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ORIGINAL PAPER

Tight Bound on Treedepth in Terms of Pathwidth and Longest Path

Meike Hatzel[1] · Gwenaël Joret[2] · Piotr Micek[3] · Marcin Pilipczuk[4] · Torsten Ueckerdt[5] · Bartosz Walczak[3]

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Abstract

We show that every graph with pathwidth strictly less than a that contains no path on 2^b vertices as a subgraph has treedepth at most $10ab$. The bound is best possible up to a constant factor.

Keywords Treewidth · Pathwidth · Treedepth

1 Introduction

Treewidth (tw), pathwidth (pw), and treedepth (td) are among the best-known and most widely studied structural width parameters of graphs. They are related by the inequalities $\mathrm{tw}(G) + 1 \leqslant \mathrm{pw}(G) + 1 \leqslant \mathrm{td}(G)$ for every graph G. Moreover, trees have treewidth 1 and arbitrarily large pathwidth, while paths have pathwidth 1 and arbitrarily large treedepth.

Treedepth is approximated by the maximum length of a path[1]: every graph containing an ℓ-vertex path has treedepth greater than $\log_2 \ell$, and every graph with no such path has treedepth less than ℓ [7, Sect. 6]. Similarly, pathwidth is approximated by the

[1] In this paper, we are concerned only about non-induced paths.

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✉ Gwenaël Joret
gwenael.joret@ulb.be

Extended author information available on the last page of the article

maximum height of a complete binary tree minor: every graph containing a complete binary tree of height h as a minor has pathwidth at least $\lfloor \frac{h}{2} \rfloor$ [9], and every graph with no such minor has pathwidth $\mathcal{O}(2^h)$ [1]. For both parameters, the exponential gap between the respective lower and upper bounds cannot be avoided, as witnessed by complete graphs. Treewidth is approximated by the maximum size of a grid minor, but here the gap is polynomial: while every graph containing a $k \times k$ grid as a minor has treewidth at least k [8], every graph with no such minor has treewidth polynomial in k [2].

Kawarabayashi and Rossman [6] showed that treedepth is approximated with polynomial gap by the three above-mentioned obstructions together: every graph with no $k \times k$ grid minor, no height k complete binary tree minor, and no 2^k-vertex path has treedepth polynomial in k. More specifically, they proved that every graph of treewidth less than k with no height k complete binary tree minor and no 2^k-vertex path has treedepth $\mathcal{O}(k^5 \log^2 k)$. Here are an improvement of this statement and an analogous result relating pathwidth and treewidth:

Theorem 1 (Czerwiński, Nadara, Pilipczuk [3][2]) *Every graph of treewidth less than t with no complete binary tree of height h as a minor and no 2^b-vertex path has treedepth $\mathcal{O}(thb)$.*

Theorem 2 (Groenland, Joret, Nadara, Walczak [5]) *Every graph of treewidth less than t with no complete binary tree of height h as a minor has pathwidth $\mathcal{O}(th)$.*

We complete the picture by proving an analogous result relating treedepth and pathwidth.

Theorem 3 *Every graph of pathwidth less than a containing no 2^b-vertex path has treedepth at most $10ab$.*

Clearly, Theorems 2 and 3 imply Theorem 1. On the other hand, Theorem 1 implies that every graph of pathwidth less than a containing no 2^b-vertex path has treedepth $\mathcal{O}(a^2 b)$. This is because every graph with pathwidth less than a has treewidth less than a and contains no complete binary tree of height $2a$ as a minor. In [5], it was conjectured that the bound on treedepth can be reduced to $\mathcal{O}(ab)$, and Theorem 3 provides a proof of this conjecture.

We remark that the bound in Theorem 3 is sharp up to a constant factor, which can be seen as follows. Let b and c be integers with $b > c \geqslant 1$, and let $a = 2^c$. Consider the graph G obtained from a path on 2^{b-c} vertices by replacing each vertex with a clique on $\frac{a}{2} = 2^{c-1}$ vertices and replacing each edge by a complete bipartite graph between the two cliques. Then $\mathrm{pw}(G) = a - 1$. Also, G has 2^{b-1} vertices, and thus it has no 2^b-vertex path. It can be checked that G has treedepth at least $\frac{a}{2}(b - c)$, which is roughly $ab/2$ when $b \gg c$. It is shown in [5] that the bound in Theorem 2 is also sharp up to a constant factor. Whether the bound in Theorem 1 can be improved remains an open problem.

