where } H(\gamma) := \sum_{\substack{\rho'\in\mathcal{Z}(T)\\ \gamma'=\gamma}} 1. \tag{6.1}$$

We show in the next section that $H(\gamma)$ defines the *horizontal multiplicity* of zeros on the line $t=\gamma$. If we assume RH, then $H(\gamma)=m_\rho=m_\gamma$ the usual multiplicity of the zero ρ. Without RH, then $H(\gamma)$ counts with multiplicity the number of zeros on the horizontal line $t=\gamma$. Thus we see, for example:

- If $H(\gamma)=1$, then on the line $t=\gamma$, we have one simple zero on the critical line.
- If $H(\gamma)=2$, then on the line $t=\gamma$, we have either a double zero on the critical line or a pair of two symmetric simple zeros.
- If $H(\gamma)=3$, then on the line $t=\gamma$, we have either a triple zero on the critical line, or a simple zero on the critical line and a pair of two symmetric simple zeros.

Note that to have at least a double zero off the critical line on the line $t=\gamma$, $H(\gamma)$ needs to be at least 4.

With this new notion of horizontal multiplicity, it becomes easy to show how to strengthen the result of Theorem 5.1. In addition, we also obtain the average of the proportion of simple zeros and the proportion of critical zeros, and also results on zeros that are either simple or critical. The latter (see (iii) in Theorem 6.1 below) was pointed out to us by Soundararajan. We summarize these results in the following theorem.

THEOREM 6.1. *Suppose there exists a constant* $\mathbf{C}\geq 1$ *and that, as* $T\to\infty$,

$$\sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}} 1 = \sum_{\rho\in\mathcal{Z}(T)} H(\gamma) \leq \big(\mathbf{C}+o(1)\big)\frac{T}{2\pi}\log T. \tag{6.2}$$

Then asymptotically,

(i) *the proportion of zeros of* $\zeta(s)$ *which are simple and on the critical line is* $\geq 2-\mathbf{C}$,

(ii) *the average of the proportions of simple zeros and of critical zeros of* $\zeta(s)$ *is* $\geq (3-\mathbf{C})/2$,

(iii) *the proportion of zeros of* $\zeta(s)$ *which are either simple or critical or both is* $\geq (4-\mathbf{C})/3$.

REMARK 6.2. Notice that (i) of Theorem 6.1 already implies Theorem 5.1.

REMARK 6.3. Note that if $\mathbf{C}=4/3$ as in Theorem 4.1, then by (ii) of Theorem 6.1 the average of the proportions of simple zeros and of critical zeros is asymptotically $\geq 5/6$. Thus at least one of these proportions is $\geq 5/6$, although the method does not determine which. This agrees with the RH result in Theorem 4.1 where the proportion of simple zeros is $\geq 2/3$ and the proportion of critical zeros is 1, thus giving an average proportion $\geq 5/6$.

We obtain in (iii) of Theorem 6.1 that $\geq 8/9$ of the zeros are either simple or critical or both. Equivalently, the proportion of zeros that are multiple zeros off the critical line is $\leq 1/9$.

Recall the multiset $\mathcal{Z}(\mathcal{T})$ from (3.1), and the set of distinct zeros $Z(T) = \{\rho : 0 < \gamma \leq T\} = \mathrm{Supp}(\mathcal{Z}(T))$. We will consider $Z(T)$ the universe in what follows and take subsets $A \subset Z(T)$, and define the complement of A' in $Z(T)$ to be $A' = \{\rho : \rho \notin A,\ 0 < \gamma \leq T\}$. Similarly, we define the multiset $\mathcal{A}$ formed from A by multiplicity and the complement $\mathcal{A}'$ in the universe $\mathcal{Z}(T)$ by

$$\mathcal{A} = \big\{\rho \in A : 0 < \gamma \leq T,\ \rho \text{ has } m_\rho \text{ copies}\big\},\ \ \mathcal{A}' = \big\{\rho \notin \mathcal{A} : 0 < \gamma \leq T\big\}. \tag{6.3}$$

