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RESEARCH PAPER

Structural and closed-form analysis of Jacobsthal and Jacobsthal Lucas p-numbers

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Abstract

This paper presents a generalization of Jacobsthal and Jacobsthal Lucas numbers, introducing a broader class known as Jacobsthal and Jacobsthal Lucas p-numbers. Each instance within this class possesses a unique Binet formula, extending the classical properties of Jacobsthal numbers and Jacobsthal-Lucas to a more flexible and comprehensive framework. By deriving individual Binet formulas for each p-number, this work lays the foundation for new analytical methods that connect integer sequences, irrational proportions, and complex numbers in novel ways. These generalized formulas aim to deepen our understanding of numerical structures and open new paths for applications across coding theory, mathematical modeling, and other fields where such recursive relationships prove essential. Also, this work provides the first known closed-form Binet formulas for Jacobsthal and Jacobsthal-Lucas p-numbers, offering a novel generalization that enriches the theory of recursive integer sequences.

Keywords Jacobsthal numbers · Jacobsthal-Lucas numbers · Binet formula · Jacobsthal p-numbers · Jacobsthal-Lucas p-numbers

Mathematics Subject Classification 11B83 · 11K31 · 05A18 · 11B37 · 11B39

Introduction

Number sequences play a crucial role in various fields of mathematics and science, as they provide structured ways to represent, analyze, and predict patterns in natural phenomena, data, and complex systems. From basic arithmetic progressions to more advanced sequences like those found in number theory and combinatorics, sequences help simplify complex relationships, making it possible to identify underlying mathematical structures. These patterns are not only essential in theoretical mathematics but are also fundamental in applications across computer science, physics, economics, and even art and nature, where they contribute to modeling growth patterns, optimizing algorithms, and revealing symmetry.

The Fibonacci sequence, one of the most famous number sequences, has roots dating back to the early thirteenth century, when Italian mathematician Leonardo of Pisa, known as Fibonacci, introduced it in his book Liber Abaci defined by the recursive formula, $F(n) = F(n-1) + F(n-2)$ with the initial terms 0 and 1, the sequence yields values like 0, 1, 1, 2, 3, 5, 8, and so on. The Fibonacci sequence has historical importance because it illustrated, among other things, the growth pattern of rabbit populations, showcasing the utility of mathematical sequences in modeling real-world phenomena.

As the Fibonacci sequence continued to evolve, mathematicians developed generalized forms by introducing variations in its terms, leading to the creation of sequences with distinct patterns known in the literature as k-sequences or p-sequences. These generalized forms allow for flexible recurrence relations, where the next term can depend on more than the previous two terms, as in the classical Fibonacci case. For example, in p-sequences, each term is derived from a recurrence that incorporates a fixed number of prior terms, broadening the possible applications and analytical depth of such sequences.

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These k-sequences and p-sequences have become invaluable in fields like cryptography, data compression, and complexity theory, as they offer novel structures for exploring self-similarity, stability, and growth in a variety of contexts. By expanding on the traditional Fibonacci model, researchers have gained powerful tools for examining the properties of recursive systems, understanding new classes of numbers, and discovering complex relationships in mathematical systems.

If we delve into these generalizations, we find that they extend the Fibonacci sequence by adjusting its recurrence relations and initial conditions, creating whole new families of sequences with unique properties.

One example: one generalization of Fibonacci numbers, introduced by Stakhov and Rozin (2006), is known as the Fibonacci p-numbers, denoted as $Fp(n)$. For any positive integer p, these numbers are defined by the recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

with the initial values $F_p(0) = 0, F_p(1) = 1, \ldots F_p(p) = 1$. When $p = 1$, this sequence reduces to the traditional Fibonacci numbers. Kiliç and Stakhov (2007) explore specific generalizations of the classical Fibonacci and Lucas numbers, referred to as generalized Fibonacci and Lucas p-numbers. They examine the relationships between the generalized Fibonacci p-numbers, $F_p(n)$, their cumulative sums, $\sum_{i=1}^{n} F_p(i)$ and the 1-factors of a certain class of bipartite graphs. Additionally, we identify particular matrices whose permanents produce the Lucas p-numbers and their cumulative sums. Also, in connection with Pascal's triangle, Kuhapatanakul (2016) examined the Fibonacci p-numbers and develop an explicit formula for them using certain properties of Pascal's triangle. Additionally, he introduced the companion matrix associated with the Fibonacci p-numbers and derive several identities for these numbers by utilizing properties of this matrix.