[2] In [3], the bound is stated in the special case $t = h = b$, but the proof works in general.

2 Preliminaries

All graphs in this paper are finite and simple, that is, they have no loops or parallel edges. All logarithms in this paper are to the base 2.

A *rooted tree* is a tree with one vertex designated as the *root*. A *rooted forest* is a disjoint union of rooted trees. We define the *height* of a rooted forest F as the maximum number of vertices on a path from a root to a leaf in F. A vertex u is an *ancestor* of a vertex v in a rooted forest F if u lies on the (unique) path from a root to v in F. A rooted forest F is an *elimination forest* of a graph G if $V(F) = V(G)$ and for every edge uv of G, one of the vertices u and v is an ancestor of the other in F. The *treedepth* of a graph G, denoted by $\mathrm{td}(G)$, is the minimum height of an elimination forest of G.

A *tree decomposition* of a graph G is a pair $(T, \mathcal{B})$ such that T is a tree, the vertices of which are called *nodes*, and $\mathcal{B}$ is a collection $\{B_t\}_{t\in V(T)}$ of subsets of $V(G)$, called *bags*, indexed by the nodes of T, such that the following conditions are satisfied:

(1) for every edge $uv \in E(G)$, there is a bag containing both u and v;
(2) for every vertex $v \in V(G)$, the set of nodes $t \in V(T)$ with $v \in B_t$ induces a non-empty subtree of T.

The *width* of a tree decomposition is the maximum size of a bag minus 1. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G. The notions of *path decomposition* and *pathwidth* are defined analogously with the extra condition that the tree T is a path. The pathwidth of G is denoted by $\mathrm{pw}(G)$.

A *k-linkage* between two subsets A and B of the vertices of a graph G is a subgraph of G that consists of k vertex-disjoint paths each starting in A and ending in B. (If A and B intersect, then a path of a k-linkage between A and B may consist of a single vertex in $A \cap B$.) A path decomposition $(P, \mathcal{B})$ of a graph G is *linked* if for any two nodes $t, t' \in V(P)$, there is a k-linkage between B_t and $B_{t'}$ where k is the minimum size of a bag B_s for nodes s on the path from t to t' in P. We use the fact that there is always a path decomposition of minimum width that is linked.

Theorem 4 (Erde [4, Theorem 5.8]) *Every graph G has a path decomposition of width $\mathrm{pw}(G)$ that is linked.*

3 Proof

We proceed with the proof of Theorem 3, that every graph of pathwidth less than a containing no 2^b-vertex path has treedepth at most $10ab$.

Let G be a graph with $\mathrm{pw}(G) < a$ and with no path on 2^b vertices. If $2^b < 2a$, then the statement of the theorem is easily seen to hold by considering a depth-first search forest of G, which is an elimination forest of G. Its height is less than 2^b, which is less than $2a$. Hence, $\mathrm{td}(G) < 2a < 10ab$. Therefore, we may assume that $2^b \geqslant 2a$. This inequality will be used at the very end of the proof.

Fix a linked path decomposition $(P, \mathcal{B})$ of G with $\mathcal{B} = \{B_t\}_{t\in V(P)}$ and $|B_t| \leqslant a$ for every node $t \in V(P)$; such a linked path decomposition exists by Theorem 4. We think of the nodes as being laid out from left to right along P. For a set of nodes $X \subseteq V(P)$, let

$$B(X) = \bigcup_{t \in X} B_t.$$

We call the node set of any subpath of P an *interval*. For an interval I, we let

$$\text{level}(I) = \min\{|B_t| \mid t \in I\} \qquad \text{and} \qquad \text{int}(I) = B(I) - B(V(P) - I).$$

Thus, $\text{int}(I)$ (the "interior" of I) is the set of vertices of G that lie only in the bags of nodes in I.

