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Orthogonal Polynomials on Bubble-Diamond Fractals

Elena Axinn[1] · Calvin Osborne[2] · Kasso A. Okoudjou[3] · Olivia Rigatti[4] · Helen Shi[3]

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Abstract

We develop a theory of polynomials and, in particular, an analog of the theory of Legendre orthogonal polynomials on the bubble-diamond fractals, a class of fractal sets that can be viewed as the completion of a limit of a sequence of finite graph approximations. In this setting, a polynomial of degree j can be viewed as a multiharmonic function, a solution of the equation $\Delta^{j+1}u = 0$. We prove that the sequence of orthogonal polynomials we construct obeys a three-term recursion formula.

Keywords Spectral graph theory · Legendre orthogonal polynomials · Pcf fractals · Bubble-diamond fractals

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✉ Kasso A. Okoudjou
kasso.okoudjou@tufts.edu

Elena Axinn
eaxinn@umich.edu

Calvin Osborne
cosborne@alumni.harvard.edu

Olivia Rigatti
origatt@ncsu.edu

Helen Shi
yues.0510@gmail.com

[1] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

[2] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

[3] Department of Mathematics, Tufts University, Medford, MA 02155, USA

[4] Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA

1 Introduction

During the last two decades, a theory of calculus on fractal sets such as the Sierpinski gasket (SG) has been developed. It is based on the spectral analysis of the fractal Laplacian [2, 8, 11]. In this context, a polynomial of degree j on SG is a solution of the equation $\Delta^{j+1}u = 0$, where Δ denoted the Laplacian on SG [5, 10]. Although most aspects of the theory of polynomials in this setting parallel their counterparts on the unit interval, several striking differences exist. In particular, there is no analog of the Weierstrass Theorem on SG; that is, the set of polynomials on SG is not complete on $L^2(SG)$ [5, Theorem 4.3.6]. Furthermore, the space of polynomials on SG is not an algebra under pointwise multiplication. However, a theory of orthogonal polynomials on SG was initiated in [9] and resulted in an analog of Legendre orthogonal polynomials on $[-1, 1]$.

This paper studies the bubble-diamond fractals, a class of fractals defined by a branching parameter $b \in \mathbb{N}$. One significant interest in exploring this class of fractals is that they present different geometrical properties, including a wide range of Hausdorff and spectral dimensions. In general, diamond-type self-similar graphs have provided an essential collection of structures with interesting physical and mathematical properties and a broad variety of geometries [1, 3, 6]. The structure of these fractals is such that they combine spectral properties of Dyson hierarchical models and transport properties of one-dimensional chains. In what follows, we will denote by K_b the bubble-diamond fractal with branching parameter $b \in \mathbb{N}$, see Definition 2.5.

In this paper, we develop a theory of polynomials and orthogonal polynomials theory on this class of bubble-diamond fractals. First, in Sect. 2, we define the bubble-diamond fractal K_b as a particular completion of a limit of self-similar bubble-diamond graphs introduced in [7]. Subsequently, we introduce the main analytic tools on K_b: the Laplacian and the Green operator. Next, in Sect. 3, we define and investigate the fundamental properties of polynomials on the bubble-diamond fractals following the approach developed in [11]. Finally, in Sect. 4, we introduce a class of monomials and construct analogues of the Legendre orthogonal polynomials on K_b, providing asymptotic results on relevant coefficients and defining a three-term recursion relation for the Legendre orthogonal polynomials.

2 Bubble-Diamond Fractals

In this section, we define the bubble-diamond fractals as the completion of the limit of finite graph approximations. To this end, we will first define these finite graph approximations, then construct a suitable metric on the union of these finite graph approximations, and finally define the bubble-diamond fractals. Throughout the process, we will develop several analytic tools on K_b that will be essential in creating a theory of polynomials on the bubble-diamond fractal.

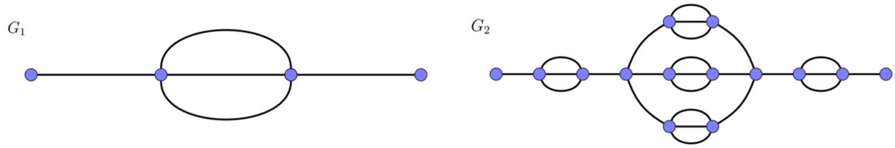

Fig. 1 Bubble-diamond graphs G_1 and G_2 for $b = 3$

2.1 Analytic Tools on Finite Bubble-Diamond Graphs

First, we will construct a sequence of finite bubble-diamond graphs G_ℓ that approximate K_b. In the process, we will develop several analytic tools, including the graph Laplacian Δ_ℓ and Green's function G_ℓ, on these finite graph approximations that we will extend to K_b in the following subsection. Note below, as will occur often throughout this paper, that we drop the dependence on the branching parameter $b \in \mathbb{N}$ to simplify our notation.

Definition 2.1 For an integer $b \geq 1$, the *bubble-diamond graphs with branching parameter* b are an inductively constructed sequence of graphs $G_\ell = (V_\ell, E_\ell)$ for $\ell \in \mathbb{N}$, where V_ℓ and E_ℓ are respectively the sets of vertices and edges of G_ℓ.

First, we will let $G_0 = (V_0, E_0)$ be the graph with two vertices $V_0 = \{q_1, q_2\}$ joined by an edge $(q_1, q_2) \in E_0$. Then, at level $\ell \in \mathbb{N}$, we construct G_ℓ by modifying each edge from $G_{\ell-1}$ by introducing two new vertices, each of which is joined to one of the two original vertices by an edge and which are joined to each another by b distinct edges. Note in particular that $V_0 \subset V_1 \subset \ldots$ for each subsequent level of bubble-diamond graphs.

For a particular instance of bubble-diamond graphs, see G_1 and G_2 for $b = 3$ in Fig. 1.

There is an equivalent construction of the vertices V_ℓ of the bubble-diamond graphs through a set of mappings that will be important in constructing polynomials on the bubble-diamond fractals. To see this, we note that we can realize the general structure of $G_{\ell+1}$ by gluing together $b + 2$ copies of G_ℓ as in Figure 2. This gives us a series of maps $F_i^\ell : V_\ell \to V_{\ell+1}$ for $i \in \{1, 2, \ldots, b+2\}$ given by

$$F_i^\ell(p) = \text{the corresponding point to } p \text{ in } G_\ell^{\text{copy } i}. \tag{2.1}$$

One important observation is that $F_i^\ell(p) = F_i^{\ell+n}(p)$ for all $n > 0$ by considering $V_\ell \subset V_{\ell+n}$, and so we will drop the dependence on ℓ in the notation $F_i : V_\ell \to V_{\ell+1}$. With this map defined, we can construct

$$V_{\ell+1} = \bigcup_{i \in \{1,2,\ldots,b+2\}} F_i(V_\ell), \tag{2.2}$$

which is equivalent to the direct geometric construction of $V_{\ell+1}$ above.