In the following years, many researchers have presented numerous studies related to these number sequences and their general forms. Some of these studies also encompass hybrids, duals, and quaternions, which are generalizations of complex numbers. For example, Prasad (2021) introduced dual complex Fibonacci p-numbers and explored several properties of these numbers, highlighting their connections to complex Fibonacci numbers and complex Fibonacci p-numbers. Some other studies and other studies related to two and three recurrence number sequences such as Padovan and Pell have also addressed (Deveci et al. 2024; Dişkaya and Menken 2024; Erdağ and Deveci 2022; Falcon 2024; Koçer and Alsan 2022; Kumari et al. 2022).

Also, Bednarz introduced new two-parameters generalization of the Fibonacci numbers (Bednarz 2021), Prasad introduced dual complex Fibonacci p-numbers and some properties of dual complex Fibonacci p-numbers (Prasad 2021), Moreover, Marin contributed to the literature by using similar techniques (Marin 1997, 2010). Also, if we touch upon the studies done with other number sequences, in (Kuloğlu and Özkan 2022; 2021) authors defined the generalized (k, r)—Gauss Pell numbers and new kinds of k-Pell and k-Pell-Lucas numbers related to the distance between numbers by using the definition of a distance between numbers. Then, they introduced incomplete Vieta-Pell and Vieta-Pell-Lucas polynomials and gave some properties these sequences (Kuloğlu et al. 2021).

In this context, the present study introduces a new class of sequences, namely the Jacobsthal and Jacobsthal-Lucas p-numbers, and derives their respective closed-form Binet formulas for arbitrary values of p. While classical Binet expressions are well established for Fibonacci-type sequences, such explicit formulas have not yet been extended to the Jacobsthal and Jacobsthal-Lucas p-families. The main scientific contribution of this work lies in the generalization of these sequences using higher-order recurrence relations, and in the development of analytical tools that can handle both positive and negative indices. These results offer a novel perspective for exploring recursive integer structures and form a foundational framework that may be applied in diverse mathematical and applied contexts, such as coding theory, combinatorics, and matrix-based modeling.

Basic properties of Jacobsthal p-number sequences

The Jacobsthal numbers are an integer sequence defined by a specific recurrence relation (Horadam 1996).

The sequence J_n is defined as:

$$J_n = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ J_{n-1} + 2J_{n-2}, & if\ n > 1 \end{cases} \tag{1}$$

Thus, each term in the Jacobsthal sequence is obtained by adding the previous term to twice the term before that. The initial terms are $0, 1, 1, 3, 5, 11, 21, \ldots$.

Jacobsthal numbers have a closed-form formula:

$$J_n = \frac{2^n - (-1)^n}{3} \tag{2}$$

This expression allows calculating any J_n directly without recursion.

Table 1 Extended Jacobsthal numbers

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$J_1(n)$	-11.2^{-5}	-5.2^{-4}	-3.2^{-3}	-2^{-2}	2^{-1}	0	1	1	3	5	11
$J_2(n)$	0	0	0	2^{-1}	0	0	1	1	1	3	5
$J_3(n)$	0	0	2^{-1}	0	0	0	1	1	1	1	3
$J_4(n)$	0	2^{-1}	0	0	0	0	1	1	1	1	1

The Jacobsthal-Lucas numbers j_n are closely related to the Jacobsthal sequence and follow a similar recurrence relation (Horadam 1996). They are defined as:

$$j_n = \begin{cases} 2, & if\ n = 0 \\ 1, & if\ n = 1 \\ j_{n-1} + 2j_{n-2}, & if\ n > 1 \end{cases} \tag{3}$$

The initial terms in this sequence are $2, 1, 5, 7, 17, 31, 65, \ldots$.

Jacobsthal-Lucas numbers have a closed-form formula:

$$j_n = 2^n + (-1)^n \tag{4}$$

These sequences share a structural resemblance to the Fibonacci numbers, making them useful in mathematical and applied contexts where similar recursive relationships are valuable.