For every $k \in \{1, \dots, a\}$ and every inclusion-maximal interval I^* with $\text{level}(I^*) \geqslant k$, we fix some k-linkage between the bags of the leftmost and the rightmost nodes in I^*, and we let $\mathcal{L}_k^*(I^*)$ be the vertex set of that k-linkage. For every $k \in \{1, \dots, a\}$ and every interval I with $\text{level}(I) \geqslant k$, we let $\mathcal{L}_k(I) = \mathcal{L}_k^*(I^*) \cap B(I)$ where I^* is the unique inclusion-maximal interval with $\text{level}(I^*) \geqslant k$ containing I. We note the following properties of the sets $\mathcal{L}_k(I)$ for further reference:

$$\mathcal{L}_k(I) \subseteq B(I), \tag{1}$$

$$\mathcal{L}_k(I') \subseteq \mathcal{L}_k(I) \qquad \text{for every interval } I' \subseteq I, \tag{2}$$

$$|\mathcal{L}_k(I) \cap B_t| \geqslant k \qquad \text{for every node } t \in I, \tag{3}$$

$$|\mathcal{L}_k(I)| < k \cdot 2^b. \tag{4}$$

We describe an iterative process in which we construct a rooted tree T whose vertices are contained in $V(G)$ except for the root, which is a special vertex $r^* \notin V(G)$. The initial tree T contains only the root r^*. We grow the tree T in rounds, in each round attaching new paths formed by some vertices of G that are not yet in T. We maintain the invariant that $T - r^*$ is an elimination forest of the corresponding induced subgraph of G, that is, for any two vertices in $V(T) - \{r^*\}$ that are adjacent in G, one is an ancestor of the other in T. The process ends when T contains all vertices of G, so that $T - r^*$ is an elimination forest of G.

A simple plan for a round would be to find a bag B_t whose removal from G would halve some measure that is proportional to the logarithm of the maximum path length. Then, after adding B_t to T (as a path), we could continue growing T independently on each of the two sides of $G - B_t$ starting from the vertex of B_t that is currently a leaf of T. This is too simple to work, but it motivates our actual approach.

For a tree T as above and an interval I, we use the following notation. Let $\ell = \text{level}(I)$. For every $k \in \{1, \dots, \ell\}$, we define

$$x_k(I, T) = |(\text{int}(I) \cap \mathcal{L}_k(I)) - V(T)| \qquad \text{and} \qquad w_k(I, T) = \sum_{i=1}^{k} \log(x_i(I, T) + 1).$$

The following "monotonicity" property is a direct consequence of (2):

$$\text{if } I' \subseteq I \text{ and } V(T') \supseteq V(T), \text{ then } x_k(I', T') \leqslant x_k(I, T) \text{ and } w_k(I', T') \leqslant w_k(I, T), \tag{5}$$

Furthermore, it follows from (4) that $x_i(I, T) + 1 \leqslant i \cdot 2^b$ for every $i \in \{1, \ldots, \ell\}$, which yields

$$w_\ell(I, T) \leqslant \sum_{i=1}^{\ell} \log(i \cdot 2^b) = b\ell + \log(\ell!), \tag{6}$$

$$w_\ell(I, T) - w_k(I, T) \leqslant \sum_{i=k+1}^{\ell} \log(i \cdot 2^b) = b(\ell - k) + \log\left(\frac{\ell!}{k!}\right) \quad \text{for every } k \in \{1, \ldots, \ell\}. \tag{7}$$

For notational convenience, we also define $w_i(\emptyset, T) = 0$.

For a vertex v of G that has been added to T at some time in the process, let depth(v) denote the number of vertices of G on the path from r^* to v in T (thus disregarding r^* in the count). Since we always augment the tree T by adding new vertices as leaves, depth(v) is determined when v is added to T and remains unchanged till the end of the process.

During the aforesaid iterative process of constructing the tree T, we maintain

- a set X of nodes of P, and the invariant that $B(X) \subseteq V(T)$;
- the family $\mathcal{I}$ of intervals contained in $V(P) - X$ that are inclusion-maximal;
- a designated vertex v_I in T for every interval $I \in \mathcal{I}$.

We also maintain the following invariants:

Inv. 1. For every interval $I \in \mathcal{I}$, the path from the root r^* to v_I in T contains every vertex of $B(I) \cap V(T)$.