We count the number of elements of $\mathcal{A}$ including multiplicity and denote it by $N(\mathcal{A}) = |\mathcal{A}|$, that is,

$$N(\mathcal{A}) := \sum_{\substack{\rho \in \mathcal{A} \\ 0<\gamma\leq T}} 1. \tag{6.4}$$

Note also that

$$N(\mathcal{A}') = N(\mathcal{Z}) - N(\mathcal{A}) = N(T) - N(\mathcal{A}).$$

We will now work with the sets of simple zeros S and critical zeros C defined by

$$\begin{aligned} S &:= \big\{\rho : 0 < \gamma \leq T,\ \rho \text{ simple } (m_\rho = 1)\big\}, \\ C &:= \big\{\rho : 0 < \gamma \leq T,\ \rho \text{ critical } (\beta = 1/2)\big\}. \end{aligned} \tag{6.5}$$

We thus have the multisets $\mathcal{S}$ and $\mathcal{C}$ as defined in (6.3); note here $\mathcal{S} = S$. With this preparation, we are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Note $\mathcal{Z}(T)$ is partitioned into disjoint multisets of zeros by

$$\mathcal{H}(k) := \big\{\rho \in \mathcal{Z}(T) : H(\gamma) = k\big\}, \quad k \in \mathbb{N}.$$

We also see from above that

$$\begin{gathered} \mathcal{H}(1) \subset \mathcal{S} \cap \mathcal{C}, \quad \mathcal{H}(2) \subset (\mathcal{S}' \cap \mathcal{C}) \cup (\mathcal{S} \cap \mathcal{C}'), \\ \mathcal{H}(3) \subset (\mathcal{S} \cap \mathcal{C}) \cup (\mathcal{S}' \cap \mathcal{C}) \cup (\mathcal{S} \cap \mathcal{C}') = \mathcal{S} \cup \mathcal{C}. \end{gathered}$$

We now assume (6.1). If $H(\gamma) = 1$, using an argument analogous to (4.8) (or (4.5) and (4.6)), we have

$$N(\mathcal{S} \cap \mathcal{C}) \geq \sum_{\substack{\rho \in \mathcal{Z}(T) \\ H(\gamma)=1}} 1 \geq \sum_{\rho \in \mathcal{Z}(T)} (2 - H(\gamma)) \geq (2 - \mathbf{C} + o(1))N(T),$$

which proves (i) of Theorem 6.1. Next,

$$(3-\mathbf{C}+o(1))N(T) \le \sum_{\rho\in\mathcal{Z}(T)} (3-H(\gamma)) \le 2 \sum_{\substack{\rho\in\mathcal{Z}(T)\\ H(\gamma)=1}} 1 + \sum_{\substack{\rho\in\mathcal{Z}(T)\\ H(\gamma)=2}} 1$$
$$\le 2N(\mathcal{S}\cap\mathcal{C}) + N(\mathcal{S}'\cap\mathcal{C}) + N(\mathcal{S}\cap\mathcal{C}') = N(\mathcal{S}) + N(\mathcal{C}),$$

which proves (ii) of Theorem 6.1. For (iii) of Theorem 6.1, we have, since $\mathcal{H}(3)$ is disjoint from $\mathcal{H}(1)$ and $\mathcal{H}(2)$ and contained in $\mathcal{S}\cup\mathcal{C}$,

$$\big(4-\mathbf{C}+o(1)\big)N(T) \le \sum_{\rho\in\mathcal{Z}(T)} \big(4-H(\gamma)\big) \le 3 \sum_{\substack{\rho\in\mathcal{Z}(T)\\ H(\gamma)=1}} 1 + 2 \sum_{\substack{\rho\in\mathcal{Z}(T)\\ H(\gamma)=2}} 1 + \sum_{\substack{\rho\in\mathcal{Z}(T)\\ H(\gamma)=3}} 1$$
$$\le 3N(\mathcal{S}\cap\mathcal{C}) + 2N(\mathcal{S}'\cap\mathcal{C}) + 2N(\mathcal{S}\cap\mathcal{C}')$$
$$= 2N(\mathcal{S}\cup\mathcal{C}) + N(\mathcal{S}\cap\mathcal{C}) \le 3N(\mathcal{S}\cup\mathcal{C}). \quad \square$$