We will now develop a number of analytic tools on the finite graph approximations of K_b. First, we define the *graph Laplacian* of a function f defined on the vertices V_ℓ

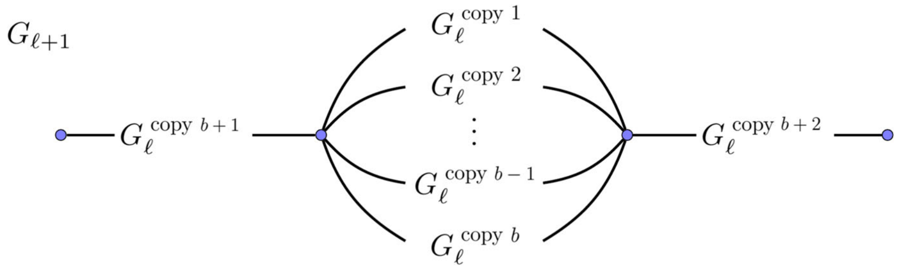

Fig. 2 Constructing $G_{\ell+1}$ by gluing together $b+2$ copies of G_ℓ

of a graph approximation G_ℓ given by

$$\Delta_\ell f(p) = \left(\frac{1}{\deg(p)} \sum_{p\sim q} f(q)\right) - f(p), \tag{2.3}$$

where the sum is taken over every edge $\{p, q\} \in E_\ell$. The finite graph Laplacian has an associated energy functional

$$\mathcal{E}_\ell(f) = \left(\frac{b}{2b+1}\right)^{-\ell} \sum_{p\sim q} |f(p) - f(q)|^2 \tag{2.4}$$

known as the *graph energy*. The constant $r = b/(2b+1)$ in the above formula is known as the *renormalization constant* for the bubble-diamond graphs, which is defined so that $\mathcal{E}_\ell$ satisfies the following energy-minimizing property.

Proposition 2.2 *For any function f defined on V_ℓ and any $\ell' \geq \ell$, we have that $\mathcal{E}_\ell(f) \leq \mathcal{E}_{\ell'}(\widetilde{f})$ for any extension $\widetilde{f}$ of f to $V_{\ell'}$, i.e., for any function $\widetilde{f}$ on $V_{\ell'}$ that satisfies $\widetilde{f}|_{V_\ell} = f$.*

Proof From the second construction of G_ℓ above, it is clear that it suffices to show that this property holds from $\ell = 0$ to $\ell' = 1$. Letting $V_1 = \{q_1, p_1, p_2, q_2\}$ be the vertices of G_1, we compute for any function f on V_0 and extension $\widetilde{f}$ to V_1 that

$$\mathcal{E}_1(\widetilde{f}) = r^{-1}\left((f(q_1) - \widetilde{f}(p_1))^2 + (f(q_2) - \widetilde{f}(p_2))^2 + b(\widetilde{f}(p_1) - \widetilde{f}(p_2))^2\right).$$

By taking partial derivatives, we find that $\mathcal{E}_1(\widetilde{f})$ is minimized when

$$\begin{cases} \widetilde{f}(p_1) = \left(\dfrac{b+1}{2b+1}\right) f(q_1) + \left(\dfrac{b}{2b+1}\right) f(q_2), \\ \widetilde{f}(p_2) = \left(\dfrac{b}{2b+1}\right) f(q_1) + \left(\dfrac{b+1}{2b+1}\right) f(q_2), \end{cases}$$

which gives us exactly the minimum graph energy $\mathcal{E}_1(\widetilde{f}) = \mathcal{E}_0(f)$ as is desired. $\square$

We further note that the graph energy induces a bilinear form through the polarization identity

$$\mathcal{E}_\ell(f, g) = \frac{1}{4}\left(\mathcal{E}_\ell(f+g) - \mathcal{E}_\ell(f-g)\right). \tag{2.5}$$

To see how the graph Laplacian Δ_ℓ and graph energy $\mathcal{E}_\ell$ on the finite bubble-diamond graphs are associated, we will turn to the Gauss-Green formula in the following subsection.

Lastly, we will construct a function G_ℓ on $V_\ell \times V_\ell$ that will serve as an analogue of *Green's function* in classical calculus. To this end, we will define

$$G_\ell(p, q) = \sum_{m=0}^{\ell} \sum_{x,y \in V_{m+1}/V_m} g(x, y)\psi_x^{(m+1)}(p)\psi_y^{(m+1)}(q), \tag{2.6}$$

where $g(x, x) = \alpha r^m$ and $g(x, y) = \beta r^m$ for $x \neq y$ with constants $\alpha = r^2(b+1)/b$ and $\beta = r^2$, and where $\psi_x^{(m)}$ is the *piecewise harmonic function* on V_m defined by $\psi_x^{(m)}(p) = \delta_{xp}$. Again, to see why G_ℓ is an analogue of Green's function on V_ℓ, we will turn to solving the Poisson equation in the following subsection.

2.2 Bubble-Diamond Fractals

We will now construct the bubble-diamond fractals from the above finite graph approximations G_ℓ. First, we will define the *graph approximation*

$$V_* = \bigcup_{\ell \geq 0} V_\ell. \tag{2.7}$$

We note that we can extend in a natural way the maps $F_i : V_\ell \to V_{\ell+1}$ that define the finite bubble-diamond graphs to maps $F_i : V_* \to V_*$ on the graph approximation. We will now extend the graph energy as defined in (2.4) to the graph approximation in the following way.

Definition 2.3 The *graph energy* of a function f defined on V_* is defined to be

$$\mathcal{E}(f) = \lim_{\ell \to \infty} \mathcal{E}_\ell(f|_{V_\ell}). \tag{2.8}$$

From this, we will define the *resistance metric* $R(p, q)$ between points $p, q \in V_*$ to be the minimum value of R such that

$$|f(p) - f(q)|^2 \leq R\mathcal{E}(f) \tag{2.9}$$

for all $f \in \operatorname{dom} \mathcal{E}$, i.e., for all functions f on V_* such that $\mathcal{E}(f) < \infty$.

As with above, we extend the graph energy on V_* to a bilinear form $\mathcal{E}(f, g)$ through the polarization identity (2.5). Both the graph energy and resistance metric play an important role for the graph approximation, as shown in the following result.

Proposition 2.4 *The graph energy $\mathcal{E}$ is an inner product on $\operatorname{dom}\mathcal{E}/(\text{constant functions})$. Furthermore, the resistance metric is a metric on V_*.*

Proof It is clear that $\mathcal{E}$ is symmetric, linear in both arguments, and satisfies $\mathcal{E}(f) \geq 0$, so it remains to show that $\mathcal{E}(f) = 0$ if and only if f is constant. We recall by Proposition 2.2 that $\mathcal{E}_\ell$ is nondecreasing as $\ell \to \infty$, and so we have that

$$\mathcal{E}(f) = 0 \Leftrightarrow \mathcal{E}_\ell(f) = 0 \text{ for all } \ell \in \mathbb{N}.$$

It is clear from (2.4) that $\mathcal{E}_\ell(f) = 0$ if and only if $f|_{V_\ell}$ is constant, and so $\mathcal{E}(f) = 0$ if and only if f is constant on V_* as is desired.