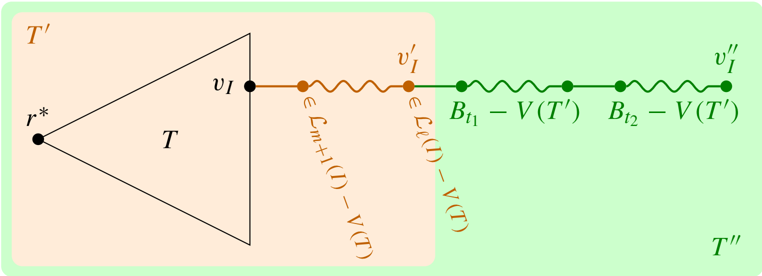
Inv. 2. For every interval $I \in \mathcal{I}$ and for $\ell = \text{level}(I)$, we have

$$\tfrac{1}{5}\,\text{depth}(v_I) + w_\ell(I, T) \leqslant (b+1)\ell + \log(\ell!).$$

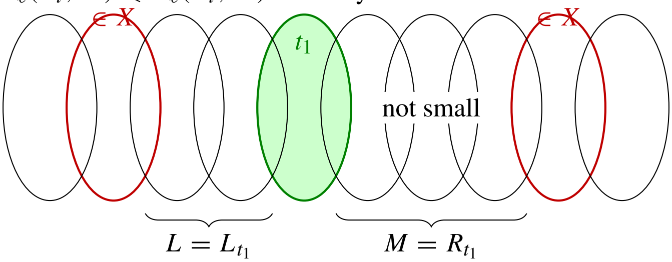
The former allows us to maintain the invariant that $T - r^*$ is an elimination forest of the corresponding induced subgraph of G, and the latter helps us bound the height of T.

Initially, the tree T contains only the root r^*, the set X is empty, $\mathcal{I} = \{I\}$ where $I = V(P)$, and $v_I = r^*$. For this initial setup, Inv. 1 holds by the fact that $B(I) \cap V(T) = \emptyset$, whereas Inv. 2 holds by (6) and the fact that depth$(v_I) = 0$.

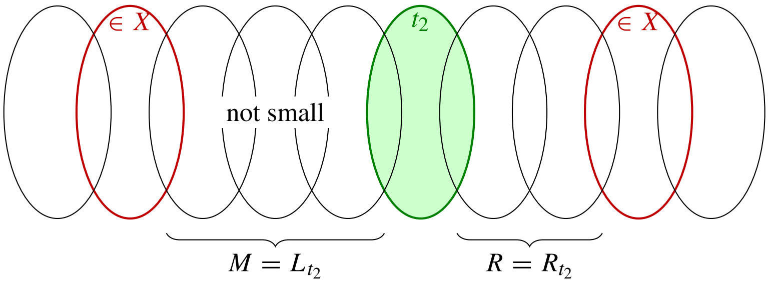
Every round consists in choosing an arbitrary interval $I \in \mathcal{I}$ and adding one or two nodes of I to X. As a result, the interval I is replaced in $\mathcal{I}$ by at most three of its proper subintervals.



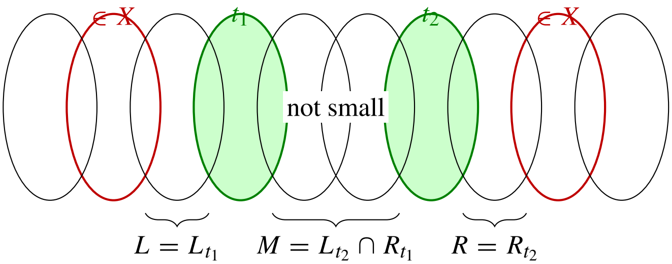
if $w_\ell(L_t, T') \leqslant w_\ell(R_t, T')$ for every small node $t \in I$:



if $w_\ell(L_t, T') > w_\ell(R_t, T')$ for every small node $t \in I$:



otherwise:



Now, we describe the details of a single round. Pick an interval $I \in \mathcal{I}$. Let

$$\ell = \text{level}(I) \qquad \text{and} \qquad m = \max(\{0\} \cup \{i \in \{1, \dots, \ell\} \mid \mathcal{L}_i(I) - V(T) = \emptyset\}).$$

It follows that

$$\text{for every node } t \in I, \text{ at least } m \text{ vertices of } B_t \text{ are already in } T. \tag{8}$$

Indeed, this is true if $m = 0$, and if $m > 0$, this follows from (3) and the fact that all vertices of $\mathcal{L}_m(I)$ are already in T. For every $i \in \{1, \dots, \ell\}$, since every vertex in $\mathcal{L}_i(I) - \text{int}(I)$ belongs to the bag of a neighbor of I in P, which belongs to X, we have

$$\mathcal{L}_i(I) - V(T) \subseteq \mathcal{L}_i(I) - B(X) \subseteq \text{int}(I). \tag{9}$$