7. What is horizontal multiplicity?

We now explain the notion of horizontal multiplicity. We also refer the reader to [1, Section 2], where this concept is first introduced. Define

$$N^{\circledast}(T) := \sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}} 1 = \sum_{\rho\in\mathcal{Z}(T)} \sum_{\substack{\rho'\in\mathcal{Z}(T)\\ \gamma'=\gamma}} 1 = \sum_{\rho\in\mathcal{Z}(T)} H(\gamma), \tag{7.1}$$

where, recalling the set of distinct zeros $Z(T) = \{\rho : 0 < \gamma \le T\}$, we have

$$H(\gamma) = \sum_{\substack{\rho'\in\mathcal{Z}(T)\\ \gamma'=\gamma}} 1 = \sum_{\substack{\rho'\in Z(T)\\ \gamma'=\gamma}} m_{\rho'}. \tag{7.2}$$

In this section we follow [8] and count zeros distinctly using $Z(T)$; thus

$$N(T) = \sum_{\rho\in\mathcal{Z}(T)} 1 = \sum_{\rho\in Z(T)} m_\rho.$$

We next define the set of distinct horizontal lines in the critical strip on which the zeros in the set $Z(T)$ lie. The set of distinct heights for these horizontal lines is $G(T) := \{g = \gamma = \Im(\rho) : \rho \in Z(T)\}$, and

$$L(g) := \big\{\sigma + ig :\ 0<\sigma<1,\ g\in G(T)\big\}. \tag{7.3}$$

For any $g \in G(T)$, we have from (7.2) that

$$H(g) = \sum_{\substack{\rho \in Z(T) \\ \gamma = g}} m_\rho = \sum_{\substack{\rho \in Z(T) \\ \rho \in L(g)}} m_\rho. \tag{7.4}$$

Thus we see that $H(g)$ is the number of zeros on the horizontal line $L(g)$ weighted with multiplicity. These distinct horizontal lines partition $Z(T)$ and thus

$$N(T) = \sum_{g \in G(T)} H(g). \tag{7.5}$$

Finally,

$$\begin{aligned} N^{\circledast}(T) &:= \sum_{\substack{\rho, \rho' \in Z(T) \\ \gamma = \gamma'}} m_\rho m_{\rho'} = \sum_{g \in G(T)} \sum_{\substack{\rho, \rho' \in Z(T) \\ \gamma = \gamma' = g}} m_\rho m_{\rho'} \\ &= \sum_{g \in G(T)} \Bigg(\sum_{\substack{\rho \in Z(T) \\ \rho \in L(g)}} m_\rho \Bigg) \Bigg(\sum_{\substack{\rho' \in Z(T) \\ \rho' \in L(g)}} m_{\rho'} \Bigg) = \sum_{g \in G(T)} H(g)^2. \end{aligned} \tag{7.6}$$

We conclude

$$H^{\circledast}(T) := \frac{N^{\circledast}(T)}{N(T)} = \frac{\sum_{g \in G(T)} H(g)^2}{\sum_{g \in G(T)} H(g)}. \tag{7.7}$$

As in [8], in the special case of this formula when RH is assumed, $H(g) = m_\rho$ for the zero $\rho = \frac{1}{2} + ig$ and this is the average multiplicity for zeros in $Z(T)$. We can think of $H(g)$ as the horizontal multiplicity of all the zeros on the line $L(g)$ and each horizontal line as a line-point with multiplicity $H(\gamma)$. Alternatively, since $H(g)$ does not depend on the position of the zeros on $L(g)$, the value of $H(g)$ is unchanged if on the line $L(g)$ we have the RH arrangement of a single zero $1/2 + ig$ of multiplicity $H(g)$. Hence $H^{\circledast}(T)$ is the actual average of $H(g)$ of these zeros on distinct horizontal lines $L(g)$ in $Z(T)$.