Now, let's define Jacobsthal p-number inspired generalization of Fibonacci numbers, introduced by Stakhov and Rozin (2006), is known as the Fibonacci p-numbers, denoted as $Fp(n)$ we define the recurrence relation for the Jacobsthal p-number. For $p = 0,1,2,\ldots$ the recurrence relation is given as follows:

$$J_p(n) = J_p(n-1) + 2J_p(n-p-1) \tag{5}$$

for initial conditions $J_p(1) = s_1, J_p(2) = s_2, \ldots, J_p(p+1) = s_{p+1}$. where $s_1, s_2, \ldots, s_{p+1}. \in \mathbb{C}$.

In particular, we can take these initial conditions as follows:

$$J_p(n) = 1, n = 1,2,\ldots,p+1.$$

Varying the value of p in the recurrence relation results in the generation of unique numerical sequences. For example:

For $p = 0$:

$$J_0(n) = J_0(n-1) + 2J_0(n-1) = 3J_0(n-1).$$

If we take $J_0(1) = 1$, then $1,3,3^2,3^3,3^4,\ldots$ is obtained. The numerical values obtained match those in the OEIS under entry A000244 (N. J. A. Sloane 2025).

For $p = 1$:

$$J_1(n) = J_1(n-1) + 2J_1(n-2)$$

for initial conditions $J_1(0) = 0, J_1(1) = 1$. This recurrence relation generates Jacobsthal numbers Table 1). For the case where we define $J_p(n) = 1, n = 1,2,\ldots,p+1$ takes values of 0 or less, we can obtain its values as follows:

in the equation $J_p(p+1) = J_p(p) + 2J_p(0)$,

since $J_p(p+1)$ and $J_p(p)$ take values of 1, $J_p(0)$ becomes 0. Similarly,

$$J_p(-1) = 0, J_p(-2) = 0, \ldots, J_p(1-p) = 0,$$
$$J_p(-p) = 2^{-1}, J_p(-p-1) = 0, \ldots, J_p(-2p+1) = 0.$$

So, if we substitute the positive and negative values of n into the expression for $p = 1, 2, 3, 4$ we obtain the following table:

Exploring root properties in the characteristic equations of Jacobsthal and Jacobsthal-Lucas p-numbers

Theorem 1.1 *For $p \in \mathbb{Z}^+$, in the equation given as $a^{p+1} - a^p - 2 = 0$, there is a relationship among its roots as shown below:*

$$a_1 + a_2 + \ldots + a_p + a_{p+1} = 1, \tag{6}$$

$$a_1 a_2 \ldots a_p a_{p+1} = 2(-1)^p, \tag{7}$$

$$\begin{aligned} &a_1a_2 + a_1a_3 + \ldots + a_1a_{p+1} + a_2a_3 + a_2a_4 + \ldots + a_2a_{p+1} \\ &+ \ldots + a_{p-1}a_p + a_{p-1}a_{p+1} + a_pa_{p+1} = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} &a_1a_2a_3a_4 + a_1a_2a_3a_5 + \ldots + a_{p-2}a_{p-1}a_pa_{p+1} = 0, \\ &\vdots \\ &a_1a_2a_3 \ldots a_{p-2}a_{p-1}a_p + a_1a_3a_4 \ldots a_{p-1}a_pa_{p+1} \\ &+ \ldots + a_2a_3a_4 \ldots a_{p-1}a_pa_{p+1} = 0 \end{aligned} \tag{9}$$

Proof There are $p+1$ complex valued roots for the characteristic equation given as $a^{p+1} - a^p - 2 = 0$. So, based on this information we can write the characteristic equation as follows:

$$a^{p+1} - a^p - 2 = (a - a_1)(a - a_2)(a - a_3)\dots(a - a_p)(a - a_{p+1}) = 0$$

For the even value of p, we get

$$\begin{aligned} a^{p+1} - a^p - 2 &= (a - a_1)(a - a_2)(a - a_3)\dots(a - a_p)(a - a_{p+1}) \\ &= a^{p+1} - a^p(a_1 + a_2 + \dots + a_p + a_{p+1}) \\ &\quad + a^{p-1}(a_1a_2 + a_1a_3 + \dots + a_1a_{p+1} + a_2a_3 + a_2a_4 + \dots \\ &\quad + a_2a_{p+1} + \dots + a_{p-1}a_p + a_{p-1}a_{p+1} + a_pa_{p+1}) \\ &\quad - a^{p-2}(a_1a_2a_3 + a_1a_3a_4 + \dots + a_1a_pa_{p+1} + a_2a_3a_4 \\ &\quad + a_2a_3a_5 + \dots + a_2a_pa_{p+1} + \dots + a_{p-1}a_pa_{p+1}) \\ &\quad + a^{p-3}(a_1a_2a_3a_4 + a_1a_2a_3a_5 + \dots + a_{p-2}a_{p-1}a_pa_{p+1}) \\ &\quad + \dots + a(a_1a_2a_3\dots a_{p-2}a_{p-1}a_p + a_1a_3a_4\dots a_{p-1}a_pa_{p+1} \\ &\quad + \dots + a_2a_3a_4\dots a_{p-1}a_pa_{p+1}) - a_1a_2\dots a_pa_{p+1} = 0 \end{aligned}$$