Let $k = \ell - m$. (We may have $m = \ell$ and $k = 0$.) For every $i \in \{m + 1, \dots, \ell\}$, by (9), we have $x_i(I, T) = |\mathcal{L}_i(I) - V(T)| \geqslant 1$. It follows that

$$w_\ell(I, T) = \sum_{i=1}^{\ell} \log(x_i(I, T) + 1) \geqslant \sum_{i=m+1}^{\ell} \log(x_i(I, T) + 1) \geqslant \ell - m = k. \tag{10}$$

Choose one vertex from the set $\mathcal{L}_i(I) - V(T)$ for each $i \in \{m + 1, \dots, \ell\}$, and add these at most k vertices into T as a path with one end attached to v_I. That is, the first vertex is added as a child of v_I and every further vertex is added as a child of the previous one. Let v'_I be the last such vertex (i.e., the other end of the path) if $k \geqslant 1$, and let $v'_I = v_I$ if $k = 0$. Let T' denote the resulting augmented tree.

Call a node $t \in I$ *small* if $|B_t| \leqslant \ell + k$. By the definition of level(I), at least one node in I is small. It follows from (8) that

$$\text{for each small node } t \in I,\ \text{at most } \ell + k - m = 2k \text{ vertices of } B_t \text{ are not yet in } T'. \tag{11}$$

Recall that we think of I as ordered from left to right. For every node $t \in I$, let L_t and R_t denote the sets of nodes of I to the left and to the right of t, respectively, so that $I = L_t \cup \{t\} \cup R_t$. If $w_\ell(L_t, T') \leqslant w_\ell(R_t, T')$ for every small node $t \in I$, then let t_1 be the rightmost small node in I, and let $L = L_{t_1}$ and $M = R_{t_1}$. Similarly, if $w_\ell(L_t, T') > w_\ell(R_t, T')$ for every small node $t \in I$, then let t_2 be the leftmost small node in I, and let $M = L_{t_2}$ and $R = R_{t_2}$. Otherwise, let t_1 be the rightmost small node in I such that $w_\ell(L_{t_1}, T') \leqslant w_\ell(R_{t_1}, T')$ and t_2 be the leftmost small node in I such that $w_\ell(L_{t_2}, T') > w_\ell(R_{t_2}, T')$. In this case, by (5), t_1 and t_2 occur in this order from left to right, and there are no small nodes between them. Now, let $L = L_{t_1}$, $R = R_{t_2}$, and $M = R_{t_1} \cap L_{t_2}$ (i.e., M is the set of nodes strictly between t_1 and t_2). See the figure.

Whenever t_1 and t_2 are defined, we add them to X. We remove I from $\mathcal{I}$, and whenever L, M, and R are defined and non-empty, we add them as new intervals to $\mathcal{I}$.

Now, we add the vertices of $B_{t_1} - V(T')$ and $B_{t_2} - V(T')$ (whenever t_1 or t_2 are defined) to T' as one path with one end attached to v'_I. That is, the first such vertex is a child of v'_I, and every further vertex is a child of the previous. Note that possibly both sets $B_{t_1} - V(T')$ and $B_{t_2} - V(T')$ are empty; in particular, this happens when $k = 0$. Let v''_I be the last vertex added if at least one vertex was added, and let $v''_I = v'_I$ otherwise. Let T'' denote the new tree. By (11), we have added at most $4k$ additional vertices, so

$$\text{depth}(v_I'') \leqslant \text{depth}(v_I') + 4k = \text{depth}(v_I) + 5k. \tag{12}$$

Whenever L, M, or R is defined and non-empty, we set the corresponding vertex v_L, v_M, or v_R to be v_I''. By Inv. 1 for I, it follows that the path from r^* to v_I'' contains every vertex of $B(I) \cap V(T'')$, which yields Inv. 1 for L, M, and R (when they are defined and non-empty).

Before verifying Inv. 2, let us capture the key properties of L, M, and R. If L is defined and non-empty (so that $L = L_{t_1}$), then let $\overline{L} = R_{t_1}$. If R is defined and non-empty (so that $R = R_{t_2}$), then let $\overline{R} = L_{t_2}$. Whenever the respective sets are defined, we have

$$w_\ell(L, T') \leqslant w_\ell(\overline{L}, T') \quad \text{and} \quad w_\ell(R, T') \leqslant w_\ell(\overline{R}, T'), \tag{13}$$

$$w_\ell(M, T'') \leqslant w_\ell(I, T), \tag{14}$$

$$\text{level}(M) \geqslant \ell + k + 1, \tag{15}$$

where (14) follows from (5), and (15) follows as there are no small nodes in M.