Next, it is clear that R is symmetric and positive-definite, so it remains to show that R satisfies the triangle inequality. To this end, we will consider distinct $p, q, z \in V_*$, and we note that there is some $\ell \in \mathbb{N}$ such that $p, q, z \in V_\ell$. It suffices to consider the case that $p \sim q$ and $q \sim z$ in V_ℓ. We will show the result when $\ell = 1$, where we will let $V_1 = \{q_1, p_1, p_2, q_2\}$ be the vertices of G_1, and we note that the result generalizes for $\ell > 1$ by the energy-minimizing algorithm in Proposition 2.2. We will assume without a loss of generality that $(p, q, z) = (q_1, p_1, p_2)$. Recall that $r = b/(2b + 1)$. First, we have from (2.4) for any function $f \in \operatorname{dom}\mathcal{E}$ that

$$\begin{aligned} r\mathcal{E}(f) \geq r\mathcal{E}_1(f) &\geq |f(p) - f(q)|^2 + b|f(q) - f(z)|^2 \\ &\geq \left(\frac{b}{b+1}\right)^2 |f(p) - f(z)|^2 + b\left(\frac{1}{b+1}\right)^2 |f(p) - f(z)|^2 \\ &\geq \left(1 + \frac{1}{b}\right)^{-1} |f(p) - f(z)|^2 \end{aligned}$$

by taking the energy-minimizing value of $f(q) = 1/(b+1)f(p) + b/(b+1)f(z)$. Then, we will consider the function $f_p \in \operatorname{dom}\mathcal{E}$ defined by extending

$$f_p|_{V_1}(q_1) = 1, \quad f_p|_{V_1}(p_1) = f_p|_{V_1}(p_2) = f_p|_{V_1}(q_2) = 0$$

to V_* by the energy-minimizing algorithm in Proposition 2.2. We compute that $\mathcal{E}(f_p) = \mathcal{E}_1(f_p|_{V_1}) = r^{-1}$ and that $|f_p(p) - f_p(q)|^2 = 1$, and so $R(p, q) \geq r$. Similarly, by extending the function $f_z \in \operatorname{dom}\mathcal{E}$ defined on V_1 by

$$f_z|_{V_1}(q_1) = f_z|_{V_1}(p_1) = 0, \quad f_z|_{V_1}(p_2) = f_z|_{V_1}(q_2) = 1,$$

we compute that $\mathcal{E}(f_z) = \mathcal{E}_1(f_z|_{V_1}) = br^{-1}$ and that $|f_z(q) - f_z(z)| = 1$, and so $R(q, r) \geq r/b$. Putting this together, we have for any $f \in \operatorname{dom}\mathcal{E}$ that

$$|f(p) - f(z)|^2 \leq \left(1 + \frac{1}{b}\right) r\mathcal{E}(f) \leq (R(p, q) + R(q, z))\,\mathcal{E}(f),$$

and so we have that $R(p, z) \leq R(p, q) + R(q, z)$ with R satisfying the triangle inequality as is desired. $\square$

We quickly remark that we can show using the above result that $\operatorname{dom}\mathcal{E}/$ (constant functions) is a Hilbert space with respect to $\mathcal{E}$. With the above result, we can finally construct the bubble-diamond fractals as the completion of the graph approximation.

Definition 2.5 The *bubble-diamond fractal* K_b *with branching parameter* b is the R-completion of V_*.

Similar to above, we can equivalently construct the bubble-diamond fractals as the unique compact set K_b that satisfies

$$K_b = \bigcup_{i\in\{1,2,\dots,b+2\}} F_i(K_b). \tag{2.10}$$

We will now extend the analytic tools from above to K_b. First, we will inductively define a *self-similar measure* μ on K_b given by $\mu(K_b) = 1$ and

$$\mu(A) = \left(\frac{1}{b+2}\right) \sum_{i\in\{1,2,\dots,b+2\}} \mu(F_i^{-1}A). \tag{2.11}$$

This scaling condition uniquely defines μ on K_b [10, Section 1.2]. With this measure, we can define the Laplacian on K_b.

Definition 2.6 Let $u \in \operatorname{dom}\mathcal{E}$ and f a continuous on K_b. We say that $u \in \operatorname{dom}\Delta$ with $\Delta u = f$ if

$$\mathcal{E}(u, g) = -\int_{K_b} f g \, d\mu \tag{2.12}$$

for all $g \in \operatorname{dom}\mathcal{E}$ with boundary conditions $g(q_1) = g(q_2) = 0$.

The connection between our formulation of the Laplacian Δ on K_b and the graph Laplacian Δ_ℓ on V_ℓ as defined in (2.3) is made clear in the following *pointwise formula*.

Proposition 2.7 *For any* $f \in \operatorname{dom}\Delta$ *and* $p \in V_* \setminus V_0$, *we have that*

$$\Delta f(p) = \left(\frac{2}{b+1}\right) \lim_{m\to\infty} \left(\frac{r}{b+2}\right)^{-m} \Delta_m f(p). \tag{2.13}$$

Conversely, suppose f *is continuous on* K_b *and the right side of* (2.13) *converges uniformly to a continuous function on* $V_* \setminus V_0$. *Then,* $u \in \operatorname{dom}\Delta$ *and* (2.13) *holds.*

Furthermore, the Laplacian satisfies the following scaling identity

$$\Delta(f \circ F_i) = \left(\frac{r}{b+2}\right) (\Delta f \circ F_i). \tag{2.14}$$

Proof By [10, Theorem 2.2.1], it suffices to compute the integral over K_b of the piecewise harmonic function $\psi_x^{(m)}$ on V_m defined by $\psi_x^{(m)}(p) = \delta_{xp}$. First, we note that each $p \in V_m$ is an element of $(b+1)$ many m-*cells* $F(K_b) = (F_{i_1} \circ \cdots \circ F_{i_m})(K_b)$. On each m-cell $F(K_b)$ with boundary points $\{F(q_1), F(q_2)\}$, we note, since there are $(b+2)^m$ many m-cells in K_b, that

$$\sum_{q \in V_*} \int_{K_b} \psi_q^{(m)} d\mu = \int_{K_b} d\mu = 1 \Rightarrow \int_{F(K_b)} \left(\psi_{F(q_1)}^{(m)} + \psi_{F(q_2)}^{(m)} \right) d\mu = \frac{1}{(b+2)^m}$$

$$\Rightarrow \int_{F(K_b)} \psi_x^{(m)} dm = \frac{1}{2(b+2)^m}.$$

By summing the above integral over the $(b+1)$ many m-cells containing $x \in V_m$ to compute the integral over K_b, we recover the above formula as is desired. The scaling identity is then an immediate consequence of the pointwise formula. □

Now that we have defined Δ on K_b, we can return to establish the correspondence between Δ and $\mathcal{E}$ that was alluded to in the above subsection. To do so, we will first define the normal derivative of a function f on K_b at a boundary point $q \in V_0$.

Definition 2.8 The *normal derivative* of a function f on K_b at a boundary point $q \in V_0$ is given by

$$\partial_n f(q) = \lim_{m \to \infty} r^{-m} \big(f(q_i) - f\big(x_m\big) \big), \tag{2.15}$$

where $x_m = (F_{b+1} \circ \cdots \circ F_{b+1})(q_2)$ if $q = q_1 \in V_0$ and $x_m = (F_{b+2} \circ \cdots \circ F_{b+2})(q_1)$ if $q = q_2 \in V_0$, both with m applications of F_i. In particular, the normal derivative exists for all $u \in \operatorname{dom} \Delta$ [10, Theorem 2.3.2].

Finally, with these definition of the graph energy, Laplacian, and normal derivative on K_b, we can finally state the Gauss-Green formula that will be essential in our development of a theory of polynomials on K_b.

Proposition 2.9 [10, Theorem 2.3.2] *For any $u \in$ dom Δ, we have that*

$$\mathcal{E}(u, g) = -\int_{K_b} (\Delta u) g d\mu + \sum_{V_0} g(\partial_n u) \tag{2.16}$$

holds for all $v \in$ dom $\mathcal{E}$.