8. Simple and critical zeros using diagonal and symmetric diagonal terms

The proof of Theorem 5.1 does not make use of the third summation on the left-hand side of (5.4). The other two terms, which are the diagonal and symmetric diagonal terms, are indeed easy to analyze in terms of the properties of "simple" and "critical". The third term there, which captures the non-symmetric horizontal terms, although included in the assumption made in both Theorem 5.1 and Theorem 6.1, does not make any contribution to the final results. In fact, we can replace the assumption in both (5.2) and (6.2) with

$$\sum_{\rho\in\mathcal{Z}(T)} m_\rho + \sum_{\substack{\rho\in\mathcal{Z}(T)\\ \beta\neq\frac{1}{2}}} m_\rho \le \big(\mathbf{C}+o(1)\big)\frac{T}{2\pi}\log T, \tag{8.1}$$

for some constant $\mathbf{C}\ge 1$, as $T\to\infty$, and obtain the exact same results (i)–(iii) of Theorem 6.1.

Proof of (i)–(iii) of Theorem 6.1 under the assumption (8.1). We retain the notation from Section 6 and in addition also considers counting the zeros with and weighted by the multiplicity m_ρ. We denote this multiplicity weighted counting by $N^*(\mathcal{A})$, which means

$$N^*(\mathcal{A}) := \sum_{\substack{\rho\in\mathcal{A}\\ 0<\gamma\le T}} m_\rho.$$

With the usual Venn diagram for the two sets S and C, we partition the zeros in $Z(T)$ into the usual four disjoint sets according to being or not being simple/critical, form the corresponding multisets, and count with multiplicity to obtain

$$N(\mathcal{Z}(T)) = N(T) = N(\mathcal{S}\cap\mathcal{C}) + N(\mathcal{S}'\cap\mathcal{C}) + N(\mathcal{S}\cap\mathcal{C}') + N(\mathcal{S}'\cap\mathcal{C}'). \tag{8.2}$$

The same decomposition in (8.1) gives on counting with multiplicity and weighting by multiplicity

$$\begin{aligned} N^*(\mathcal{S}\cap\mathcal{C}) + N^*(\mathcal{S}'\cap\mathcal{C}) + 2N^*(\mathcal{S}\cap\mathcal{C}') + 2N^*(\mathcal{S}'\cap\mathcal{C}') \\ \le (\mathbf{C}+o(1))N(T). \end{aligned} \tag{8.3}$$

Since $N^*(\mathcal{S}\cap\mathcal{C}) = N(\mathcal{S}\cap\mathcal{C})$ and $N^*(\mathcal{S}\cap\mathcal{C}') = N(\mathcal{S}\cap\mathcal{C}')$, this equation becomes

$$N(\mathcal{S}\cap\mathcal{C}) + N^*(\mathcal{S}'\cap\mathcal{C}) + 2N(\mathcal{S}\cap\mathcal{C}') + 2N^*(\mathcal{S}'\cap\mathcal{C}') \le (\mathbf{C}+o(1))N(T).$$

We eliminate the last two N^*'s in this equation using

$$N^*(\mathcal{S}'\cap\mathcal{C}) \ge 2N(\mathcal{S}'\cap\mathcal{C}) \quad \text{and} \quad N^*(\mathcal{S}'\cap\mathcal{C}') \ge 2N(\mathcal{S}'\cap\mathcal{C}'),$$

where these are equalities if $\mathcal{S}'$ only contains double zeros. Thus we obtain

$$N(\mathcal{S}\cap\mathcal{C}) + 2N(\mathcal{S}'\cap\mathcal{C}) + 2N(\mathcal{S}\cap\mathcal{C}') + 4N(\mathcal{S}'\cap\mathcal{C}') \le (\mathbf{C}+o(1))N(T). \tag{8.4}$$