While a bound analogous to (14) holds also for $w_\ell(L, T'')$ and $w_\ell(R, T'')$, we need a stronger one. First we focus on the interval L, and the argument is symmetric for the interval R. For $i \in \{1, \dots, \ell\}$, we compare $x_i(I, T)$ with $x_i(L, T')$ and $x_i(\overline{L}, T')$. Note that $\text{int}(L)$ and $\text{int}(\overline{L})$ are vertex-disjoint and are both contained in $\text{int}(I)$. For each $i \in \{1, \dots, \ell\}$, we have $\mathcal{L}_i(L) \subseteq \mathcal{L}_i(I)$ and $\mathcal{L}_i(\overline{L}) \subseteq \mathcal{L}_i(I)$, by (2). For each $i \in \{m+1, \dots, \ell\}$, we have put one vertex of $\mathcal{L}_i(I) - V(T)$ into T'; this vertex belongs to $\text{int}(I)$ by (9). This, the property (1), and the definition of x_i imply that for each $i \in \{1, \dots, \ell\}$, we have

$$x_i(I, T) \geqslant \begin{cases} x_i(L, T') + x_i(\overline{L}, T') & \text{if } i \leqslant m, \\ x_i(L, T') + x_i(\overline{L}, T') + 1 & \text{if } i > m. \end{cases}$$

Since $x_i(L, T')$ and $x_i(\overline{L}, T')$ are non-negative, the above implies

$$\begin{aligned} & x_i(I, T) + 1 \\ & \quad \geqslant \begin{cases} x_i(L, T') + x_i(\overline{L}, T') + 1 \geqslant \frac{1}{2}(x_i(L, T') + x_i(\overline{L}, T') + 2) & \text{if } i \leqslant m, \\ x_i(L, T') + x_i(\overline{L}, T') + 2 & \text{if } i > m. \end{cases} \end{aligned} \tag{16}$$

Recalling that $k = \ell - m$, we calculate

$$\begin{aligned} w_\ell(I, T) &= \sum_{i=1}^{\ell} \log(x_i(I, T) + 1) \\ &\geqslant \sum_{i=1}^{\ell} \log(x_i(L, T') + x_i(\overline{L}, T') + 2) - m && \text{by (16)} \end{aligned}$$

$$\begin{aligned}
&\geqslant \sum_{i=1}^{\ell} \frac{\log(x_i(L,T')+1)+\log(x_i(\overline{L},T')+1)}{2} + \ell - m && (*)\\
&= \tfrac{1}{2}(w_\ell(L,T') + w_\ell(\overline{L},T')) + k \\
&\geqslant w_\ell(L,T') + k && \text{by (13)}\\
&\geqslant w_\ell(L,T'') + k && \text{by (5)},
\end{aligned}$$

where in $(*)$, we use the inequality $\log(x+y) = \log\frac{x+y}{2} + 1 \geqslant \frac{1}{2}(\log x + \log y) + 1$ that follows from the concavity of log. From this and the analogous argument for R, we conclude that

$$w_\ell(L,T'') + k \leqslant w_\ell(I,T) \quad \text{and} \quad w_\ell(R,T'') + k \leqslant w_\ell(I,T). \tag{17}$$

Now, we are set to verify Inv. 2 for the intervals L, R, and M (when they are defined and non-empty). We have

$$\begin{aligned}
&\tfrac{1}{5}\operatorname{depth}(v_L) + w_{\text{level}(L)}(L,T'')\\
&\quad\leqslant \tfrac{1}{5}\operatorname{depth}(v_I) + k + w_\ell(L,T'') + b(\text{level}(L) - \ell)\\
&\qquad + \log\left(\frac{\text{level}(L)!}{\ell!}\right) && \text{by (12) and (7)}\\
&\quad\leqslant \tfrac{1}{5}\operatorname{depth}(v_I) + w_\ell(I,T) + b(\text{level}(L) - \ell)\\
&\qquad + \log\left(\frac{\text{level}(L)!}{\ell!}\right) && \text{by (17)}\\
&\quad\leqslant (b+1)\ell + \log(\ell!) + b(\text{level}(L) - \ell)\\
&\qquad + \log\left(\frac{\text{level}(L)!}{\ell!}\right) && \text{by Inv. 2 for } I\\
&\quad\leqslant (b+1)\,\text{level}(L) + \log(\text{level}(L)!).
\end{aligned}$$