The last analytic tool that we will develop on K_b is finishing our construction of Green's formula. We will extend the construction from (2.6) of Green's formula on $V_\ell \times V_\ell$ to $K_b \times K_b$ in the following way.

Definition 2.10 We define *Green's function* on $V_* \times V_*$ to be

$$G(p, q) = \lim_{\ell \to \infty} G_\ell(p, q), \tag{2.17}$$

which we then continuously extend to be a function on $K_b \times K_b$.

This construction of Green's function will allow us to solve the *Dirichlet problem* $\Delta u = f$ with the boundary conditions $u(q_1) = u(q_2) = 0$ for any continuous function f on K_b, or, equivalently, to show that the restriction of the Laplacian Δ to the domain $u \in \text{dom}\,\Delta$ with $u(q_1) = u(q_2) = 0$ is invertible.

Proposition 2.11 [10, Theorem 2.6.1] *The Dirichlet problem $\Delta u = f$ with $u(q_1) = u(q_2) = 0$ has a unique solution $u \in \text{dom}\,\Delta$ for any continuous function f on K_b given by*

$$u(p) = -\int_{K_b} G(p,q)f(q)d\mu(q). \tag{2.18}$$

One important observation that we will make here is that, when $b = 1$, the bubble-diamond fractal K_b reduces to the unit interval. In this case, we have further that the Laplacian, normal derivative, and Green's function respectively reduce to the second derivative, first derivative, and Green's function from classical calculus. This will continue for the rest of the paper, where our construction of the monomials and the Legendre polynomials will reduce down similarly to the classical case.

3 Polynomials on the Bubble-Diamond Fractals

We now define and investigate the class of polynomials on K_b. As motivation, we first define and investigate the properties of harmonic functions, i.e., polynomials of degree zero as are defined below. To construct these harmonic functions, we propose an extension algorithm. Subsequently, we use an approach developed in [11] to construct higher-order polynomials. In particular, as seen below, a *polynomial* on K_b can be viewed as a multiharmonic function.

Definition 3.1 For each $j \geq 0$, we let $\mathcal{H}_j = \{f : \Delta^{j+1} f = 0\}$ be the space of all *polynomials* of degree at most j.

The next result proves that $\mathcal{H}_j$ is a vector space of dimension $2j+2$. More specifically, the following statements motivated by [11, Lemma 2.2, Lemma 2.3] hold.

Proposition 3.2 *For any $j \geq 0$ and $k = 1$ or 2, let f_{jk} be the solution to $\Delta^{j+1} f_{jk} = 0$ that satisfies the boundary conditions*

$$\Delta^m f_{jk}(q_n) = \delta_{jm}\delta_{kn} \quad \textit{for} \quad n = 1, 2.$$

(a) For each $j \geq 0$, the set $\{f_{mk} : m = 0, 1, \ldots, j;\ k = 1, 2\}$ is a basis for $\mathcal{H}_j$. Furthermore, $f \in \mathcal{H}_j$ if and only if

$$f = \sum_{m=0}^{j} \sum_{k=1}^{2} (\Delta^m f(q_k)) f_{mk}. \tag{3.1}$$

(b) For $i \in \{1, 2, \dots, b+2\}$ *and* $k \in \{1, 2\}$*, we have that*

$$f_{jk} \circ F_i = \sum_{m=0}^{j} \sum_{n=1}^{2} \left(\frac{r}{b+2} \right)^m f_{(j-m)k}(F_i q_n) f_{mn}. \tag{3.2}$$

Proof The first part follows by observing that both sides of (3.1) belong to $\mathcal{H}_j$. The second part is an application of (2.14). □

The scaling identity (3.2) will be particularly important in constructing the multiharmonic functions f_{jk}, as it will suffice to compute the values of $f_{jk}(F_i q_n)$ to construct the multiharmonic functions on all of V_*.

3.1 Harmonic Functions

To motivate the construction of polynomials on K_b, we first describe the basis for the space $\mathcal{H}_0$ of harmonic functions by explicitly computing the harmonic functions f_{01}, f_{02}, which we will do in the sequel.

We recall from Proposition 2.2, given a function f on V_0, that the energy minimizing extension $\widetilde{f}$ of f to V_1 is given by

$$\begin{cases} \widetilde{f}(F_{b+1}q_2) = \left(\dfrac{b+1}{2b+1} \right) f(q_1) + \left(\dfrac{b}{2b+1} \right) f(q_2), \\ \widetilde{f}(F_{b+2}q_1) = \left(\dfrac{b}{2b+1} \right) f(q_1) + \left(\dfrac{b+1}{2b+1} \right) f(q_2). \end{cases}$$

From this observation, we iteratively compute the values of the harmonic function f_{01} on V_* by letting $f_{01}(q_1) = 1$ and $f_{01}(q_2) = 0$ and then defining

$$\begin{pmatrix} f_{01}(F_i q_1) \\ f_{01}(F_i q_2) \end{pmatrix} = A_i \begin{pmatrix} f_{01}(q_1) \\ f_{01}(q_2) \end{pmatrix}, \tag{3.3}$$

where A_i are the *harmonic extension matrices*

$$A_{b+1} = \begin{pmatrix} 1 & 0 \\ \dfrac{b+1}{2b+1} & \dfrac{b}{2b+1} \end{pmatrix}, \quad A_{b+2} = \begin{pmatrix} \dfrac{b}{2b+1} & \dfrac{b+1}{2b+1} \\ 0 & 1 \end{pmatrix},$$

$$A_i = \begin{pmatrix} \dfrac{b+1}{2b+1} & \dfrac{b}{2b+1} \\ \dfrac{b}{2b+1} & \dfrac{b+1}{2b+1} \end{pmatrix}$$

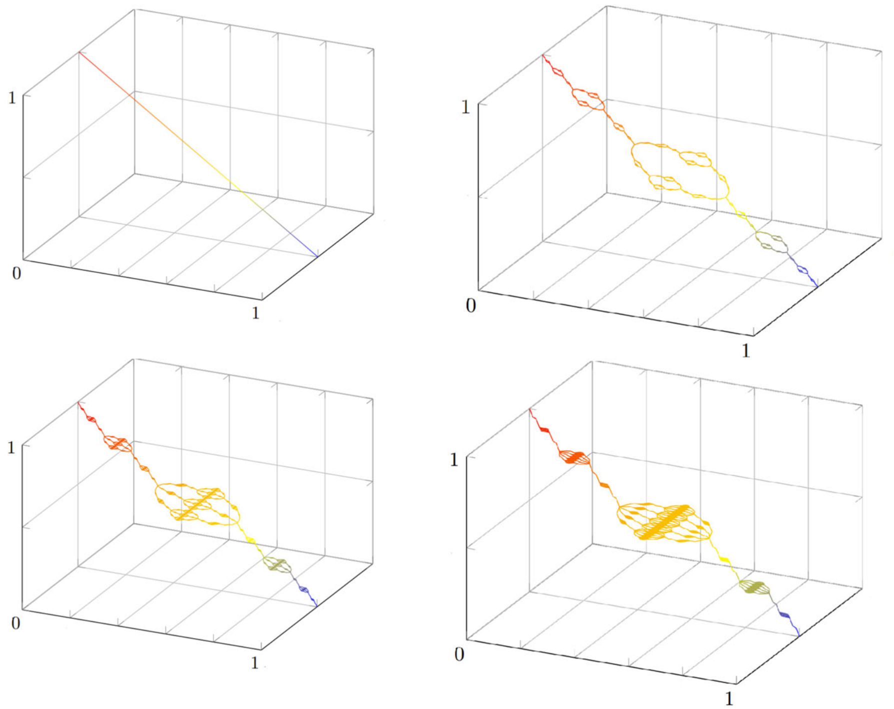

Fig. 3 The harmonic function f_{01} on the bubble-diamond fractals with branching parameter $b = 1, 2, 3, 5$

for $i \in \{1, 2, \ldots, b\}$. We repeat this *harmonic extension algorithm* iteratively to compute the values of f_{01} on the rest of V_*, and we perform a similar computation to construct f_{02}.