Now by (8.2) we see

$$N(\mathcal{S}' \cap \mathcal{C}) + N(\mathcal{S} \cap \mathcal{C}') + N(\mathcal{S}' \cap \mathcal{C}') = N(T) - N(\mathcal{S} \cap \mathcal{C}),$$

and substituting this into (8.4), we obtain

$$N(\mathcal{S} \cap \mathcal{C}) + 2(N(T) - N(\mathcal{S} \cap \mathcal{C})) + 2N(\mathcal{S}' \cap \mathcal{C}') \le (\mathbf{C} + o(1))N(T).$$

Thus we obtain

$$N(\mathcal{S} \cap \mathcal{C}) \ge \big(2 - \mathbf{C} + o(1)\big)N(T) + 2N(\mathcal{S}' \cap \mathcal{C}'). \tag{8.5}$$

From this we immediately have $N(\mathcal{S} \cap \mathcal{C}) \ge (2 - \mathbf{C} + o(1))N(T)$, which proves (i) of Theorem 6.1.

To prove (ii) of Theorem 6.1, we begin with the equalities

$$\begin{gathered} N(\mathcal{S} \cup \mathcal{C}) = N(\mathcal{S}) + N(\mathcal{C}) - N(\mathcal{S} \cap \mathcal{C}), \\ N(\mathcal{S} \cup \mathcal{C}) = N(\mathcal{S} \cap \mathcal{C}') + N(\mathcal{S} \cap \mathcal{C}) + N(\mathcal{S}' \cap \mathcal{C}), \end{gathered} \tag{8.6}$$

which on combining gives

$$N(\mathcal{S}) + N(\mathcal{C}) = 2N(\mathcal{S} \cap \mathcal{C}) + N(\mathcal{S}' \cap \mathcal{C}) + N(\mathcal{S} \cap \mathcal{C}').$$

On applying (8.2), this gives

$$N(\mathcal{S}) + N(\mathcal{C}) = N(\mathcal{S} \cap \mathcal{C}) + N(T) - N(\mathcal{S}' \cap \mathcal{C}'),$$

and by (8.5) produces

$$N(\mathcal{S}) + N(\mathcal{C}) \ge \big(3 - \mathbf{C} + o(1)\big)N(T) + N(\mathcal{S}' \cap \mathcal{C}') \ge \big(3 - \mathbf{C} + o(1)\big)N(T).$$

To prove (iii) of Theorem 6.1, first note by (8.4) and the second equation in (8.6) that

$$2N(\mathcal{S} \cup \mathcal{C}) - N(\mathcal{S} \cap \mathcal{C}) + 4N(\mathcal{S}' \cap \mathcal{C}') \le (\mathbf{C} + o(1))N(T).$$

Next, since $(\mathcal{S} \cup \mathcal{C})' = \mathcal{S}' \cap \mathcal{C}'$, we have $N(\mathcal{S}' \cap \mathcal{C}') = N(T) - N(\mathcal{S} \cup \mathcal{C})$, and substituting this gives

$$\big(4 - \mathbf{C} + o(1)\big)N(T) \le 2N(\mathcal{S} \cup \mathcal{C}) + N(\mathcal{S} \cap \mathcal{C}) \le 3N(\mathcal{S} \cup \mathcal{C}). \quad \square$$

9. Two results on critical zeros

Using Theorem 6.1, we proved in [1] the following result.