Thus,

$$a_1 + a_2 + \dots + a_p + a_{p+1} = 1$$
$$a_1a_2\dots a_pa_{p+1} = 2$$

$$a_1a_2 + a_1a_3 + \dots + a_1a_{p+1} + a_2a_3 + a_2a_4 + \dots + a_2a_{p+1} + \dots + a_{p-1}a_p + a_{p-1}a_{p+1} + a_pa_{p+1} = 0$$

$$a_1a_2a_3a_4 + a_1a_2a_3a_5 + \dots + a_{p-2}a_{p-1}a_pa_{p+1} = 0$$

$$\vdots$$

$$a_1a_2a_3\dots a_{p-2}a_{p-1}a_p + a_1a_3a_4\dots a_{p-1}a_pa_{p+1} + \dots + a_2a_3a_4\dots a_{p-1}a_pa_{p+1} = 0.$$

For the odd value of we get

$$\begin{aligned} a^{p+1} - a^p - 2 &= (a - a_1)(a - a_2)(a - a_3)\dots(a - a_p)(a - a_{p+1}) \\ &= a^{p+1} - a^p(a_1 + a_2 + \dots + a_p + a_{p+1}) \\ &\quad + a^{p-1}(a_1a_2 + a_1a_3 + \dots + a_1a_{p+1} + a_2a_3 + a_2a_4 + \dots \\ &\quad + a_2a_{p+1} + \dots + a_{p-1}a_p + a_{p-1}a_{p+1} + a_pa_{p+1}) \\ &\quad - a^{p-2}(a_1a_2a_3 + a_1a_3a_4 + \dots + a_1a_pa_{p+1} + a_2a_3a_4 \\ &\quad + a_2a_3a_5 + \dots + a_2a_pa_{p+1} + \dots + a_{p-1}a_pa_{p+1}) \\ &\quad + a^{p-3}(a_1a_2a_3a_4 + a_1a_2a_3a_5 + \dots + a_{p-2}a_{p-1}a_pa_{p+1}) \\ &\quad + \dots + a(a_1a_2a_3\dots a_{p-2}a_{p-1}a_p + a_1a_3a_4\dots a_{p-1}a_pa_{p+1} \\ &\quad + \dots + a_2a_3a_4\dots a_{p-1}a_pa_{p+1}) - a_1a_2\dots a_pa_{p+1} = 0 \end{aligned}$$

Thus,

$$a_1 + a_2 + \dots + a_p + a_{p+1} = 1,$$

$$a_1a_2\dots a_pa_{p+1} = -2,$$

$$a_1a_2 + a_1a_3 + \dots + a_1a_{p+1} + a_2a_3 + a_2a_4 + \dots + a_2a_{p+1} + \dots + a_{p-1}a_p + a_{p-1}a_{p+1} + a_pa_{p+1} = 0,$$

$$a_1a_2a_3a_4 + a_1a_2a_3a_5 + \dots + a_{p-2}a_{p-1}a_pa_{p+1} = 0,$$

$$\vdots$$

$$a_1a_2a_3\dots a_{p-2}a_{p-1}a_p + a_1a_3a_4\dots a_{p-1}a_pa_{p+1} + \dots + a_2a_3a_4\dots a_{p-1}a_pa_{p+1} = 0.$$

As desired.

Theorem 1.2 *For $p \in \mathbb{Z}^+$ and $k \in \{1, 2, 3, \dots p\}$, in the equation given as $a^{p+1} - a^p - 2 = 0$, there is a relationship among its roots as shown below:*

$$(a_1 + a_2 + \dots + a_p + a_{p+1})^k = a_1^k + a_2^k + \dots + a_{p+1}^k = 1^k = 1 \tag{10}$$

Proof Since the sums of the double, triple, quadruple,... n-fold products of the terms given in the expansion of the characteristic equation are equal to 0, the proof of the theorem is clear.