We illustrate several harmonic functions f_{01} for different branching parameters, see Fig. 3. Of particular interest, note that the harmonic extension algorithm generates f_{01} as a linear function on the real line when $b = 1$.

3.2 Higher Order Polynomials

In this section, for all $j \geq 0$, we evaluate $f_{jk}(F_i q_n)$ for all $i \in \{1, 2, \ldots, b+2\}$ and $k \in \{1, 2\}$. These values together with (3.2) are needed to construct the multiharmonic basis $\{f_{01}, f_{02}, f_{11}, f_{12}, \ldots f_{j1}, f_{j2}\}$ for $\mathcal{H}_j$. Motivated by the results proved in [11], we need to evaluate the inner products of the basis elements, i.e.,

$$I(jk, j'k') = \int_{K_b} f_{jk} f_{j'k'} d\mu. \tag{3.4}$$

for all $j, j' \geq 0$ and $k, k' \in \{1, 2\}$. Subsequently, we provide two recursive formulas for $f_{jk}(F_i q_n)$ and $I(jk, j'k')$ that can be simultaneously solved. In particular, we will define the quantities to be solved for as

$$\begin{cases} a_\ell = I(\ell k, 0k), & \\ b_\ell = I(\ell k, 0n) & \text{for } k \neq n, \\ p_\ell = \left(\frac{r}{b+2}\right)^\ell f_{\ell k}(F_i q_k)) = \left(\frac{r}{b+2}\right)^\ell f_{\ell k}(F_k q_i) & \text{for } i \neq k, \\ q_\ell = \left(\frac{r}{b+2}\right)^\ell f_{\ell k}(F_i q_n) & \text{for } i, k, n \text{ distinct.} \end{cases} \tag{3.5}$$

We begin by evaluating the initial value of these quantities. We note by the self-similarity of μ that

$$I(0k, 0k') = \int_{K_b} f_{0k} f_{0k'} d\mu = \frac{1}{b+2} \sum_{i=1}^{b+2} \int_{K_b} (f_{0k} \circ F_i)(f_{0k'} \circ F_i) d\mu.$$

It follows from (3.2) that the following equation holds

$$\begin{pmatrix} I(01,01) \\ I(01,02) \\ I(02,01) \\ I(02,02) \end{pmatrix} = \begin{pmatrix} A(11,11) & A(11,12) & A(11,21) & A(11,22) \\ A(12,11) & A(12,12) & A(12,21) & A(12,22) \\ A(21,11) & A(21,12) & A(21,21) & A(21,22) \\ A(22,11) & A(22,12) & A(22,21) & A(22,22) \end{pmatrix} \begin{pmatrix} I(01,01) \\ I(01,02) \\ I(02,01) \\ I(02,02) \end{pmatrix}, \tag{3.6}$$

where the values of $A(kk', nn')$ are computed as

$$A(kk', nn') = \frac{1}{b+2} \sum_{i=1}^{b+2} f_{0k}(F_i(q_n)) f_{0k'}(F_i(q_{n'}))$$

from the harmonic extension algorithm.

We observe by symmetry that $a_0 = I(01, 01) = I(02, 02)$ and $b_0 = I(01, 02) = I(02, 01)$ and that

$$\sum_{i=1}^{2} \sum_{j=1}^{2} I(0i, 0j) = \int_{K_b} f_{01}(f_{01} + f_{02}) d\mu + \int_{K_b} f_{02}(f_{01} + f_{02}) d\mu$$
$$= \int_{K_b} (f_{01} + f_{02}) d\mu = 1.$$

Consequently, we have that (3.6) reduces to

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} A(11,11) + A(11,22) & A(11,12) + A(11,21) \\ A(12,11) + A(12,22) & A(12,12) + A(12,21) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.$$

With this new matrix representation, we can now directly compute the initial values of a_ℓ, b_ℓ as

$$a_0 = \frac{b+1}{2+4b}, \qquad b_0 = \frac{b}{2+4b}, \tag{3.7}$$

since $a_0 + b_0 = 1/2$ and $\begin{pmatrix} a_0 & b_0 \end{pmatrix}^\top$ is an eigenvector of the above matrix with eigenvalue 1. Furthermore, we compute the initial values of p_ℓ, q_ℓ as

$$p_0 = \frac{b+1}{2b+1}, \qquad q_0 = \frac{b}{2b+1} \tag{3.8}$$

from the construction of the harmonic functions above and (3.2).

We will now state our two recursive formulas for computing a_ℓ, b_ℓ, p_ℓ, and q_ℓ. The first recursive formula for a_ℓ, b_ℓ is an application of [11, Lemma 2.4].

Theorem 3.3 *For any $j > 0$, the quantities a_j and b_j in (3.5) satisfy the recursive formula*

$$\begin{cases} \dfrac{(b+2)^{j+1}(2b+1)^{j+1}}{b^j} a_j & = \nu_1 a_j + \nu_2 b_j + (b+1) \displaystyle\sum_{\ell=0}^{j-1} (a_\ell + b_\ell)\left((b+1)p_{j-\ell} + b q_{j-\ell}\right), \\ \dfrac{(b+2)^{j+1}(2b+1)^{j+1}}{b^j} b_j & = \upsilon_1 a_j + \upsilon_2 b_j + (b+1) \displaystyle\sum_{\ell=0}^{j-1} (a_j + b_j)(b p_{j-\ell} + (b+1) q_{j-\ell}), \end{cases} \tag{3.9}$$

where

$$\nu_1 = \frac{2b^3 + 8b^2 + 7b + 2}{2b+1}, \quad \nu_2 = \frac{2b^3 + 6b^2 + 6b + 2}{2b+1}, \quad \upsilon_1 = \frac{2b^3 + 4b^2 + 2b}{2b+1},$$
$$\upsilon_2 = \frac{2b^3 + 6b^2 + 3b}{2b+1}$$

and the initial conditions a_0, b_0, p_0, and q_0 are given by (3.7) *and* (3.8).