Theorem 9.1 (Baluyot, Goldston, Suriajaya, Turnage-Butterbaugh). *Assume that all of the zeros of* $\zeta(s)$ *with* $T < \gamma \le 2T$ *lie in the region*

$$B_b = \Big\{ s = \sigma + it : \Big|\sigma - \frac{1}{2}\Big| < \frac{b}{2\log T},\ T < t \le 2T \Big\}. \tag{9.1}$$

Then if $b = 0.3185$, *we have for the zeros of* $\zeta(s)$ *with* $T < \gamma \le 2T$ *that at least* $2/3$ *are simple and on the critical line.*

We define the *average spacing between consecutive zeros* by

$$\frac{\text{Length of } [0,T]}{N(T)} \sim \frac{T}{\frac{T}{2\pi}\log T} \sim \frac{2\pi}{\log T} \quad \text{as } T \to \infty. \tag{9.2}$$

Thus in Theorem 9.1 we are assuming all the zeros are close to the critical line within a horizontal distance of $b/(4\pi)$ times the average spacing between consecutive zeros at height T.

The following conjecture gives an asymptotic formula for the number of pairs of zeros counted with multiplicity within a distance λ times the average spacing.

Pair Correlation Conjecture (PCC). *For fixed* $\lambda > 0$ *and* $T \to \infty$,

$$\sum_{\substack{\rho,\rho' \in \mathcal{Z}(T) \\ 0 < \gamma - \gamma' \le \frac{2\pi\lambda}{\log T}}} 1 \sim \left(\int_0^\lambda \left(1 - \Big(\frac{\sin \pi u}{\pi u}\Big)^2 \right) du \right) \frac{T}{2\pi} \log T.$$

Montgomery first made this conjecture in [9]. From PCC sprung the discovery that the zeros of L-functions obey a random matrix eigenvalue model; for $\zeta(s)$ this is the Gaussian Unitary Ensemble (GUE) model.

Since PCC only concerns the non-zero vertical distance between zeros, it is unaffected by RH and nothing in PCC by itself implies directly that any zeros are on the critical line. Montgomery's reasoning to support PCC used RH, but Gallagher and Mueller [8] proved that PCC implies asymptotically 100% of the zeros are simple using a different method based on unconditional results of Selberg and Fujii. However, since RH was assumed and used in many parts of [8], we clarified this in [6] to show the results on PCC hold without RH. As a consequence, by Theorem 5.1, we also obtain that asymptotically 100% of the zeros are on the critical line.

Theorem 9.2 (Goldston, Lee, Suriajaya, Schettler). *The Pair Correlation Conjecture implies asymptotically* 100% *of the zeros of the* $\zeta(s)$ *are simple and on the critical line.*

The proof of Theorem 9.2 actually depends on the following property of PCC.

Essential Simplicity Hypothesis (ESH). *If $\lambda \to 0$ as $T \to \infty$, then we have*

$$\sum_{\substack{\rho,\rho' \in \mathcal{Z}(T) \\ |\gamma-\gamma'| \le \frac{2\pi\lambda}{\log T}}} 1 = \Big(1 + o(1)\Big)\frac{T}{2\pi}\log T. \tag{9.3}$$

ESH was introduced while assuming RH by Mueller [10]. Clearly (9.3) in ESH implies $\mathbf{C} = 1$, and thus Theorems 5.1 and 6.1 hold with $\mathbf{C} = 1$. Thus also in (7.7) we have $N^{\circledast}(T)/N(T) \sim 1$ as $T \to \infty$. This reduces the proof of Theorem 9.2 to proving without RH that PCC implies ESH.

Other pair correlation conjectures besides PCC also imply ESH. In [7] we proved that the Alternative Hypothesis (AH) gives a different pair correlation density function from PCC, which in one form implies ESH and thus Theorem 9.1 with PCC replaced by that form of AH.

We mention that Selberg [13] and Bombieri and Hejhal [2] have both made use of ESH in their work on linear combinations of L-functions. In particular, the latter paper made use of Montgomery's Fejér kernel method used in Theorem 4.1 to obtain partial results without using ESH.

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