The exact same bounds hold with L replaced by R. Finally, for M, we have

$$\begin{aligned}
&\tfrac{1}{5}\operatorname{depth}(v_M) + w_{\text{level}(M)}(M,T'')\\
&\quad\leqslant \tfrac{1}{5}\operatorname{depth}(v_I) + k + w_\ell(M,T'') + b(\text{level}(M) - \ell)\\
&\qquad + \log\left(\frac{\text{level}(M)!}{\ell!}\right) && \text{by (12) and (7)}\\
&\quad\leqslant \tfrac{1}{5}\operatorname{depth}(v_I) + k + w_\ell(I,T) + b(\text{level}(M) - \ell)\\
&\qquad + \log\left(\frac{\text{level}(M)!}{\ell!}\right) && \text{by (14)}\\
&\quad\leqslant (b+1)\ell + \log(\ell!) + k + b(\text{level}(M) - \ell)\\
&\qquad + \log\left(\frac{\text{level}(M)!}{\ell!}\right) && \text{by Inv. 2 for } I
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (b+1)\ell + \operatorname{level}(M) - \ell + b(\operatorname{level}(M) - \ell) \\
&\quad + \log(\operatorname{level}(M)!) && \text{by (15)} \\
&= (b+1)\operatorname{level}(M) + \log(\operatorname{level}(M)!).
\end{aligned}
$$

This completes the round of our process for the interval I, with T'' becoming the new tree T. We have shown that both invariants, Inv. 1 and Inv. 2 are preserved.

The process ends when all vertices of G have been added to T. It remains to show that $T - r^*$ is an elimination forest of G with height at most $10ab$. To see that it is an elimination forest, observe that whenever a vertex $v \in \operatorname{int}(I)$ is added to T when considering an interval I, Inv. 1 guarantees that all neighbors of v in G that are already in T lie on the path from r^* to v in T, as the neighbors of v in G belong to $B(I)$ by the definition of path decomposition.

The height of the forest $T - r^*$ is equal to $\max\{\operatorname{depth}(v) \mid v \in V(G)\}$. Let v be a vertex of G. Consider the moment in the process when v has been added to T. Say, it happened when processing an interval I with $\operatorname{level}(I) = \ell$. Let m and k be the values fixed when processing I. Clearly, we have $\operatorname{depth}(v) \leqslant \operatorname{depth}(v_I'')$ and $\ell \leqslant a$. Therefore,

$$
\begin{aligned}
\operatorname{depth}(v) &\leqslant \operatorname{depth}(v_I'') \\
&\leqslant \operatorname{depth}(v_I) + 5k && \text{by (12)} \\
&\leqslant \operatorname{depth}(v_I) + 5w_\ell(I, T) && \text{by (10)} \\
&\leqslant 5(b+1)\ell + 5\log(\ell!) && \text{by Inv. 2} \\
&\leqslant 5(b+1)a + 5\log(a!) \\
&\leqslant 5ab + 5a\log a + 5a.
\end{aligned}
$$

Recall that $2^b \geqslant 2a$ and thus $b \geqslant \log(2a) = \log a + 1$. It follows that

$$\operatorname{depth}(v) \leqslant 5ab + 5a\log a + 5a \leqslant 10ab.$$

We conclude that $T - r^*$ is an elimination forest of G with height at most $10ab$, as desired.

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Authors and Affiliations

Meike Hatzel[1] · Gwenaël Joret[2] · Piotr Micek[3] · Marcin Pilipczuk[4] · Torsten Ueckerdt[5] · Bartosz Walczak[3]

Meike Hatzel
meikehatzel@nii.ac.jp

Piotr Micek
piotr.micek@uj.edu.pl

Marcin Pilipczuk
malcin@mimuw.edu.pl

Torsten Ueckerdt
torsten.ueckerdt@kit.edu

Bartosz Walczak
bartosz.walczak@uj.edu.pl

[1] National Institute of Informatics, Tokyo, Japan

[2] Computer Science Department, Université libre de Bruxelles, Brussels, Belgium

[3] Department of Theoretical Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland

[4] Institute of Informatics, University of Warsaw, Warsaw, Poland

[5] Computer Science Department, Karlsruhe Institute of Technology, Karlsruhe, Germany