Proof Let $\lambda = 1/(b+2)$ and $r = b/(2b+1)$. We compute (3.4) for $j' = 0$ to find that

$$\begin{aligned} \left(\frac{2b+1}{b[b+1]}\right)\left(\frac{(r\lambda)^{-j}}{\lambda}\right) a_j = & \sum_{\ell=0}^{j} \Bigg[a_\ell \left(\left[1 + \frac{1}{b}\right] p_{j-\ell} + q_{j-\ell} \right) \\ & + b_\ell \left(\left[1 - \frac{1}{b+1}\right] p_{j-\ell} + q_{j-\ell} \right) \Bigg] \\ & + \left(\frac{2b+1}{b[b+1]}\right) \sum_{\ell=0}^{j} \left(b_\ell p_{j-\ell}\right) + \left(\frac{2b+1}{b[b+1]}\right) a_j + \frac{b_j}{b} \\ = & \sum_{\ell=0}^{j} (a_\ell + b_\ell) \left(\left[1 + \frac{1}{b}\right] p_{j-\ell} + q_{j-\ell} \right) \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{2b+1}{b[b+1]}\right)a_j+\frac{b_j}{b}\\
&=\sum_{\ell=0}^{j-1}(a_\ell+b_\ell)\left(\left[1+\frac{1}{b}\right]p_{j-\ell}+q_{j-\ell}\right)\\
&+\frac{2b^3+8b^2+7b+2}{b(b+1)(2b+1)}a_j+\frac{2(b+1)^2}{b(2b+1)}b_j,
\end{aligned}
$$

which after simplifications yield the first equation as is desired. The second equation is obtained through a comparable method. □

The second recursive formula for p_ℓ, q_ℓ is an application of [11, Lemma 2.6].

Theorem 3.4 *For any $j > 0$, the quantities p_j and q_j in (3.5) satisfy the recursive formula*

$$
\begin{cases}
(2b+1)p_j &= -\sum_{\ell=0}^{j-1}\left[p_{j-\ell-1}((b+1)^2a_\ell+b^2b_\ell)+b(b+1)q_{j-\ell-1}(a_\ell+b_\ell)\right]-(b+1)b_{j-1},\\
(2b+1)q_j &= -\sum_{\ell=0}^{j-1}\left[b(b+1)p_{j-\ell-1}(a_\ell+b_\ell)+q_{j-\ell-1}((b+1)^2a_\ell+b^2b_\ell)\right]-bb_{j-1},
\end{cases}
\tag{3.10}
$$

where the initial conditions a_0, b_0, p_0, and q_0 are given by (3.7) and (3.8).

By solving (3.9) and (3.10) simultaneously, we are able to compute $f_{jk}(F_i q_n)$, and we then apply (3.2) to compute the values of f_{jk} on V_* so that f_{jk} can be continuously extended to functions on K_b. This completes our construction of a first basis for polynomials on the bubble-diamond fractals.

Figure 4 displays the basis function f_{jk} on the bubble-diamond fractal with branching parameter $b = 2$.

4 Orthogonal Polynomials on the Bubble–Diamond Fractals

Finally, in this section, we construct a sequence of orthogonal polynomials on the bubble-diamond fractals in analogy to the Legendre polynomials. To do so, we consider a different basis for $\mathcal{H}_j$, whose members we call monomials on the bubble-diamond fractals. When applying the Gram-Schmidt orthogonalization process to these monomials, we obtain the desired sequence of orthogonal polynomials on K_b. For comparison to the Sierpinski, we refer to [4, 9].

We will first define the monomial functions on K_b that we will later construct.

Definition 4.1 For each $j \geq 0$ and $k \in \{1, 2\}$, we will let $P_{jk} \in \mathcal{H}_j$ be the *monomials* defined by the boundary conditions

$$
\begin{cases}
\Delta^m P_{jk}(q_1) = \delta_{mj}\delta_{k1},\\
\partial_n \Delta^m P_{jk}(q_1) = \delta_{mj}\delta_{k2}
\end{cases}
$$

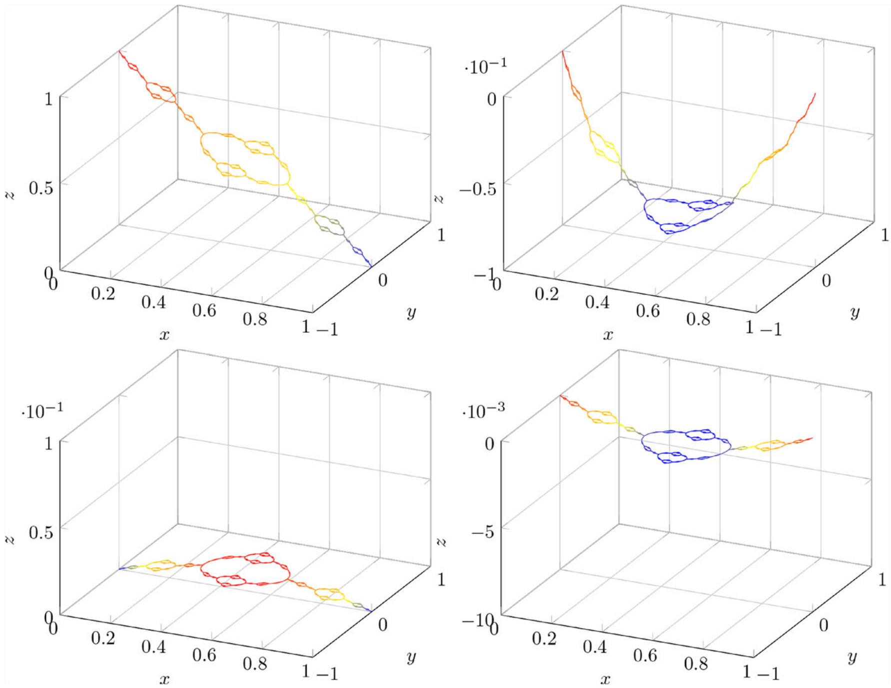

Fig. 4 The harmonic functions f_{01}, f_{11}, f_{21}, f_{31} on the bubble-diamond fractal with branching parameter $b = 2$

for all $m \in \{0, 1, \ldots, j\}$.

We observe that the set $\{P_{01}, P_{02}, \ldots, P_{j1}, P_{j2}\}$ in $\mathcal{H}_j$ is analogous to the set of monmials $x^{2j+k}/(2j+k)!$ on the real line. Furthermore, we note that this construction could have similarly been performed by using the boundary conditions at $q_2 \in V_0$.

As with the multiharmonic functions f_{jk}, we have a comparable scaling identity for P_{jk} that follows from Proposition 3.3.

Proposition 4.2 *For $i \in \{1, 2, \ldots, b+2\}$ and $k \in \{1, 2\}$, we have that*

$$P_{jk} \circ F_i = \begin{cases} \left(\dfrac{r}{b+2}\right)^j P_{j1} & k = 1, \\ r\left(\dfrac{r}{b+2}\right)^j P_{j2} & k = 2. \end{cases} \tag{4.1}$$

Proof Let $\lambda = 1/(b+2)$ and $r = b/(2b+1)$. By taking Laplacians, we compute using (2.14) that

$$\Delta^j(P_{j1} \circ F_1^m) = (r\lambda)^m P_{01} \circ F_1^m = (r\lambda)^m P_{01}$$

since $P_{01} = 1$ is a constant map. We finally note that $\Delta P_{(j+1)k} = P_{jk}$, which gives us the desired result for $k = 1$. The result for $k = 2$ is obtained through a comparable method. □

Similar to our application of Proposition 3.2 above, we can now use (4.1) to construct P_{jk} on all of V_* from just the value of $P_{jk}(q_2)$, which we compute by solving a pair of recursive formulas. To this end, we will define the quantities

$$\begin{cases} \alpha_j = P_{j1}(q_2), \\ \beta_j = P_{j2}(q_2), \\ \eta_j = \partial_n P_{j1}(q_2), \\ \gamma_j = \partial_n P_{j2}(q_2) \end{cases} \tag{4.2}$$

with initial values $\alpha_0 = 1$, $\alpha_1 = 1/2$, $\beta_0 = -1$, $\eta_0 = -1$, and $\gamma_0 = 1$ computed by noting that $P_{00} = f_{01} + f_{02}$ and $P_{01} = f_{02}$.