The Binet approach for Jacobsthal and Jacobsthal-Lucas 2-number solutions

The equation, recurrence relation, and other relationships for $p = 2$ and the characteristic equation, recurrence relation and initial conditions are given as follows:

$$a^3 - a^2 - 2 = 0 \tag{11}$$

$$J_2(n) = J_2(n-1) + 2J_2(n-3),\ J_2(0) = 0,\ J_2(1) = 1,\ J_2(2) = 1$$

The roots are

$$a_1 \cong 1{,}6959$$

$$a_2 \cong -0{,}34781 - 1{,}02885i$$

$$a_3 \cong -0{,}34781 + 1{,}02885i$$

Binet formula for the Jacobsthal 2-numbers is

$$J_2(n) = k_1(a_1)^n + k_2(a_2)^n + k_3(a_3)^n$$

The numerical values of k_1, k_2, k_3 are solutions of the following system:

$$J_2(0) = 0 = k_1 + k_2 + k_3$$

$$J_2(1) = 1 = k_1(a_1)^1 + k_2(a_2)^1 + k_3(a_3)^1$$

$$J_2(2) = 1 = k_1(a_1)^2 + k_2(a_2)^2 + k_3(a_3)^2$$

The following values are obtained through basic algorithmic operations.

$k_1 = 0{,}323883, k_2 = -0{,}161942 + 0{,}164298i, k_3 = 0{,}161942 - 0{,}164298i$

Therefore, the Binets formula for the Jacobsthal 2-number is

$$J_2(n) = 0{,}323883(1{,}6959)^n + (-0{,}161942 + 0{,}164298i)(-0{,}34781 - 1{,}02885i)^n + (0{,}161942 - 0{,}164298i)(-0{,}34781 + 1{,}02885i)^n$$

Taking $k_1 = k_2 = k_3 = 1$, we obtain the Binets formula for the Jacobsthal Lucas 2-numbers.

$$j_2(n) = (1{,}6959)^n + (-0{,}34781 - 1{,}02885i)^n + (-0{,}34781 + 1{,}02885i)^n.$$

For $n = 0$ we obtain $j_2(0) = 3$, Accordingly to the following recurrence relation Jacobsthal Lucas 2-numbers.

$$j_2(n) = j_2(n-1) + 2j_2(n-3), j_2(0) = 3, j_2(1) = 1, j_2(2) = 1.$$

The Binet approach for Jacobsthal and Jacobsthal-Lucas 3-number solutions

The equation, recurrence relation, and other relationships for $p = 3$ are as follows:

$$a^4 - a^3 - 2 = 0 \tag{12}$$

$$J_3(n) = J_3(n-1) + 2J_3(n-4),$$
$$J_3(0) = 0, J_3(1) = 1,$$
$$J_3(2) = 1, J_3(3) = 1$$

the roots are

$$a_1 = -1$$

$$a_2 \cong 1,5437$$

$$a_3 \cong 0{,}22816 - 1{,}11514i$$

$$a_4 \cong 0,22816 + 1,11514i.$$

The roots of Eq. (12) were obtained with the help of the Mathematica program.

The equation that allows us to obtain the terms of the number Jacobsthal 3-numbers are as follows:

$$J_3(n) = k_1(a_1)^n + k_2(a_2)^n + k_3(a_3)^n + k_4(a_4)^n.$$

The numerical values of k_1, k_2, k_3, k_4 are solutions of the following system

$$J_3(0) = 0 = k_1 + k_2 + k_3 + k_4,$$

$$J_3(1) = 1 = k_1a_1 + k_2a_2 + k_3a_3 + k_4a_4,$$

$$J_3(2) = 1 = k_1(a_1)^2 + k_2(a_2)^2 + k_3(a_3)^2 + k_4(a_4)^2,$$

$$J_3(3) = 1 = k_1(a_1)^3 + k_2(a_2)^3 + k_3(a_3)^3 + k_4(a_4)^3.$$

The following values are obtained through basic algorithmic operations.