We will now state the two recursive formulas for α_j, β_j and η_j, γ_j, which are derived from the same method as [8, Theorem 2.3, Theorem 2.12].

Theorem 4.3 *For any $j > 0$, the quantities α_j, β_j, η_j, and γ_j in (4.2) satisfy the recursive formula*

$$\begin{cases} \alpha_j &= \zeta_j \cdot \displaystyle\sum_{\ell=1}^{j-1} \alpha_{j-\ell} \left(2\alpha_\ell + \sum_{\ell'=1}^{\ell} \alpha_{\ell-\ell'}\alpha_{\ell'} \right), \\ \beta_j &= \iota_j \cdot \displaystyle\sum_{\ell=0}^{j-1} \sum_{\ell'=0}^{j-\ell} \beta_\ell \alpha_{\ell'} \alpha_{j-\ell-\ell'}, \\ \eta_j &= b_{j-1} + \displaystyle\sum_{\ell=0}^{j} \alpha_{j-\ell}\alpha_{\ell-1}, \\ \gamma_j &= \displaystyle\sum_{\ell=0}^{j} \beta_{j-\ell}\alpha_{\ell-1}, \end{cases} \tag{4.3}$$

where

$$\zeta_j = \frac{(b+1)^2}{(b+2)(2b+1)((b+2)^{j-1}(2b+1)^{j-1}/b^{j-1} - 1)},$$
$$\iota_j = \frac{(b+1)^2}{(2b+1)((b+2)^j(2b+1)^j/b^j - 1)},$$

and the initial conditions α_0, α_1, β_0, η_0, and γ_0 are given above.

By applying the scaling identity (4.1), we once again compute the values of P_{jk} on V_* so that P_{jk} can be continuously extended to functions on K_b. This completes our construction of a second basis of monomials on the bubble-diamond fractals. For

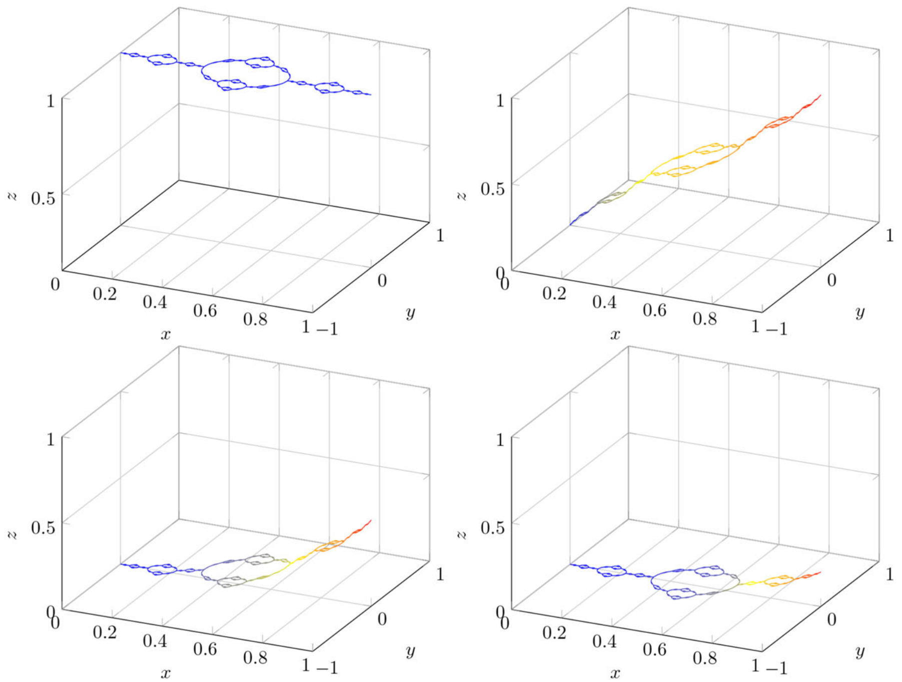

Fig. 5 The monomials $P_{01}, P_{02}, P_{11}, P_{12}$ on the bubble-diamond fractal with branching parameter $b = 2$

particular instances of P_{jk} on the bubble-diamond fractal with branching parameter $b = 2$, see Fig. 5.

Finally, we will construct the Legendre polynomials on K_b by performing the Gram-Schmidt orthogonalization process on the above monomial basis. To this end, the next result allows us to compute the inner products for the monomials defined in Definition 4.1

Theorem 4.4 *For all $j, j' \geq 0$, we have that*

$$\begin{cases} \langle P_{j1}, P_{j'1} \rangle &= \sum_{\ell=0}^{j} (\alpha_{j-\ell}\eta_{j'+\ell+1} - \alpha_{j'+\ell+1}\eta_{j-\ell}), \\ \langle P_{j1}, P_{j'2} \rangle &= \sum_{\ell=0}^{j} (\alpha_{j-\ell}\gamma_{j'+\ell+1} - \beta_{j'+\ell+1}\eta_{j-\ell}), \\ \langle P_{j2}, P_{j'2} \rangle &= \sum_{\ell=0}^{j} (\beta_{j-\ell}\gamma_{j'+\ell+1} - \beta_{j'+\ell+1}\gamma_{j-\ell}). \end{cases} \tag{4.4}$$

Proof Following the procedure of [9, Lemma 2.1], we find that

$$\langle P_{ji}, P_{ki'} \rangle = \sum_{\ell=0}^{j} \sum_{n=1}^{2} \left(P_{(j-\ell)i}(q_n)\partial_n P_{(k+\ell+1)i'}(q_n) - P_{(k+\ell+1)i'}(q_n)\partial_n P_{(j-\ell)i}(q_n) \right).$$

From the above definitions, this simplifies to

$$\langle P_{j1}, P_{k1}\rangle = \sum_{\ell=0}^{j} (\alpha_{j-\ell}\eta_{k+\ell+1} - \alpha_{k+\ell+1}\eta_{j-\ell}),$$

where we note that $P_{(k+\ell+1)i'}(q_1) = \partial_n P_{(k+\ell+1)i'}(q_1) = 0$ for all $k, \ell \geq 0$. The rest of the identities hold through a comparable method. □

We quickly note that we can alternatively express

$$\langle P_{j2}, P_{j'1}\rangle = \sum_{\ell=0}^{j} (\beta_{j-\ell}\eta_{j'+\ell+1} - \alpha_{j'+\ell+1}\gamma_{j-\ell})$$

by examining the symmetry of the coefficients.

We now apply the Gram-Schmidt orthogonalization process to the monomials $\{P_{j1}, P_{j2}\}_{j\geq 0}$. To make more clear the association with the Legendre polynomials from classical calculus, we will make the change in notation $P_{(2j+k)} = P_{jk}$ which makes $P_{(2j+k)} \in \mathcal{H}_j$. We will then construct a family of orthogonal polynomials as follows.

Definition 4.5 The *orthogonal polynomials* $\{p_j\}_{j\geq 0}$ are obtained by applying the Gram-Schmidt process to $\{P_j\}_{j\geq 1}$, where $p_0 = P_0$ and for $j \geq 1$ we have that

$$p_j(x) = P_j(x) - \sum_{\ell=0}^{j-1} d_\ell^2 \langle P_j, p_\ell\rangle p_\ell(x) \tag{4.5}$$

for coefficients $d_j = ||p_j||^{-1}$.