$k_1 = -0{,}142856, k_2 = 0{,}314979, k_3 = -0{,}0860619 + 0{,}183915i,$

$k_4 = -0{,}0860619 - 0{,}183915i$

Therefore, the Binets formula for the Jacobsthal 3-number is

$$J_3(n) = (-0{,}142856)(-1)^n + (0{,}314979)(1{,}5437)^n + (-0{,}0860619 + 0{,}183915i)(0{,}22816 - 1{,}11514i)^n + (-0{,}0860619 - 0{,}183915i)(0{,}22816 + 1{,}11514i)^n.$$

Taking $k_1 = k_2 = k_3 = k_4 = 1$, we obtain the Binets formula for the Jacobsthal Lucas 3-numbers. $j_2(n) = (-1)^n + (1{,}5437)^n + (0{,}22816 - 1{,}11514i)^n + (0{,}22816 + 1{,}11514i)^n$.

For $n=0$ we obtain $j_3(0) = 4$, accordingly to the following recurrence relation we obtain Jacobsthal Lucas 3-numbers.

$$j_3(n) = j_3(n-1) + 2j_3(n-4), j_3(0) = 4,$$
$$j_3(1) = 1, j_3(2) = 1 j_3(3) = 1.$$

The generalized Binet approach for Jacobsthal and Jacobsthal-Lucas p-number solutions

Theorem 1.3 *For $p > 0$, $p \in \mathbb{Z}$ any Jacobsthal p-number $J_p(n)$ can be represent as follows:*

$$J_p(n) = k_1(a_1)^n + k_2(a_2)^n + \cdots + k_p(a_p)^n + k_{p+1}(a_{p+1})^n \tag{13}$$

where $a_1, a_2, \ldots, a_{p+1}$ are the roots of characteristic equation $a^{p+1} - a^p - 2 = 0$ and $k_1, k_2, \ldots, k_{p+1}$ which are constant coefficients.

Proof Let us represent the Jacobsthal p-number $J_p(p+1)$ using (13):

$$J_p(p+1) = k_1(a_1)^{p+1} + k_2(a_2)^{p+1} + \cdots + k_p(a_p)^{p+1} + k_{p+1}(a_{p+1})^{p+1}.$$

The roots $a_1, a_2, \ldots, a_{p+1}$ of the characteristic equation have the following property:

We can perform the following transformation using the equation $a^{p+1} - a^p - 2 = 0$.

$$a_k^n - a_k^{n-1} - 2a_k^{n-p-1} = 0\,(for n = p+1)\,,$$
$$a_k^n = a_k^{n-1} + 2a_k^{n-p-1}$$

where $k = 1,2,\ldots,p+1$ and $n = 0, \pm 1, \pm 2, \ldots$.

We have

$$J_p\,(p+1) = 2\left[k_1 a_1^0 + k_2 a_2^0 + \ldots + k_{p+1} a_{p+1}^0\right] + \left[k_1 a_1^p + k_2 a_2^p + \ldots + k_{p+1} a_{p+1}^p\right].$$

Therefore, $J_p\,(p+1) = J_p\,(p) + 2J_p\,(0)$.

The basic recurrence relation holds for $J_p\,(p+1)$, and it can be easily shown that this formula applies to all positive values of n. Next, we demonstrate that the formula also holds for negative values of n. By substituting $n = -1$ into Eq. (1), we obtain

$$J_p\,(-1) = k_1 (a_1)^{-1} + k_2 (a_2)^{-1} + \ldots + k_p (a_p)^{-1} + k_{p+1} (a_{p+1})^{-1}.$$

Since $a_k^n - a_k^{n-1} = 2a_k^{n-p-1}$ for $n = p$ we obtain.

$$a_k^p - a_k^{p-1} = 2a_k^{-1} \text{ and } \frac{a_k^p - a_k^{p-1}}{2} = a_k^{-1}.$$

Therefore,

$$J_p\,(-1) = k_1\left(a_1^p - a_1^{p-1}\right)2^{-1} + k_2\left(a_2^p - a_2^{p-1}\right)2^{-1} + \ldots + k_{p+1}\left(a_{p+1}^p - a_{p+1}^{p-1}\right)2^{-1}$$

$$= 2^{-1}\left[k_1 a_1^p + k_2 a_2^p + \ldots + k_{p+1} a_{p+1}^p\right] - 2^{-1}\left[k_1 a_1^{p-1} + k_2 a_2^{p-1} + \ldots + k_{p+1} a_{p+1}^{p-1}\right]$$

and then

$$J_p\,(-1) = \left(J_p\,(p) - J_p\,(p-1)\right)2^{-1} = 0.$$

Similarly, the validity of formula (1) for all negative n values is straightforward to prove.

Theorem 1.4 *For $p > 0$, $p \in \mathbb{Z}$ the Binet's formula;*

$$j_p\,(n) = (a_1)^n + (a_2)^n + \cdots + (a_p)^n + (a_{p+1})^n.$$

Given that $a_1, a_2, \ldots, a_{p+1}$ are the roots for $a^{p+1} - a^p - 2 = 0$, they define the Jacobsthal-Lucas p-sequences $j_p\,(n)$ for $n = 0, \pm 1, \pm 2, \ldots$, which can be represented by the following recurrence relation:

$$j_p\,(n) = j_p\,(n-1) + 2j_p\,(n-p-1)\,,$$

$$j_p\,(0) = p+1, j_p\,(1) = j_p\,(2) = \ldots = j_p\,(p) = 1.$$

Proof To verify the validity of Theorem 1.4 for Jacobsthal-Lucas p-sequences, let us consider the case when $n = 0$. In this case, we express this formula as follows:

$$j_p\,(0) = (a_1)^0 + (a_2)^0 + \ldots + (a_p)^0 + (a_{p+1})^0 = p+1.$$

For $n = 1, 2, 3, \ldots, p$ we write;

$$j_p\,(1) = (a_1)^1 + (a_2)^1 + \ldots + (a_p)^1 + (a_{p+1})^1 = 1,$$

$$j_p\,(2) = (a_1)^2 + (a_2)^2 + \ldots + (a_p)^2 + (a_{p+1})^2 = 1^2,$$

$$\vdots$$

$$j_p\,(p) = (a_1)^p + (a_2)^p + \ldots + (a_p)^p + (a_{p+1})^p = 1^p.$$

Based on Theorem 1.2, the expressions we analyzed earlier simplify to 1, confirming that the formula is valid for $n = 1, 2, 3, \ldots, p$. To establish the correctness of this formula for any Jacobsthal-Lucas p-number $j_p\,(n)$ (where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$), derived from the recurrence relation with initial conditions, we employ a technique analogous to the one used in Theorem 1.3, utilizing identities that connect to the roots of the characteristic equation.

The primary contribution of Theorem 1.4, therefore, is the introduction of a novel family of recursive sequences governed by the same recurrence relation and initial conditions. This new family naturally extends the classical Lucas numbers, with the specific case of $p = 1$ corresponding to the standard Lucas sequence. Using Theorem 1.2, we derive the first terms of the Jacobsthal-Lucas p-numbers, as shown below:

$$j_p\,(0) = p+1, j_p\,(1) = j_p\,(2) = \ldots = j_p\,(p) = 1,$$

$$j_p\,(n) = j_p\,(n-1) + 2j_p\,(n-p-1)\,.$$

Conclusions

In this paper, we observe that extending these formulas to Jacobsthal numbers allows for a deeper exploration of numerical relationships. By incorporating complex numbers into this framework, we uncover connections similar linking key mathematical concepts. These relationships span integers (such as the Jacobsthal numbers), irrational numbers (including generalized golden ratios).

Although it is laborious to predict the exact applications of these new Binet formulas for Jacobsthal numbers, their introduction marks a significant advancement in mathematical research. This work builds upon the well-established theory of Binet formulas in Fibonacci and Lucas numbers, a field that continues to grow and expand in contemporary mathematics.

Furthermore, by introducing these Jacobsthal specific Binet formulas, along with related Jacobsthal functions and matrices, we present a novel mathematical framework for scientific inquiry, one that resonates with the principles of "Harmony Mathematics." This new framework shows potential for diverse applications, including in theoretical physics, where the Fibonacci sequence and golden ratio are already of great interest. We anticipate that these Jacobsthal-based Binet formulas will find use in coding theory, especially in systems that involve matrices of Jacobsthal numbers, much like the applications of Fibonacci matrices in similar fields.

Also, this study is primarily theoretical, Jacobsthal and Jacobsthal-Lucas p-numbers due to their structured recursive nature and closed-form expressions have potential relevance in several domains. These include coding theory, cryptographic algorithms, and recurrence-based models in computer science and discrete systems. The unique Binet formulas presented here can serve as mathematical tools in the development of matrix-based representations, especially in systems where generalized sequences improve performance or robustness, much like their Fibonacci counterparts.

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