By normalizing this family of polynomials as $\pi_j = d_j p_j$, we obtain the analog for the Legendre orthogonal polynomials on K_b.

Definition 4.6 For each $j \geq 0$, we will call $\pi_j \in \mathcal{H}_j$ the *Legendre polynomial of degree* j as constructed above, which satisfies

$$\langle \pi_j, \pi_{j'}\rangle = \delta_{jj'} \tag{4.6}$$

for all $j, j' \geq 0$.

Figure 6 displays some of these orthogonal polynomials on a bubble-diamond fractal with branching parameter $b = 1, 2$. In particular, when $b = 1$, we observe that this construction recovers the classical Legendre orthogonal polynomials on the unit interval.

We show that, like in the classical case, the family of orthogonal polynomials $\{p_j\}_{j\geq 0}$ satisfies a three-term recursion formula. To this end, consider for all $j \geq 0$

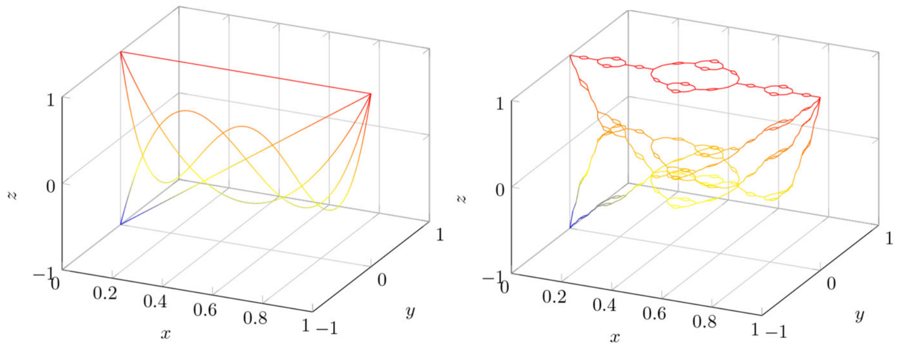

Fig. 6 The first five Legendre polynomials on the bubble-diamond fractals with branching parameter $b = 1, 2$

the auxiliary polynomial g_j given by

$$g_{j+1}(x) = -\int_{K_b} G(x, y) p_j(y)\, d\mu(y), \tag{4.7}$$

where G is Green's function defined in Proposition 2.10. It follows from construction that

$$\Delta g_{j+1} = p_j, \tag{4.8}$$

where $g_{j+1} \in \mathcal{H}_{j+1}$ is a polynomial of degree at most $j + 1$. As such, we have the following three-term recursion formula.

Theorem 4.7 *For all $j \geq 0$, we have that*

$$g_{j+1}(x) = p_{j+1}(x) + s_j p_j(x) + t_j p_{j-1}(x), \tag{4.9}$$

where $p_{-1} = g_0 = 0$. Furthermore, the coefficients $\{s_j\}_{j\geq 0}$ and $\{t_j\}_{j\geq 0}$ are given by

$$\begin{cases} s_j = d_j^2 \langle g_{j+1}, p_j \rangle, \\ t_j = d_{j-1}^2 d_j^{-2}. \end{cases} \tag{4.10}$$

In particular, we have that $d_j^{-2} = d_0^{-2} t_1 t_2 \ldots t_j$.

Proof For all $j \geq 0$ and $\ell \in \{0, 1, \ldots, j\}$, we see that

$$\langle g_{j+1}, p_\ell \rangle = -\iint_{K_b \times K_b} G(x, y) p_j(y) p_\ell(x) d\mu(y) d\mu(x) = \langle p_j, g_{\ell+1} \rangle. \tag{4.11}$$

Because $g_{\ell+1}$ is a polynomial of degree at most $\ell + 1$, we have that $g_{\ell+1}$ is orthogonal to all p_j for which $\ell + 1 < j$. Thus, the expansion of $g_{j+1} \in \mathcal{H}_{j+1}$ with respect to the set of orthogonal polynomials $\{p_\ell\}_{\ell=0}^{j+1}$ is of the form

$$g_{j+1} = ap_{j+1} + bp_j + cp_{j-1}. \tag{4.12}$$

We note that p_j can be expressed as a sum of P_j and lower-order terms, which implies that g_{j+1} can be expressed as a sum of P_{j+1} and lower order terms. We thus have that $g_{j+1} - p_{j+1}$ is orthogonal to p_{j+1}, and so $a = 1$. Taking the inner product of g_{j+1} with p_j, we find the coefficients s_j. Furthermore, taking the inner product of g_{j+1} with p_{j-1} yields

$$\begin{aligned} t_j\langle p_{j-1}, p_{j-1}\rangle &= \langle g_{j+1}, p_{j-1}\rangle \\ &= \langle p_j, g_j\rangle \\ &= \langle p_j, p_j + s_{j-1}p_{j-1} + t_{j-1}p_{j-2}\rangle \\ &= \langle p_j, p_j\rangle, \end{aligned}$$

which gives us the desired formula for the coefficients t_j. □

The following result about the orthonormal sequence $\{\pi_j\}_{j\geq 0}$ is a consequence of Theorem 4.7.

Corollary 4.8 *Let $\{\pi_j\}_{j=0}^{\infty}$ be the sequence of orthogonal polynomials given in Definition 4.6. Let $\tilde{g}_0 = 0$ and for $j \geq 0$ define*

$$\tilde{g}_{j+1}(x) = -\int_{K_b} G(x, y)\pi_j(y)d\mu(y).$$

The following three-term recursion formula holds.

$$\tilde{g}_{j+1}(x) = \sqrt{t_{j+1}}\pi_{j+1}(x) + s_j\pi_j(x) + \sqrt{t_j}\pi_{j-1}(x), \tag{4.13}$$

where the sequence $\{s_j\}_{j\geq 0}$ and $\{t_j\}_{j\geq 0}$ are defined in Theorem 4.7.

The following result shows that the sequences in the three-term recursion formula (4.9) are bounded.

Corollary 4.9 *Let $\{s_j\}_{j\geq 0}$ and $\{t_j\}_{j\geq 0}$ be defined as in Theorem 4.7. For each $j \geq 0$, we have that*

$$\begin{cases} 0 \leq t_j \leq \|G\|_2^2, \\ -\|G\|_2 \leq s_j \leq 0, \end{cases} \tag{4.14}$$

where $\|G\|_2$ is the L^2 norm of the Green function. Furthermore,

$$\|p_j\| = d_j^{-1} \leq d_0^{-1}\|G\|_2^j.$$

Proof The fact that $t_j \geq 0$ follows from the second equation in (4.10). In addition,

$$t_j d_{j-1}^{-2} = |\langle g_{j+1}, p_{j-1} \rangle| \leq \|g_{j+1}\|_2 d_{j-1} \leq \|G\|_2 d_j^{-1} d_{j-1}^{-1}.$$

Consequently, $t_j \leq \|G\|_2 d_{j-1} d_j^{-1} = \|G\|_2 t_j^{1/2}$, from which the first inequality in (4.14) follows.

The second inequality follows from writing the Green function in terms of the eigenvalues and eigenvectors of the Laplacian, see [9, Theorem 2.1]. □

5 Numerics

The figures and numerics included in this paper were generated by an implementation of the above constructions in a `Python` script by embedding K_b in $[0, 1]^2$, made available as a GitHub repository here. The included `README.md` file contains more specific details on the implementation of the code.

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Author Contributions All authors contributed equally to this work

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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